Received 13/09/16, Revised 04/10/16

SOME PROPERTIES OF QUADRATIC WEIGHTED GEOMETRIC MEAN OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES VIA KATO'S INEQUALITY

S. S. DRAGOMIR[1,2]

Abstract. In this paper we consider the *quadratic weighted geometric mean*

$$T\circledS_{\nu}V:=\left|\left|VT^{-1}\right|^{\nu}T\right|^{2}$$

for bounded linear operators T, V in the Hilbert space H with T invertible and $\nu\in[0,1]$. Using the celebrated Kato's inequality we give some operator inequalities such as

$$\left|\left\langle T\circledS Vx,y\right\rangle\right|^{2}\leq\left\langle T\circledS_{\nu}Vx,x\right\rangle\left\langle T\circledS_{1-\nu}Vy,y\right\rangle$$

for any x, $y\in H$ and $\nu\in[0,1]$, where $T\circledS V:=T\circledS_{1/2}V$. Applications for n-tuples of invertible operators and norm inequalities are provided as well.

1. Introduction

Assume that A, B are positive operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. The *weighted operator arithmetic mean* for the pair (A,B) is defined by

$$A\nabla_{\nu}B:=(1-\nu)A+\nu B.$$

In 1980, Kubo & Ando, [21] introduced the *weighted operator geometric mean* for the pair (A,B) with A positive and invertible and B positive by

$$A\sharp_{\nu}B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}.$$

If A, B are positive invertible operators then we can also consider the *weighted operator harmonic mean* defined by (see for instance [21])

$$A!_{\nu}B:=\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}.$$

We have the following fundamental operator means inequalities

$$A!_{\nu}B\leq A\sharp_{\nu}B\leq A\nabla_{\nu}B,\ \nu\in[0,1]\tag{1.1}$$

for any A, B positive invertible operators. For $\nu=\frac{1}{2}$, we denote the above means by $A\nabla B$, $A\sharp B$ and $A!B$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [17, p. 240]: *If the operator $A\in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B:=\sqrt{A}\in B(H)$ such that $B^{2}=A$. If A is invertible, then so is B.*

If $A\in\mathcal{B}(H)$, then the operator $A^{*}A$ is selfadjoint and positive. Define the *"absolute value"* operator by $|A|:=\sqrt{A^{*}A}$.

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Weighted geometric mean, Weighted harmonic mean, Kato's inequality, Operator modulus, Arithmetic mean-geometric mean-harmonic mean inequality.

In the recent paper [6] we generalized the concept of weighted operator geometric mean as follows.

We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T, V) by

$$T \circledS_\nu V := \left| \left| VT^{-1} \right|^\nu T \right|^2 \tag{1.2}$$

for $\nu \geq 0$. For $V \in \mathcal{B}^{-1}(H)$ we can also extend the definition (1.2) for $\nu < 0$.

By the definition of modulus, we also have

$$T \circledS_\nu V = T^* \left| VT^{-1} \right|^{2\nu} T = T^* \left((T^*)^{-1} V^* V T^{-1} \right)^\nu T \tag{1.3}$$

for any $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$.

For $\nu = \frac{1}{2}$ we denote

$$T \circledS V := \left| \left| VT^{-1} \right|^{1/2} T \right|^2 = T^* \left| VT^{-1} \right| T = T^* \left((T^*)^{-1} V^* V T^{-1} \right)^{1/2} T.$$

It has been shown in [6] that the following representation holds

$$T \circledS_\nu V = |T|^2 \sharp_\nu |V|^2 \tag{1.4}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and any real ν.

We have the following fundamental inequalities extending (1.1):

$$|T|^2 \nabla_\nu |V|^2 \geq T \circledS_\nu V \geq |T|^2 !_\nu |V|^2 \tag{1.5}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and for $\nu \in [0, 1]$. In particular, we have

$$|T|^2 \nabla |V|^2 \geq T \circledS V \geq |T|^2 ! |V|^2 \tag{1.6}$$

for T, $V \in \mathcal{B}^{-1}(H)$.

We can define the related weighted operator means for $\nu \in [0, 1]$ and the operators T, V as above by

$$T \circledS_\nu^{1/2} V := (T \circledS_\nu V)^{1/2} = \left| \left| VT^{-1} \right|^{1/2} T \right|,$$

$$T \nabla_\nu^{1/2} V := \left(|T|^2 \nabla_\nu |V|^2 \right)^{1/2} = \left((1 - \nu) |T|^2 + \nu |V|^2 \right)^{1/2}$$

and

$$T !_\nu^{1/2} V := \left(|T|^2 !_\nu |V|^2 \right)^{1/2} = \left((1 - \nu) |T|^{-2} + \nu |V|^{-2} \right)^{-1/2},$$

then by taking the square root in (1.5) we get [6]

$$T \nabla_\nu^{1/2} V \geq T \circledS_\nu^{1/2} V \geq T !_\nu^{1/2} V \tag{1.7}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0, 1]$.

Using the representation (1.4) and the fact that $B \sharp_{1-\nu} A = A \sharp_\nu B$ for any positive invertible operators A and B we can state that

$$T \circledS_{1-\nu} V = V \circledS_\nu T \tag{1.8}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0, 1]$.

By using the celebrated Kato's inequality we give in what follows another proof for the equality (1.8). Some operator inequalities such as

$$|\langle T \circledS V x, y \rangle|^2 \leq \langle T \circledS_\nu V x, x \rangle \langle T \circledS_{1-\nu} V y, y \rangle$$

for any $x,\ y\in H$ and $\nu\in[0,1]$, where $T\circledS V:=T\circledS_{1/2}V$ are also established. Applications for n-tuples of invertible operators and norm inequalities are given as well.

2. Some Results Via Kato's Inequality

In 1952, Kato [18] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator U on H:

$$|\langle Ux,y\rangle|^2\le\langle (U^*U)^\alpha x,x\rangle\left\langle (UU^*)^{1-\alpha}y,y\right\rangle, \tag{2.1}$$

for any $x,\ y\in H$, $\alpha\in[0,1]$. Utilizing the modulus notation introduced before, we can write (2.1) as follows

$$|\langle Ux,y\rangle|^2\le\left\langle |U|^{2\alpha}x,x\right\rangle\left\langle |U^*|^{2(1-\alpha)}y,y\right\rangle \tag{2.2}$$

for any $x,\ y\in H$, $\alpha\in[0,1]$.

For various interesting generalizations, extension and Kato related results, see the papers [1]-[5], [7]-[16], [22]-[25] and [26].

We observe that for any $T,\ V\in\mathcal{B}^{-1}(H)$ and $\nu\in[0,1]$ we have

$$(T\circledS_\nu V)^{-1}=\left(T^*\left((T^*)^{-1}V^*VT^{-1}\right)^\nu T\right)^{-1}=T^{-1}\left(TV^{-1}(V^*)^{-1}T^*\right)^\nu (T^*)^{-1}$$

and

$$\begin{aligned}&(T^*)^{-1}\circledS_\nu (V^*)^{-1}\\&=\left((T^*)^{-1}\right)^*\left(\left(\left((T^*)^{-1}\right)^*\right)^{-1}\left((V^*)^{-1}\right)^*(V^*)^{-1}\left((T^*)^{-1}\right)^{-1}\right)^\nu (T^*)^{-1}\\&=T^{-1}\left(TV^{-1}(V^*)^{-1}T^*\right)^\nu (T^*)^{-1}\end{aligned}$$

showing that

$$(T\circledS_\nu V)^{-1}=(T^*)^{-1}\circledS_\nu (V^*)^{-1} \tag{2.3}$$

for any $T,\ V\in\mathcal{B}^{-1}(H)$ and $\nu\in[0,1]$.

In order to prove the result (1.8) we need the following lemma that holds via Kato's inequality.

Lemma 1. *For any $T,\ V\in\mathcal{B}^{-1}(H)$ and $\nu\in[0,1]$ we have the inequality*

$$|\langle x,y\rangle|^2\le\langle T\circledS_\nu Vx,x\rangle\left\langle (V\circledS_{1-\nu}T)^{-1}y,y\right\rangle \tag{2.4}$$

for any $x,\ y\in H$.

Proof. Let $T,\ V\in\mathcal{B}^{-1}(H)$ and take $U=VT^{-1}$ in (2.2) to get

$$\begin{aligned}\left|\langle VT^{-1}u,v\rangle\right|^2&\le\left\langle \left|VT^{-1}\right|^{2\nu}u,u\right\rangle\left\langle \left|\left(VT^{-1}\right)^*\right|^{2(1-\nu)}v,v\right\rangle\\&=\left\langle \left|VT^{-1}\right|^{2\nu}u,u\right\rangle\left\langle \left|(T^*)^{-1}V^*\right|^{2(1-\nu)}v,v\right\rangle\\&=\left\langle \left|VT^{-1}\right|^{2\nu}u,u\right\rangle\left\langle \left|(T^*)^{-1}\left((V^*)^{-1}\right)^{-1}\right|^{2(1-\nu)}v,v\right\rangle\end{aligned} \tag{2.5}$$

for any $u,\ v\in H$.

Let x, $y \in H$. If we take in (2.5) $u = Tx$ and $v = (V^*)^{-1} y$ then we get

$$|\langle x,y\rangle|^2 \le \left\langle \left|VT^{-1}\right|^{2\nu} Tx, Tx \right\rangle \times \left\langle \left|(T^*)^{-1}\left((V^*)^{-1}\right)^{-1}\right|^{2(1-\nu)} (V^*)^{-1} y, (V^*)^{-1} y \right\rangle = \left\langle T^* \left|VT^{-1}\right|^{2\nu} Tx, x \right\rangle \times \left\langle \left((V^*)^{-1}\right)^* \left|(T^*)^{-1}\left((V^*)^{-1}\right)^{-1}\right|^{2(1-\nu)} (V^*)^{-1} y, y \right\rangle = \langle T \circledS_\nu V x, x\rangle \left\langle (V^*)^{-1} \circledS_{1-\nu} (T^*)^{-1} y, y \right\rangle \tag{2.6}$$

for any $\nu \in [0,1]$.

Since, by (2.3) we have

$$(V^*)^{-1} \circledS_{1-\nu} (T^*)^{-1} = (V \circledS_{1-\nu} T)^{-1}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$, then by (2.6) we get the desired result (2.4). □

Theorem 1. *For any T, $V \in \mathcal{B}^{-1}(H)$ and $t \in [0,1]$ we have*

$$T \circledS_{1-t} V = V \circledS_t T. \tag{2.7}$$

Proof. If in (2.4) we take $y = V \circledS_{1-\nu} Tu$, $u \in H$ and $x = w$, then we get

$$|\langle w, V \circledS_{1-\nu} Tu\rangle|^2 \le \langle T \circledS_\nu V w, w\rangle \langle u, V \circledS_{1-\nu} Tu\rangle$$

and by putting $t = 1-\nu$ we get

$$|\langle V \circledS_t Tu, w\rangle|^2 \le \langle V \circledS_t Tu, u\rangle \langle T \circledS_{1-t} V w, w\rangle$$

for any T, $V \in \mathcal{B}^{-1}(H)$, $t \in [0,1]$ and u, $w \in H$.

In particular, we have

$$|\langle V \circledS_t Tu, u\rangle|^2 \le \langle V \circledS_t Tu, u\rangle \langle T \circledS_{1-t} V u, u\rangle \tag{2.8}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $u \in H$.

Since $V \circledS_t T > 0$ then the inequality (2.8) is equivalent to

$$\langle V \circledS_t Tu, u\rangle \le \langle T \circledS_{1-t} V u, u\rangle \tag{2.9}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $u \in H$.

If we replace in (2.9) V by T and t by $1-t$ we also get

$$\langle T \circledS_{1-t} V u, u\rangle \le \langle V \circledS_t Tu, u\rangle \tag{2.10}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $u \in H$.

Therefore, by (2.9) and (2.10) we can conclude that, for $t \in [0,1]$,

$$\langle T \circledS_{1-t} V u, u\rangle = \langle V \circledS_t Tu, u\rangle \tag{2.11}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $u \in H$.

Using the polarization identity for a positive operator P, namely

$$\langle Px, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \left\langle P\left(\left(x + i^k y\right), x + i^k y\right\rangle, \ x, y \in H$$

and by the identity (2.11) we conclude that for $t \in [0,1]$ we have

$$\langle T \circledS_{1-t} V x, y \rangle = \langle V \circledS_t T x, y \rangle$$

for any $T,\ V \in \mathcal{B}^{-1}(H)$ and $x,\ y \in H$, which proves the desired result (2.7). □

We have the following inequalities:

Theorem 2. *For any $T,\ V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$ we have the inequality*

$$\left| \left\langle |V|^2 x, y \right\rangle \right|^2 \le \langle T \circledS_{2\nu} V x, x \rangle \left\langle T \circledS_{2(1-\nu)} V y, y \right\rangle \tag{2.12}$$

and the inequality

$$|\langle T \circledS V x, y \rangle|^2 \le \langle T \circledS_\nu V x, x \rangle \langle T \circledS_{1-\nu} V y, y \rangle \tag{2.13}$$

for any $x,\ y \in H$.

Proof. If we write Kato's inequality (2.2) for $U = \left|VT^{-1}\right|^2$, which is an invertible positive operator for any $T,\ V \in \mathcal{B}^{-1}(H)$, then we get

$$\left| \left\langle \left|VT^{-1}\right|^2 u, v \right\rangle \right|^2 \le \left\langle \left|VT^{-1}\right|^{4\alpha} u, u \right\rangle \left\langle \left|VT^{-1}\right|^{4(1-\alpha)} v, v \right\rangle \tag{2.14}$$

for any $u,\ v \in H$.

If we take in (2.14) $u = Tx$ and $v = Ty$, then we get

$$\left| \left\langle \left|VT^{-1}\right|^2 Tx, Ty \right\rangle \right|^2 \le \left\langle \left|VT^{-1}\right|^{4\alpha} Tx, Tx \right\rangle \left\langle \left|VT^{-1}\right|^{4(1-\alpha)} Ty, Ty \right\rangle$$

for any $x,\ y \in H$, namely

$$\left| \left\langle T^* \left|VT^{-1}\right|^2 Tx, y \right\rangle \right|^2 \le \left\langle T^* \left|VT^{-1}\right|^{4\alpha} Tx, x \right\rangle \left\langle T^* \left|VT^{-1}\right|^{4(1-\alpha)} Ty, y \right\rangle \tag{2.15}$$

for any $x,\ y \in H$ and $\nu \in [0,1]$.

Since

$$T^* \left|VT^{-1}\right|^2 T = T^* (T^*)^{-1} V^* V T^{-1} T = |V|^2$$

and

$$T^* \left|VT^{-1}\right|^{4\alpha} T = T \circledS_{2\nu} V,\ T^* \left|VT^{-1}\right|^{4(1-\alpha)} T = T \circledS_{2(1-\nu)} V$$

then by (2.15) we get (2.12).

Now, if we take in Kato's inequality $U = \left|VT^{-1}\right|$, which is an invertible positive operator for any $T,\ V \in \mathcal{B}^{-1}(H)$, then we also get

$$\left| \left\langle \left|VT^{-1}\right| x, y \right\rangle \right|^2 \le \left\langle \left|VT^{-1}\right|^{2\alpha} x, x \right\rangle \left\langle \left|VT^{-1}\right|^{2(1-\alpha)} y, y \right\rangle \tag{2.16}$$

for any $x,\ y \in H$ and $\nu \in [0,1]$.

Further, if we take in (2.16) $u = Tx$ and $v = Ty$, then we get

$$\left| \left\langle T^* \left|VT^{-1}\right| Tx, y \right\rangle \right|^2 \le \left\langle T^* \left|VT^{-1}\right|^{2\alpha} Tx, x \right\rangle \left\langle T^* \left|VT^{-1}\right|^{2(1-\alpha)} Ty, y \right\rangle$$

for any $x,\ y \in H$ and $\nu \in [0,1]$, which proves (2.13). □

Remark 1. *Since $T \circledS_{1-\nu} V = V \circledS_\nu T$, then inequality (2.13) also can be written as*

$$|\langle T \circledS V x, y \rangle|^2 \le \langle T \circledS_\nu V x, x \rangle \langle V \circledS_\nu T y, y \rangle \tag{2.17}$$

for any $x,\ y \in H$.

If we use the mean $T\circledS_{\nu}^{1/2}V := (T\circledS_{\nu}V)^{1/2}$*, then we have the equivalent inequalities*

$$\left|\left\langle |V|^2 x, y\right\rangle\right| \leq \left\|T\circledS_{2\nu}^{1/2}Vx\right\| \left\|T\circledS_{2(1-\nu)}^{1/2}Vy\right\| \tag{2.18}$$

and

$$|\langle T\circledS Vx, y\rangle| \leq \left\|T\circledS_{\nu}^{1/2}Vx\right\| \left\|T\circledS_{1-\nu}^{1/2}Vy\right\| \tag{2.19}$$

for any T*,* $V \in \mathcal{B}^{-1}(H)$*,* $\nu \in [0,1]$ *and* x*,* $y \in H$*.*

Taking the supremum over $\|x\| = \|y\| = 1$ *in (2.12) and (2.13) we get the norm inequalities*

$$\|V\|^4 \leq \|T\circledS_{2\nu}V\| \left\|T\circledS_{2(1-\nu)}V\right\| \tag{2.20}$$

and

$$\|T\circledS V\|^2 \leq \|T\circledS_{\nu}V\| \, \|T\circledS_{1-\nu}V\| \tag{2.21}$$

for any T*,* $V \in \mathcal{B}^{-1}(H)$ *and* $\nu \in [0,1]$*.*

If A, B are invertible positive operators, then by taking $T = A^{1/2}$ and $V = B^{1/2}$ in Theorem 2 we get

$$|\langle Bx, y\rangle|^2 \leq \langle A\sharp_{2\nu}Bx, x\rangle \left\langle A\sharp_{2(1-\nu)}By, y\right\rangle \tag{2.22}$$

and the inequality

$$|\langle A\sharp Bx, y\rangle|^2 \leq \langle A\sharp_{\nu}Bx, x\rangle \langle A\sharp_{1-\nu}By, y\rangle \tag{2.23}$$

for any x, $y \in H$.

We also have the norm inequalities

$$\|B\|^2 \leq \|A\sharp_{2\nu}B\| \left\|A\sharp_{2(1-\nu)}B\right\| \tag{2.24}$$

and

$$\|A\sharp B\|^2 \leq \|A\sharp_{\nu}B\| \, \|A\sharp_{1-\nu}B\| \tag{2.25}$$

for any A, B invertible positive operators and $\nu \in [0,1]$.

3. Inequalities for n-Tuples of Operators

Consider the Cartesian product $\mathcal{B}^{-1,(n)}(H) := \mathcal{B}^{-1}(H) \times \cdots \times \mathcal{B}^{-1}(H)$, where $\mathcal{B}^{-1}(H)$ denotes the class of all bounded linear invertible operators on H.

Theorem 3. *Let* $(T_1, \ldots, T_n)$*,* $(V_1, \ldots, V_n)$*,* $\in \mathcal{B}^{-1,(n)}(H)$ *and* $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ *be an* n*-tuple of nonnegative weights not all of them equal to zero. Then we have*

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left|\left\langle |V_j|^2 x, y\right\rangle\right| \\ &\leq \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j y, y\right\rangle^{1/2} \end{aligned} \tag{3.1}$$

and

$$\sum_{j=1}^{n} p_j \left|\langle T_j \circledS V_j x, y\rangle\right| \leq \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{\nu} V_j x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{1-\nu} V_j y, y\right\rangle^{1/2} \tag{3.2}$$

for any x*,* $y \in H$ *and* $\alpha \in [0,1]$*.*

In particular, we have

$$\left\langle \sum_{j=1}^{n} p_j \left|V_j\right|^2 x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j x, x \right\rangle^{1/2} \tag{3.3}$$

and

$$\left\langle \sum_{j=1}^{n} p_j T_j \circledS V_j x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{\nu} V_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{1-\nu} V_j x, x \right\rangle^{1/2} \tag{3.4}$$

for any $x \in H$ and $\alpha \in [0,1]$.

Proof. From inequality (2.12) we have

$$\left|\left\langle \left|V_j\right|^2 x, y \right\rangle\right| \leq \left\langle T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2} \tag{3.5}$$

for any $j \in \{1, ..., n\}$, for any $x, y \in H$ and $\alpha \in [0,1]$.

If we multiply (3.5) by $p_j \geq 0$, $j \in \{1, ..., n\}$ and sum from 1 to n, then we get

$$\sum_{j=1}^{n} p_j \left|\left\langle \left|V_j\right|^2 x, y \right\rangle\right| \leq \sum_{j=1}^{n} p_j \left\langle T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2} \tag{3.6}$$

for any x, $y \in H$ and $\alpha \in [0,1]$.

Now, on making use of the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^2 \right)^{1/2} \left(\sum_{j=1}^{n} p_j b_j^2 \right)^{1/2}$$

where $(a_1, ..., a_n)$, $(b_1, ..., b_n) \in \mathbb{R}_+^n$, and choose $a_j = \left\langle T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2}$ and $b_j = \left\langle T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2}$, then we get

$$\begin{aligned} &\sum_{j=1}^{n} p_j \left\langle T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2} \\ &\leq \left(\sum_{j=1}^{n} p_j \left\langle T_j \circledS_{2\nu} V_j x, x \right\rangle \right)^{1/2} \left(\sum_{j=1}^{n} p_j \left\langle T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle \right)^{1/2} \\ &= \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2} \end{aligned} \tag{3.7}$$

for any x, $y \in H$.

On making use of (3.6) and (3.7) we get (3.1).

From (2.13) we also have

$$\left|\left\langle T_j \circledS V_j x, y \right\rangle\right| \leq \left\langle T_j \circledS_{\nu} V_j x, x \right\rangle^{1/2} \left\langle T_j \circledS_{1-\nu} V_j y, y \right\rangle^{1/2} \tag{3.8}$$

for any $j \in \{1, ..., n\}$, for any x, $y \in H$ and $\alpha \in [0,1]$.

By making use of a similar argument as above we deduce the desired result (3.2). The details are omitted. □

We have the norm inequalities:

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\left\| \sum_{j=1}^{n} p_j |V_j|^2 \right\| \le \left\| \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j \right\|^{1/2} \left\| \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j \right\|^{1/2} \tag{3.9}$$

and

$$\left\| \sum_{j=1}^{n} p_j T_j \circledS V_j \right\| \le \left\| \sum_{j=1}^{n} p_j T_j \circledS_{\nu} V_j \right\|^{1/2} \left\| \sum_{j=1}^{n} p_j T_j \circledS_{1-\nu} V_j \right\|^{1/2} \tag{3.10}$$

for any $\alpha \in [0,1]$.

Proof. By the generalized triangle inequality for modulus, we have

$$\left| \left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, y \right\rangle \right| \le \sum_{j=1}^{n} p_j \left| \left\langle |V_j|^2 x, y \right\rangle \right|$$

for any x, $y \in H$.

By (3.1) we then have

$$\left| \left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, y \right\rangle \right| \le \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j y, y \right\rangle^{1/2}$$

for any x, $y \in H$ and $\alpha \in [0,1]$.

By taking the supremum over $\|x\| = \|y\| = 1$ in this inequality and since the operators $\sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j$ and $\sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j$ are selfadjoint, we get the desired result (3.9).

The inequality (3.10) follows in a similar way by utilising (3.2). □

We have the inequalities in the operator order:

Corollary 2. *With the assumptions of Theorem 3 we have*

$$\sum_{j=1}^{n} p_j |V_j|^2 \le \sum_{j=1}^{n} p_j \left(\frac{T_j \circledS_{2\nu} V_j + T_j \circledS_{2(1-\nu)} V_j}{2} \right) \tag{3.11}$$

and

$$\sum_{j=1}^{n} p_j T_j \circledS V_j \le \sum_{j=1}^{n} p_j \left(\frac{T_j \circledS_{\nu} V_j + T_j \circledS_{1-\nu} V_j}{2} \right) \tag{3.12}$$

for any $\alpha \in [0,1]$.

Proof. Using the elementary inequality

$$\sqrt{ab} \le \frac{1}{2}(a+b), \ a, \ b \ge 0,$$

we have

$$\left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j x, x \right\rangle^{1/2}$$
$$\leq \frac{1}{2} \left(\left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j x, x \right\rangle + \left\langle \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j x, x \right\rangle \right)$$
$$= \left\langle \frac{1}{2} \left(\sum_{j=1}^{n} p_j T_j \circledS_{2\nu} V_j + \sum_{j=1}^{n} p_j T_j \circledS_{2(1-\nu)} V_j \right) x, x \right\rangle$$

for any $x \in H$ and $\alpha \in [0,1]$.

Making use of this inequality and (3.3) we deduce the desired result (3.11). □

Remark 2. *If A_j, B_j are invertible positive operators for $j \in \{1, ..., n\}$, then by taking $T_j = A_j^{1/2}$ and $V_j = B_j^{1/2}$ in Theorem 2 we get*

$$\sum_{j=1}^{n} p_j |\langle B_j x, y \rangle| \leq \left\langle \sum_{j=1}^{n} p_j A_j \sharp_{2\nu} B_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j A_j \sharp_{2(1-\nu)} B_j y, y \right\rangle^{1/2} \tag{3.13}$$

and

$$\sum_{j=1}^{n} p_j |\langle A_j \sharp B_j x, y \rangle| \leq \left\langle \sum_{j=1}^{n} p_j A_j \sharp_{\nu} B_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j A_j \sharp_{1-\nu} B_j y, y \right\rangle^{1/2} \tag{3.14}$$

for any $x, y \in H$ and $\alpha \in [0,1]$.

From (3.9) and (3.10) we get the norm inequalities

$$\left\| \sum_{j=1}^{n} p_j B_j \right\| \leq \left\| \sum_{j=1}^{n} p_j A_j \sharp_{2\nu} B \right\|^{1/2} \left\| \sum_{j=1}^{n} p_j A_j \sharp_{2(1-\nu)} B_j \right\|^{1/2} \tag{3.15}$$

and

$$\left\| \sum_{j=1}^{n} p_j A_j \sharp B_j \right\| \leq \left\| \sum_{j=1}^{n} p_j A_j \sharp_{\nu} B_j \right\|^{1/2} \left\| \sum_{j=1}^{n} p_j A_j \sharp_{1-\nu} B_j \right\|^{1/2} \tag{3.16}$$

for any $\alpha \in [0,1]$.

From (3.11) and (3.12) we have the operator inequalities

$$\sum_{j=1}^{n} p_j B_j \leq \sum_{j=1}^{n} p_j \left(\frac{A_j \sharp_{2\nu} B + A_j \sharp_{2(1-\nu)} B_j}{2} \right) \tag{3.17}$$

and

$$\sum_{j=1}^{n} p_j A_j \sharp B_j \leq \sum_{j=1}^{n} p_j \left(\frac{A_j \sharp_{\nu} B_j + A_j \sharp_{1-\nu} B_j}{2} \right) \tag{3.18}$$

for any $\alpha \in [0,1]$.

We have the following generalization of Schwarz inequality:

Corollary 3. *Let* $(T_1,\dots,T_n)$, $(V_1,\dots,V_n)$, $\in \mathcal{B}^{-1,(n)}(H)$ *with* $\sum_{j=1}^n |V_j|^2 = 1_H$ *and* $\nu \in [0,1]$. *Then we have*

$$|\langle x,y\rangle| \le \left\langle \sum_{j=1}^n T_j \circledS_{2\nu} V_j x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^n T_j \circledS_{2(1-\nu)} V_j y, y\right\rangle^{1/2} \tag{3.19}$$

for any $x,\ y \in H$ *and* $\alpha \in [0,1]$.

In particular, if $|V|^2 = 1_H$, *then for any* $T \in \mathcal{B}^{-1}(H)$ *we have*

$$|\langle x,y\rangle| \le \left\langle \left||T^*|^{-2\nu} T\right|^2 x, x\right\rangle^{1/2} \left\langle \left||T^*|^{-2(1-\nu)} T\right|^2 y, y\right\rangle^{1/2} \tag{3.20}$$

for any $x,\ y \in H$ *and* $\alpha \in [0,1]$.

Proof. We have by (3.1) for $p_j = 1,\ j \in \{1,...,n\}$

$$\begin{aligned} |\langle x,y\rangle| &= \left|\left\langle \sum_{j=1}^n |V_j|^2 x, y\right\rangle\right| \le \sum_{j=1}^n \left|\left\langle |V_j|^2 x, y\right\rangle\right| \\ &\le \left\langle \sum_{j=1}^n T_j \circledS_{2\nu} V_j x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^n T_j \circledS_{2(1-\nu)} V_j y, y\right\rangle^{1/2} \end{aligned} \tag{3.21}$$

for any $x,\ y \in H$ and $\alpha \in [0,1]$.

For $n = 1$, namely when $V_1 = V$ with $|V|^2 = 1_H$ then we have

$$|\langle x,y\rangle| \le \langle T \circledS_{2\nu} V x, x\rangle^{1/2} \left\langle T \circledS_{2(1-\nu)} V y, y\right\rangle^{1/2} \tag{3.22}$$

for any $x,\ y \in H$ and $\alpha \in [0,1]$.

Since

$$\begin{aligned} T \circledS_{2\nu} V &= T^*\left((T^*)^{-1} V^* V T^{-1}\right)^{2\nu} T = T^*\left((T^*)^{-1} T^{-1}\right)^{2\nu} T \\ &= T^*\left((TT^*)^{-1}\right)^{2\nu} T = T^*\left(|T^*|^{-2}\right)^{2\nu} T = T^*\left(|T^*|^{-2\nu}\right)^2 T \\ &= \left||T^*|^{-2\nu} T\right|^2 \end{aligned}$$

and

$$T \circledS_{2(1-\nu)} V = \left||T^*|^{-2(1-\nu)} T\right|^2,$$

then by (3.22) we get (3.20). □

Remark 3. *The inequality (3.20) can be written in an equivalent form as*

$$|\langle x,y\rangle| \le \left\||T^*|^{-2\nu} Tx\right\| \left\||T^*|^{-2(1-\nu)} Ty\right\| \tag{3.23}$$

for any $x,\ y \in H$ *and* $\alpha \in [0,1]$. *If we use Schwarz's inequality*

$$|\langle u,v\rangle| \le \|u\|\,\|v\|$$

for the choices $u = |T^*|^{-2\nu} Tx$ *and* $v = |T^*|^{-2(1-\nu)} Ty$ *and since*

$$\begin{aligned}\langle u, v\rangle &= \left\langle |T^*|^{-2\nu} Tx, |T^*|^{-2(1-\nu)} Ty \right\rangle = \left\langle \left(|T^*|^{-2(1-\nu)} T\right)^* |T^*|^{-2\nu} Tx, y \right\rangle \\ &= \left\langle T^* |T^*|^{-2(1-\nu)} |T^*|^{-2\nu} Tx, y \right\rangle = \left\langle T^* |T^*|^{-2} Tx, y \right\rangle \\ &= \left\langle T^* (TT^*)^{-1} Tx, y \right\rangle = \langle x, y\rangle ,\end{aligned}$$

then we also obtain inequality (3.23).

If we consider the Hölder's numbers p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then, for instance, the equality (2.7) can be written as

$$T \circledS_{1/q} V = V \circledS_{1/p} T \tag{3.24}$$

while the inequality (2.13), as

$$|\langle T \circledS V x, y\rangle|^2 \le \left\langle T \circledS_{1/p} V x, x \right\rangle \left\langle T \circledS_{1/q} V y, y \right\rangle , \; x, y \in H \tag{3.25}$$

for any T, $V \in \mathcal{B}^{-1}(H)$.

Theorem 4. *Let* $(T_1, \ldots, T_n)$, $(V_1, \ldots, V_n)$, $\in \mathcal{B}^{-1,(n)}(H)$, $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ *an n-tuple of nonnegative weights not all of them equal to zero and* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. Then we have*

$$\begin{aligned}&\sum_{j=1}^{n} p_j \left| \left\langle |V_j|^2 x, y \right\rangle \right|^2 \\ &\le \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q y, y \right\rangle^{1/q}\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}&\sum_{j=1}^{n} p_j \left|\langle T_j \circledS V_j x, y\rangle\right|^2 \\ &\le \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{1/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{1/q} V_j\right)^q y, y \right\rangle^{1/q}\end{aligned} \tag{3.27}$$

for any x, $y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned}&\sum_{j=1}^{n} p_j \left\langle |V_j|^2 x, x \right\rangle^2 \\ &\le \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q x, x \right\rangle^{1/q}\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}&\sum_{j=1}^{n} p_j \langle T_j \circledS V_j x, x\rangle^2 \\ &\le \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{1/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{1/q} V_j\right)^q x, x \right\rangle^{1/q}\end{aligned} \tag{3.29}$$

for any $x \in H$ *with with* $\|x\| = 1$.

Proof. Recall the Hölder-McCarthy inequality $\langle P^r x, x\rangle \leq \langle Px, x\rangle^r$ that holds for any positive operator P and any power $r \in (0,1)$.

Using the inequality (2.12) for V_j, T_j, $j \in \{1,...,n\}$ and $\nu = \frac{1}{p}$, we have

$$\left|\left\langle |V_j|^2 x, y\right\rangle\right|^2 \leq \left\langle T_j \circledS_{2/p} V_j x, x\right\rangle \left\langle T_j \circledS_{2/q} V_j y, y\right\rangle \tag{3.30}$$

for any x, $y \in H$.

If x, $y \in H$ with $\|x\| = \|y\| = 1$, then by Hölder-McCarthy inequality we have

$$\left\langle T_j \circledS_{2/p} V_j x, x\right\rangle = \left\langle \left[\left(T_j \circledS_{2/p} V_j\right)^p\right]^{1/p} x, x\right\rangle \leq \left\langle \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle^{1/p}$$

and

$$\left\langle T_j \circledS_{2/q} V_j y, y\right\rangle = \left\langle \left[\left(T_j \circledS_{2/q} V_j\right)^q\right]^{1/q} y, y\right\rangle \leq \left\langle \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle^{1/q}$$

for any $j \in \{1,...,n\}$.

Then by (3.2) and these inequalities, we have

$$\sum_{j=1}^n p_j \left|\left\langle |V_j|^2 x, y\right\rangle\right|^2 \leq \sum_{j=1}^n p_j \left\langle \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle^{1/p} \left\langle \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle^{1/q} \tag{3.31}$$

for any x, $y \in H$ with $\|x\| = \|y\| = 1$.

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^n p_j a_j b_j \leq \left(\sum_{j=1}^n p_j a_j^p\right)^{1/p} \left(\sum_{j=1}^n p_j b_j^q\right)^{1/q}, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1,...,a_n), (b_1,...,b_n) \in \mathbb{R}_+^n$, and choose $a_j = \left\langle \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle^{1/p}$ while $b_j = \left\langle \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle^{1/q}$, then we get

$$\begin{aligned} &\sum_{j=1}^n p_j \left\langle \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle^{1/p} \left\langle \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle^{1/q} \\ &\leq \left[\sum_{j=1}^n p_j \left\langle \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle\right]^{1/p} \left[\sum_{j=1}^n p_j \left\langle \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle\right]^{1/q} \\ &= \left\langle \sum_{j=1}^n p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x\right\rangle^{1/p} \left\langle \sum_{j=1}^n p_j \left(T_j \circledS_{2/q} V_j\right)^q y, y\right\rangle^{1/q} \end{aligned} \tag{3.32}$$

for any x, $y \in H$ with $\|x\| = \|y\| = 1$.

By making use of (3.31) and (3.32) we deduce the desired inequality (3.26).

The inequality (3.27) follows in a similar way by (2.13) and we omit the details. □

We have the following norm inequalities:

Corollary 4. *Let* $(T_1,\ldots,T_n)$, $(V_1,\ldots,V_n)$, $\in \mathcal{B}^{-1,(n)}(H)$ *and* $(p_1,...,p_n) \in \mathbb{R}_+^{*n}$ *a probability distribution, i.e.* $\sum_{j=1}^n p_j = 1$. *If* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then we* *have*

$$\left\|\sum_{j=1}^{n}p_j\left|V_j\right|^2\right\|\leq\left\|\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/p}V_j\right)^p\right\|^{2/p}\left\|\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/q}V_j\right)^q\right\|^{2/q}\tag{3.33}$$

and

$$\left\|\sum_{j=1}^{n}p_jT_j\circledS V_j\right\|\leq\left\|\sum_{j=1}^{n}p_j\left(T_j\circledS_{1/p}V_j\right)^p\right\|^{2/p}\left\|\sum_{j=1}^{n}p_j\left(T_j\circledS_{1/q}V_j\right)^q\right\|^{2/q}.\tag{3.34}$$

Proof. By the generalized triangle inequality and the Cauchy-Bunyakovsky-Schwarz discrete inequality we have

$$\sum_{j=1}^{n}p_j\left|\left\langle\left|V_j\right|^2x,y\right\rangle\right|^2\geq\left(\sum_{j=1}^{n}p_j\left|\left\langle\left|V_j\right|^2x,y\right\rangle\right|\right)^2\geq\left|\left\langle\sum_{j=1}^{n}p_j\left|V_j\right|^2x,y\right\rangle\right|^2\tag{3.35}$$

for any x, $y\in H$ with $\|x\|=\|y\|=1$.

If we use the inequality (3.26) and (3.35), then we have

$$\begin{aligned}&\left|\left\langle\sum_{j=1}^{n}p_j\left|V_j\right|^2x,y\right\rangle\right|^2\\&\leq\left\langle\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/p}V_j\right)^px,x\right\rangle^{1/p}\left\langle\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/q}V_j\right)^qy,y\right\rangle^{1/q},\end{aligned}\tag{3.36}$$

which is equivalent to

$$\left|\left\langle\sum_{j=1}^{n}p_j\left|V_j\right|^2x,y\right\rangle\right|\leq\left\langle\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/p}V_j\right)^px,x\right\rangle^{2/p}\left\langle\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/q}V_j\right)^qy,y\right\rangle^{2/q}$$

for any x, $y\in H$ with $\|x\|=\|y\|=1$.

By taking the supremum over $\|x\|=\|y\|=1$ in this inequality and since the operators $\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/p}V_j\right)^p$ and $\sum_{j=1}^{n}p_j\left(T_j\circledS_{2/q}V_j\right)^q$ are selfadjoint, then we get the desired result (3.33).

The inequality (3.34) follows from (3.27) in a similar way and we omit the details. □

Corollary 5. *With the assumptions of Corollary 4 we have the inequalities*

$$\left\|\sum_{j=1}^{n}p_j\left|V_j\right|^2\right\|^2\leq\left\|\sum_{j=1}^{n}p_j\left(\frac{1}{p}\left(T_j\circledS_{2/p}V_j\right)^p+\frac{1}{q}\left(T_j\circledS_{2/q}V_j\right)^q\right)\right\|\tag{3.37}$$

and

$$\left\|\sum_{j=1}^{n}p_jT_j\circledS V_j\right\|^2\leq\left\|\sum_{j=1}^{n}p_j\left(\frac{1}{p}\left(T_j\circledS_{1/p}V_j\right)^p+\frac{1}{q}\left(T_j\circledS_{1/q}V_j\right)^q\right)\right\|.\tag{3.38}$$

Proof. From (3.36) we have for $y=x$ that

$$\left|\left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, x \right\rangle\right|^2 \le \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q x, x \right\rangle^{1/q} \tag{3.39}$$

for any x, $y \in H$ with $\|x\| = \|y\| = 1$.

By the elementary inequality

$$a^{1/p} b^{1/q} \le \frac{1}{p} a + \frac{1}{q} b,\ \ a, b > 0,\ \ \frac{1}{p} + \frac{1}{q} = 1$$

we have

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q x, x \right\rangle^{1/q} \\ &\le \frac{1}{p} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p x, x \right\rangle + \frac{1}{q} \left\langle \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q x, x \right\rangle \\ &= \left\langle \left[\frac{1}{p} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p + \frac{1}{q} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q \right] x, x \right\rangle \end{aligned} \tag{3.40}$$

for any $x \in H$ with $\|x\| = 1$.

By (3.39) we then have

$$\left|\left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, x \right\rangle\right|^2 \le \left\langle \left[\frac{1}{p} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p + \frac{1}{q} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q \right] x, x \right\rangle$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ and taking into account that $\sum_{j=1}^{n} p_j |V_j|^2$ and

$$\frac{1}{p} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/p} V_j\right)^p + \frac{1}{q} \sum_{j=1}^{n} p_j \left(T_j \circledS_{2/q} V_j\right)^q$$

are selfadjoint operators, we obtain the desired result (3.37).

The inequality (3.38) follows in a similar way and the details are omitted. □

If A_j, B_j are invertible positive operators for $j \in \{1, ..., n\}$, then by taking $T_j = A_j^{1/2}$ and $V_j = B_j^{1/2}$ in Theorem 4 and the subsequent inequalities, we can derive various results for the weighted geometric mean of positive operators. The details are not presented here.

References

[1] S. S. Dragomir, Some inequalities of Kato type for sequences of operators in Hilbert spaces. *Publ. Res. Inst. Math. Sci.* **48** (2012), no. 4, 937–955.

[2] S. S. Dragomir, Y. J. Cho and Y.-H. Kim, Applications of Kato's inequality for n-tuples of operators in Hilbert spaces, (I). *J. Inequal. Appl.* **2013**, 2013:21, 16 pp

[3] S. S. Dragomir, Applications of Kato's inequality to operator-valued integrals on Hilbert spaces. *Asian-Eur. J. Math.* **6** (2013), no. 4, 1350059, 18 pp.

[4] S. S. Dragomir, Y. J. Cho and Y.-H. Kim, Applications of Kato's inequality for n-tuples of operators in Hilbert spaces, (II). *J. Inequal. Appl.* **2013**, 2013:464, 20 pp.

[5] S. S. Dragomir, Some inequalities generalizing Kato's and Furuta's results. *Filomat* **28** (2014), no. 1, 179–195.

[6] S. S. Dragomir, The quadratic weighted geometric mean for bounded linear operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 145. [Online http://rgmia.org/papers/v19/v19a145.pdf].

[7] M. Fujii, C.-S. Lin and R. Nakamoto, Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2** (1999), no. 2, 215–221.

[8] M. Fujii and T. Furuta, Löwner-Heinz, Cordes and Heinz-Kato inequalities. *Math. Japon.* **38** (1993), no. 1, 73–78.

[9] M. Fujii, E. Kamei, C. Kotari and H. Yamada, Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Japon.* **40** (1994), no. 2, 259–267

[10] M. Fujii, Y.O. Kim, and Y. Seo, Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1** (2003), no. 2, 275–283

[11] M. Fujii, Y.O. Kim and M. Tominaga, Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.* (FJMS) **6** (2002), no. 3, 225–238

[12] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 223–228.

[13] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3** (1999), no. 3, 293–302,

[14] T. Furuta, Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equations Operator Theory* **29** (1997), no. 1, 1–9.

[15] T. Furuta, Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Amer. Math. Soc.* **120** (1994), no. 1, 223–231.

[16] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.

[17] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[18] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.

[19] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[20] F. Kittaneh, Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27** (1993), no. 4, 279–285.

[21] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **246** (1979/80), no. 3, 205–224.

[22] C.-S. Lin, On Heinz-Kato-Furuta inequality with best bounds. J. *Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15** (2008), no. 1, 93–101.

[23] C.-S. Lin, On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2** (2007), no. 37-40, 1849–1858,

[24] C.-S. Lin, On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Japon.* **50** (1999), no. 3, 463–468.

[25] C.-S. Lin, On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2** (1999), no. 2, 283–287.

[26] M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **127** (1999), no. 10, 2899–2904.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa