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AN EXTENSION OF A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract

In this paper the authors introduce and investigate certain characterization properties of the class $\overline{H}_\lambda(\alpha,\beta)$ of analytic functions with negative coefficients as an extension of the subclass $\overline{H}(\alpha,\beta)$ due to Lashin.

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1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (1.1)$$

which are analytic in the unit disk $U = \{z: |z| \leq 1\}$. And let S denote the subclass of A consisting of univalent functions $f(z)$ in U.

A function $f(z)$ in S is said to be starlike of order α if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in U),$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all functions in S which are starlike of order α. Also, it is well known that

$$S^*(\alpha) \subseteq S^*(0) \equiv S^*.$$

A function $f(z)$ in S is said to be convex of order α in U if and only if

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in U),$$

For some α $(0 \le \alpha < 1)$. We denote by $K(\alpha)$ the class of all functions in S which are convex of order α.

The classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [1], and later studied by Schild [2], MacGregor [3] and Pinchuk [4].

Let T denote the subclass of S whose elements can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \ge 0). \tag{1.2}$$

Next we denote by $T^*(\alpha)$ and $C(\alpha)$ respectively, the classes obtained by taking the intercessions of $S^*(\alpha)$ and $K(\alpha)$ with T,

$$T^*(\alpha) = S^*(\alpha) \cap T \text{ and } C(\alpha) = K(\alpha) \cap T.$$

The classes $T^*(\alpha)$ and $C(\alpha)$ were introduced by silverman [5].

Let $H(\alpha, \beta)$ denote the class of functions $f(z) \in A$ which satisfy the condition

$$Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}\right) > \beta \tag{1.3}$$

for some $(\alpha \ge 0)$, $0 \le \beta < 1, \frac{f(z)}{z} \ne 0$ and $z \in U$.

The classes $H(\alpha, \beta)$ and $H(\alpha, 0)$ were introduced and studied by Obraddovic and Joshi [6], Padmanabhan [7], Li and Owa [8], Xu and Yang [9], Singh and Gupta [10] and others.

Further, we denote by $\overline{H}(\alpha, \beta)$ the class obtained by taking intercessions of the class $H(\alpha, \beta)$ with T, that is

$$\overline{H}(\alpha, \beta) = H(\alpha, \beta) \cap T. \tag{1.4}$$

We note that

$$\overline{H}(0, \beta) = T^*(\beta)$$

The class $\overline{H}(\alpha, \beta)$ was introduced by Lashin [11].

We note that the Binomial expansions of the functions $f(z)$ of forms (1.1) and (1.2) are respectively

$$h(z) = (z + \sum_{k=2}^{\infty} a_k z^k)^{\lambda} = z^{\lambda} + \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1} \tag{1.5}$$

and

$$F(z) = (z - \sum_{k=2}^{\infty} a_k z^k)^{\lambda} = z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1} \quad . \tag{1.6}$$

Following (1.3) and (1.4) by making use of (1.5) and (1.6) we obtain respectively the classes $H_{\lambda}(\alpha,\beta)$ and $\overline{H}_{\lambda}(\alpha,\beta)$:

$$(i) \quad H_{\lambda}(\alpha,\beta) \coloneqq \left\{ h(z) \in A: Re\left(\frac{\alpha z^2 h''(z)}{h(z)} + \frac{zh'(z)}{h(z)}\right) > \beta \right\} \quad z \in U \tag{1.7}$$

for some $(\alpha \geq 0)$, $0 \leq \beta < 1, \frac{h(z)}{z} \neq 0$ and $\lambda \in \mathbb{N}_0$.

(ii)

$$\overline{H}_{\lambda}(\alpha,\beta) = H_{\lambda}(\alpha,\beta) \cap T. \tag{1.8}$$

We note that

(i) $\overline{H}_1(\alpha,\beta) = \overline{H}(\alpha,\beta)$ (Lashin [11]),

(ii) $\overline{H}_1(0,\beta) = T^*(\beta)$ (Silverman [5]).

2. PRELIMINARIES

We shall need the following definition and lemma in the course of this work:

Following the earlier investigations of Goodman [12] and Ruscheweyh [13], we give the following definition:

Definition 1. The δ −neighborhood of function $h(z) \in T$ by:

$$N_{\delta}(f) = \left\{ F \in T: F(z) = z - \sum_{k=2}^{\infty} b_k(\lambda) z^{k+\lambda-1}, \sum_{k=2}^{\infty} k|a_k(\lambda) - b_k(\lambda)| \leq \delta \right\}.$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_\delta(f)=\left\{F\in T: F(z)=z-\sum_{k=2}^{\infty} b_k(\lambda)z^{k+\lambda-1}, \sum_{k=2}^{\infty} k|b_k(\lambda)|\le\delta\right\}. \qquad (2.1)$$

Where

$$F(z)=g^\lambda(z)=\left(z-\sum_{k=2}^{\infty} b_k z^k\right)^\lambda.$$

Lemma 1. ([14]). If h and F are analytic in U with $h\prec F$, then

$$\int_0^{2\pi}\left|F\left(re^{i\theta}\right)\right|^\delta d\theta\le\int_0^{2\pi}\left|h\left(re^{i\theta}\right)\right|^\delta d\theta.$$

Where $\delta>0$, $z=re^{i\theta}$ and $0<r<1$.

3. COEFFICIENT ESTIMATES FOR THE CLASS $\overline{H}_\lambda(\alpha,\beta)$

Our first theorem in this section is a necessary and sufficient condition for $h(z)\in T$ to belong to the class $\overline{H}_\lambda(\alpha,\beta)$.

Theorem 3.1 *A function $h(z)\in T$ is in the class $\overline{H}_\lambda(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}[(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\beta)]a_k(\lambda)$$

$$\le 1-\beta-(\alpha\lambda+1)(1-\lambda) \qquad (3.1)$$

Proof. Assume that the inequality (3.1) holds and let $|z|<1$. Then

$$Re\left(\frac{\alpha z^2h''(z)}{h(z)}+\frac{zh'(z)}{h(z)}\right)>\beta.$$

This implies that

$$\left|\frac{\alpha z^2h''(z)}{h(z)}+\frac{zh'(z)}{h(z)}-1\right|>\beta-1 \qquad (3.2)$$

Therefore by substituting for the first and the second derivatives of $h(z)$ in rhs of (3.2) we have

$$\left|\frac{\alpha z^2h''(z)+zh'(z)-h(z)}{h(z)}\right|$$

$$= \left| \frac{\alpha z^2 \lambda(\lambda-1) z^{\lambda-2} - \alpha z^2 \sum_{k=2}^{\infty} (k+\lambda-1)(k+\lambda-2) a_k(\lambda) z^{k+\lambda-3} + \lambda z^{\lambda-1} z - z \sum_{k=2}^{\infty} (k+\lambda-1) a_k(\lambda) z^{k+\lambda-2} - z^{\lambda} + \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k+\lambda-1}}{z^{\lambda} - \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k+\lambda-1}} \right|$$

$$= \left| \frac{\alpha \lambda(\lambda-1) z^{\lambda} - \alpha \sum_{k=2}^{\infty} (k+\lambda-1)(k+\lambda-2) a_k(\lambda) z^{k+\lambda-1} + \lambda z^{\lambda} - \sum_{k=2}^{\infty} (k+\lambda-1) a_k(\lambda) z^{k+\lambda-1} - z^{\lambda} + \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k+\lambda-1}}{z^{\lambda} - \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k+\lambda-1}} \right|$$

$$= \left| \frac{\alpha \lambda(\lambda-1) + \alpha \sum_{k=2}^{\infty} (k+\lambda-1)(k+\lambda-2) a_k(\lambda) z^{k-1} + \lambda - \sum_{k=2}^{\infty} (k+\lambda-1) a_k(\lambda) z^{k-1} - 1 + \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k-1}}{1 - \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k-1}} \right|$$

after dividing through by z^{λ} and collecting like terms.

So that a simple computation now yields

$$\left| \frac{(\alpha\lambda+1)(1-\lambda) + \{\sum_{k=2}^{\infty} [(k+\lambda-2)(\alpha k + \alpha\lambda - \alpha + 1)] a_{k(\lambda)} z^{k-1}\}}{1 - \sum_{k=2}^{\infty} a_{k(\lambda)} z^{k-1}} \right| > \beta - 1$$

$$\Rightarrow \left(\frac{(\alpha\lambda+1)(1-\lambda) + \{\sum_{k=2}^{\infty} [(k+\lambda-2)(\alpha k + \alpha\lambda - \alpha + 1)] a_{k(\lambda)}\}}{1 - \sum_{k=2}^{\infty} a_{k(\lambda)}} \right) \leq 1 - \beta$$

That is that,

$$\sum_{k=2}^{\infty} [(k+\lambda-2)(\alpha k + \alpha\lambda - \alpha + 1) + (1-\beta)] a_{k(\lambda)} \leq 1 - \beta - (\alpha\lambda+1)(1-\lambda)$$

This completes the proof of the theorem.

This shows that the values of $\left(\frac{\alpha z^2 h''(z)}{h(z)} + \frac{z h'(z)}{h(z)}\right)$ lie in the circle centred at $w = 1$ whose radius is $1 - \beta - (\alpha\lambda+1)(1-\lambda)$. Hence, $h(z)$ is in the class $\overline{H}_{\lambda}(\alpha, \beta)$.

To prove the converse, we assume that h(z) defined by (1.6) is in the class $\overline{H}_{\lambda}(\alpha, \beta)$. Then,

$$Re\left(\frac{\alpha z^2 h''(z)}{h(z)} + \frac{z h'(z)}{h(z)}\right)$$

$$= Re\left(\frac{\alpha z^2\left(\lambda z^{\lambda-1} - \sum_{k=2}^{\infty}(k+\lambda-1)a_k(\lambda)z^{k+\lambda-2}\right)}{z-\sum_{k=2}^{\infty}a_k z^k}\right.$$

$$\left. + \frac{z\left(\lambda(\lambda-1)z^{\lambda-2} - \sum_{k=2}^{\infty}(k+\lambda-1)(k+\lambda-2)a_k(\lambda)z^{k+\lambda-3}\right)}{z-\sum_{k=2}^{\infty}a_k z^k}\right) > \beta, \qquad z \in U.$$

Choose value of z on the real axis so that $\frac{\alpha z^2 h''(z)}{h(z)} + \frac{zh'(z)}{h(z)}$ is real such that upon rearranging and by letting $z \to 1^-$ through real values, we have

$$\left(\frac{\alpha\lambda(\lambda-1) + \lambda - \sum_{k=2}^{\infty}(k+\lambda-1)(\alpha k + \alpha\lambda - 2\alpha + 1)a_k(\lambda)}{1-\sum_{k=2}^{\infty}a_{k(\lambda)}}\right) > \beta.$$

Therefore, a little computation yields

$$(\alpha\lambda+1)(1-\lambda) + \sum_{k=2}^{\infty}\{(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1) + (1-\ \beta)\}a_{k(\lambda)} \le 1-\ \beta\ .$$

Which obviously is the required result (3.1). Finally, we note that assertion (3.1) of Theorem 3.1 has the extremal function

$$h_1(z) = z - \frac{1-\beta-(\alpha\lambda+1)(1-\lambda)}{[(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\ \beta)\ ]}z^{k+\lambda-1} \qquad (k \ge 2) \qquad (3.3)$$

Corollary 3.*2* *Let* $h(z) \in T$ *be in the class* $\overline{H}_{\lambda}(\alpha,\beta)$*then we have*

$$a_k(\lambda) \le \frac{1-\beta-(\alpha\lambda+1)(1-\lambda)}{[(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\ \beta)\ ]} \qquad (k \le 2) \qquad (3.4)$$

Equality in (3.4) holds true for the function h(z) given by (3.3).

By taking $\lambda = 1$ in the Theorem 2.1 we have the following:

Corollary 3.*3*..

A function $f(z)$ of the form (1.2) is in the class $\overline{H}(\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty}[(k-1)(\alpha k+1)+(1-\ \beta)]a_k \le 1-\beta \qquad \text{Lashin [11]}$$

4. INCLUSION PROPERTY OF THE FUNCTIONS $f^{\lambda}(z) \in \overline{H}_{\lambda}(\alpha,\beta)$

Theorem 4.1 *Let* $0 \le \alpha_1 < \alpha_2$ *and* $0 \le \beta < 1$. *Then* $\overline{H}_{\lambda}(\alpha_2,\beta) \subset \overline{H}_{\lambda}(\alpha_1,\beta)$.

Proof. It follows from Theorem 3.1 that

$$\sum_{k=2}^{\infty}[(k+\lambda-2)(\alpha_1 k+\alpha_1\lambda-\alpha_1+1)+(1-\beta)]a_k < \sum_{k=2}^{\infty}[(k+\lambda-2)(\alpha_2 k+\alpha_2\lambda-\alpha_2+1)+(1-\beta)]a_k \le 1-\beta-(\alpha\lambda+1)(1-\lambda)$$

for $h(z) \in \overline{H}(\alpha_2,\beta)$. Hence $h(z) \in \overline{H}(\alpha_1,\beta)$.

5. NEIGHBOURHOOD RESULTS

Theorem 5.1 $\overline{H}_{\lambda}(\alpha,\beta) \subseteq N_{\delta}(e)$, where $\delta = \frac{2[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta}$.

Proof. Let $h(z) \in \overline{H}_{\lambda}(\alpha,\beta)$. Then, in view of Theorem 3.1, since $(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\beta)$ is an increasing function of k $(k \geqslant 2)$, we have

$$(\alpha\lambda^2+\alpha\lambda+\lambda+1-\beta)\sum_{k=2}^{\infty}a_k(\lambda) \le \sum_{k=2}^{\infty}[(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\beta)]a_k$$

$$\le 1-\beta-(\alpha\lambda+1)(1-\lambda) \text{ and}$$

hence,

$$\sum_{k=2}^{\infty}\alpha_k(\lambda) \le \frac{1-\beta-(\alpha\lambda+1)(1-\lambda)}{(\alpha\lambda+1)(\lambda+1)-\beta}. \qquad (5.1)$$

We find from (3.1) on the other hand that

$$\frac{(\alpha\lambda+1)(\lambda+1)}{2}\sum_{k=2}^{\infty}ka_k(\lambda)-\beta\sum_{k=2}^{\infty}a_k(\lambda) \le 1-\beta-(\alpha\lambda+1)(1-\lambda) \qquad (5.2)$$

From (5.2) and (5.1), we have

$$\frac{(\alpha\lambda+1)(\lambda+1)}{2}\sum_{k=2}^{\infty}ka_k(\lambda) \le [1-\beta-(\alpha\lambda+1)(1-\lambda)]+\beta\sum_{k=2}^{\infty}a_k(\lambda)$$

$$\le [1-\beta-(\alpha\lambda+1)(1-\lambda)]+\beta\left(\frac{1-\beta-(\alpha\lambda+1)(1-\lambda)}{(\alpha\lambda+1)(\lambda+1)-\beta}\right)$$

$$\le [1-\beta-(\alpha\lambda+1)(1-\lambda)]\left[1+\frac{\beta}{(\alpha\lambda+1)(\lambda+1)-\beta}\right]$$

$$\leq [1-\beta-(\alpha\lambda+1)(1-\lambda)]\left[\frac{(\alpha\lambda+1)(\lambda+1)}{(\alpha\lambda+1)(\lambda+1)-\beta}\right]$$

That is

$$\sum_{k=2}^{\infty} k a_k(\lambda) \leq \frac{2[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} = \delta, \qquad (5.3)$$

End of proof in view of the definition (2.1) of the definition 1.

6. INTEGRAL MEANS INEQUALITIES

Applying lemma 1 and (3.1), we prove the following theorem.

Theorem 6.1. *Let* $\delta > 0$. *If* $h(z) \in \overline{H}_\lambda(\alpha,\beta)$, *then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left|h\left(re^{i\theta}\right)\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|h_2\left(re^{i\theta}\right)\right|^{\delta} d\theta,$$

Where

$$h_2(z) = z^{\lambda} - \frac{[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} z^{\lambda+1} \qquad (6.1)$$

Proof. Let $h(z)$ defined by (1.6) and $h_2(z)$ be given by (6.1). We must show that

$$\int_0^{2\pi} \left|1-\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1}\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|1-\frac{[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} z\right|^{\delta} d\theta.$$

By Lemma 1, it suffices to show that

$$1-\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1} \prec 1-\frac{[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} z.$$

Setting

$$1-\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1} = 1-\frac{[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} z.$$

we have

$$\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1} = \frac{[1-\beta-(\alpha\lambda+1)(1-\lambda)]}{(\alpha\lambda+1)(\lambda+1)-\beta} w(z) \qquad (6.2)$$

$$w(z) = \sum_{k=2}^{\infty} \frac{(\alpha\lambda + 1)(\lambda + 1) - \beta}{[1 - \beta - (\alpha\lambda + 1)(1 - \lambda)]} a_k(\lambda)\, z^{k-1}$$

From (6.2) and (3.1), we obtain

$$|w(z)| = \left| \sum_{k=2}^{\infty} \frac{(\alpha\lambda + 1)(\lambda + 1) - \beta}{[1 - \beta - (\alpha\lambda + 1)(1 - \lambda)]} a_k(\lambda)\, z^{k-1} \right|$$

$$\leq |z| \left| \frac{(\alpha\lambda + 1)(\lambda + 1) - \beta}{[1 - \beta - (\alpha\lambda + 1)(1 - \lambda)]} a_k(\lambda) \right| \leq |z|.$$

This completes the proof of the theorem.

Letting $\lambda = 1$ in the Theorem 6.1 above, we have:

Corollary 6.2.

Let $\delta > 0$. *If* $f(z) \in \overline{H}(\alpha, \beta)$, *then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\delta} d\theta \leq \int_0^{2\pi} \left| f_2\left(re^{i\theta}\right) \right|^{\delta} d\theta,$$

Where

$$f_2(z) = \frac{(1 - \beta)}{(2\alpha + 2 - \beta)} z^2$$

Letting $\alpha = 0$ in the above corollary, we have the following:

Corollary 6.3.

Let $\delta = 0$. *If* $f(z) \in T^*(\beta)$, *then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\delta} d\theta \leq \int_0^{2\pi} \left| f_2\left(re^{i\theta}\right) \right|^{\delta} d\theta,$$

Where

$$f_2(z) = \frac{(1 - \beta)}{(2 - \beta)} z^2.$$

(See [11]).

7. PARTIAL SUMS

In this section, we will examine the ratio of a function of the form (1.6) to its sequence of partial sums defined by $h_1(z) = z^\lambda$ and $h_n(z) = z - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}$ when the coefficients of h are sufficiently small to satisfy the condition (3.1) we will determine sharp lower bounds for $Re\left(\frac{h(z)}{h_n(z)}\right) Re\left(\frac{h_n(z)}{h(z)}\right)$.

In what follows, we will use the well-known result that

$$Re\frac{1-w(z)}{1+w(z)} > 0, \qquad z \in U,$$

if and only if

$$w(z) = \sum_{k=2}^{\infty} c_k z^{k+\lambda-1}$$

satisfy the inequality

$$|w(z)| \leq |z|.$$

Theorem 7.1. *If* $f^\lambda(z) \in \overline{H}_\lambda(\alpha,\beta)$, *then*

$$Re\left(\frac{h(z)}{h_n(z)}\right) \geq 1 - \frac{1}{c_{n+1}} \qquad (z \in U, n \in N) \tag{7.1}$$

and

$$Re\left(\frac{h_n(z)}{h(z)}\right) \geq \frac{c_{n+1}}{1+c_{n+1}} \qquad (z \in U, n \in N), \tag{7.2}$$

where $\left(c_k =: \frac{(k+\lambda-2)(\alpha k+\alpha\lambda-\alpha+1)+(1-\beta)}{1-\beta-(\alpha\lambda+1)(1-\lambda)}\right)$. *The estimates in (7.1) and (7.2) are sharp.*

Proof. We employ the same technique used by Silverman [15]. From (7.1), we may write

$$c_{n+1}\left\{\frac{f^\lambda(z)}{f_n^\lambda(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\}$$

$$= c_{n+1}\left(\frac{f^\lambda(z)}{f_n^\lambda(z)} - \frac{c_{n+1}}{c_{n+1}} + \frac{1}{c_{n+1}}\right)$$

$$= c_{n+1}\left(\frac{c_{n+1}\left(z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}\right) - c_{n+1}\left(z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}\right) + z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}}{c_{n+1}\left(z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}\right)}\right)$$

$$= \frac{z^{\lambda} c_{n+1} - c_{n+1}\sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1} - z^{\lambda} c_{n+1} + c_{n+1}\sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} + z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}}{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}}$$

$$= \frac{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} + c_{n+1}\sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} - c_{n+1}\sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}}{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}}$$

$$= \frac{1 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{1 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1}} = \frac{1 + H(z)}{1 + I(z)}.$$

Where

$$H(z) = -\sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}$$

and

$$I(z) = -\sum_{k=2}^{n} a_k(\lambda) z^{k-1}$$

Set

$$\frac{1 + H(z)}{1 + I(z)} = \frac{1 - w(z)}{1 + w(z)}.$$

So that

$$w(z) = \frac{I(z) - H(z)}{2 + H(z) + I(z)}$$

Then

$$w(z) = \frac{-\sum_{k=2}^{n} a_k(\lambda) z^{k-1} + \sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1} - \sum_{k=2}^{n} a_k(\lambda) z^{k-1}}$$

$$w(z) = \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - 2\sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}$$

$$|w(z)| \le \left|\frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - 2\sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}\right|$$

and

$$|w(z)| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}{2-2\sum_{k=2}^{n} a_k(\lambda)\ -c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}$$

To see that $|w(z)| \leq 1$ if and only if

$$c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) + \sum_{k=2}^{n} a_k(\lambda)\ \leq 1$$

we note that

$$\sum_{k=2}^{n} a_k(\lambda)\ \leq 1 - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)$$

and

$$|w(z)| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}{2-2(1-c_{n+1}\sum_{k=n+1}^{\infty} a_k\,(\lambda)) - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}$$

$$|w(z)| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}{2-2+2\sum_{k=n+1}^{\infty} a_k\,(\lambda) - c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}$$

$$|w(z)| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)}{c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda)} \leq 1$$

This readily yields assertion (7.1) of Theorem 7.1

Similarly, we take $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $f_n(z) = z - \sum_{k=2}^{n} a_k z^k$ to get

$$1 + c_{n+1}\left(\frac{f_n^{\lambda}(z)}{f^{\lambda}(z)} - \frac{c_{n+1}}{1+c_{n+1}}\right)$$

$$= 1 + c_{n+1}\left(\frac{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1}}{z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}} - \frac{c_{n+1}}{1+c_{n+1}}\right)$$

$$= (1+c_{n+1})\left(\frac{(1+c_{n+1})(\,z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1})\ - c_{n+1}(z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}}{(1+c_{n+1})(z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}\,)}\right)$$

$$= \frac{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} + z c_{n+1} - c_{n+1}\sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} - z c_{n+1} + c_{n+1}\sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}}{(z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}\,)}$$

$$= \frac{z^{\lambda} - \sum_{k=2}^{n} a_k(\lambda) z^{k+\lambda-1} + c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k+\lambda-1}}{(z^{\lambda} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k+\lambda-1}\,)}$$

$$= \frac{1 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1}\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1})}{(1 - \sum_{k=2}^{\infty} a_k(\lambda) z^{k-1}\,)}$$

$$= \frac{1 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1})}{(1 - \sum_{k=2}^{\infty} a_k(\lambda) z^{k-1})} = \frac{1 + J(z)}{1 + L(z)}$$

Where

$$J(z) = -\sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}$$

and

$$L(z) = -\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1}$$

Again, set

$$\frac{1 + J(z)}{1 + L(z)} = \frac{1 - w(z)}{1 + w(z)},$$

We have

$$w(z) = \frac{L(z) - J(z)}{2 + J(z) + L(z)}$$

$$w(z) = \frac{-\sum_{k=2}^{\infty} a_k(\lambda) z^{k-1} + \sum_{k=2}^{n} a_k(\lambda) z^{k-1} - c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1} - \sum_{k=2}^{\infty} a_k(\lambda) z^{k-1}}$$

$$w(z) = \frac{-\sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1} - c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - \sum_{k=2}^{n} a_k(\lambda) z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1} - \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1} - \sum_{k=2}^{n} a_k(\lambda) z^{k-1}}$$

and

$$|w(z)| = \left| -\frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}}{2 - 2\sum_{k=2}^{n} a_k(\lambda) z^{k-1} + (1 - c_{n+1}) \sum_{k=n+1}^{\infty} a_k(\lambda) z^{k-1}} \right|$$

$$|w(z)| = \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k(\lambda)}{2 - 2\sum_{k=2}^{n} a_k(\lambda) + (1 - c_{n+1})) \sum_{k=n+1}^{\infty} a_k(\lambda)}$$

To see that $|w(z)| \leq 1$ if and only if

$$c_{n+1} \sum_{k=n+1}^{\infty} a_k((\lambda)) + \sum_{k=2}^{n} a_k(\lambda) \leq 1$$

We note that

$$c_{n+1} \sum_{k=n+1}^{\infty} a_k(\lambda) \leq 1 - \sum_{k=2}^{n} a_k(\lambda)$$

and

$$|w(z)| = \frac{\sum_{k=n+1}^{\infty} a_k(\lambda) + 1 - \sum_{k=2}^{n} a_k(\lambda)}{2 - 2\sum_{k=2}^{n} a_k(\lambda) - 1 + \sum_{k=2}^{n} a_k(\lambda) + \sum_{k=n+1}^{\infty} a_k(\lambda)}$$

$$|w(z)| = \frac{\sum_{k=n+1}^{\infty} a_k(\lambda) + 1 - \sum_{k=2}^{n} a_k(\lambda)}{\sum_{k=n+1}^{\infty} a_k(\lambda) + 1 - \sum_{k=2}^{n} a_k \ (\lambda)} \leq 1$$

This immediately yields assertion (7.2) of Theorem 7.1.

Following similar argument, the ratios , $Re\left(\frac{h'(z)}{h_n'(z)}\right)$ and $Re\left(\frac{h_n'(z)}{h'(z)}\right)$ involving derivatives may be obtained

as well.

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