Received 04/02/16

REFINEMENTS AND REVERSES OF HÖLDER'S INEQUALITY FOR ISOONIC FUNCTIONALS VIA A RESULT OF CARTWRIGHT AND FIELD

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Abstract. In this paper we establish some refinements and reverses of Hölder's inequality for isotonic linear functionals by the use of a Cartwright and Field result from 1978. Applications for integrals and sequences of real numbers are also given.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [3], [15] and [16]). For other inequalities for isotonic functionals see [2], [4]-[14] and [17]-[19].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0$, $k \in E$).

One of the most important inequalities for positive linear functionals is *Hölder's inequality*, namely

$$A(fg) \leq A^{1/p}(f^p) A^{1/q}(g^q) \tag{1.1}$$

provided $f, g \geq 0$ on E, $f^p, g^q, fg \in L$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if $f, g \geq 0$ on E with $f^2, g^2, fg \in L$ then we have the *Cauchy-Bunyakovsky-Schwarz (CBS) inequality* for isotonic functionals

$$A(fg) \leq A^{1/2}(f^2) A^{1/2}(g^2). \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Isotonic Functionals, Hölder's inequality, Schwarz's inequality, Integral inequalities, Discrete inequalities.

We have the following inequality that provides a refinement and a reverse for the celebrated *Young's inequality*

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.3}$$

for any $a,\ b>0$ and $\nu \in [0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Motivated by the above facts, we establish in this paper some refinements and reverses of Hölder's inequality for isotonic linear functionals by the use of Cartwright and Field inequality (1.3). Applications for integrals and sequences of real numbers are also given.

2. Refinements and Reverses of Hölder's Inequality

We have the following reverse of Hölder's inequality for isotonic functionals:

Theorem 1. *Let $A: L \to \mathbb{R}$ be a normalised isotonic functional and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,\ g: E \to \mathbb{R}$ are such that $fg,\ f^p,\ g^q,\ g^{2q},\ f^{2p},\ g^q f^p \in L$ and*

$$0 < m_1 \le f \le M_1 < \infty,\ 0 < m_2 \le g \le M_2 < \infty, \tag{2.1}$$

for some constants $m_1,\ M_1,\ m_2,\ M_2$, then by putting

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\} \tag{2.2}$$

we have

$$\begin{aligned} 0 &\le \frac{1}{2qpM_{p,q}}\left(\frac{A(g^{2q})}{A^2(g^q)} - 2\frac{A(g^q f^p)}{A(g^q)A(f^p)} + \frac{A(f^{2p})}{A^2(f^p)}\right) \\ &\le 1 - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \frac{M_{p,q}}{2qp}\left(\frac{A(g^{2q})}{A^2(g^q)} - 2\frac{A(g^q f^p)}{A(g^q)A(f^p)} + \frac{A(f^{2p})}{A^2(f^p)}\right). \end{aligned} \tag{2.3}$$

Proof. Observe that, by (2.1) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p$$

and

$$\left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$m_{p,q} \le \frac{f^p}{A(f^p)}, \frac{g^q}{A(g^q)} \le M_{p,q},$$

where

$$\begin{aligned} m_{p,q} \quad &: \quad = \min\left\{\left(\frac{m_1}{M_1}\right)^p, \left(\frac{m_2}{M_2}\right)^q\right\} = \min\left\{\frac{1}{\left(\frac{M_1}{m_1}\right)^p}, \frac{1}{\left(\frac{M_2}{m_2}\right)^q}\right\} \\ &= \quad \frac{1}{\max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}} = \frac{1}{M_{p,q}}. \end{aligned}$$

Using the inequality (1.3) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$ we get

$$\begin{aligned} 0 &\leq \frac{1}{2qpM_{p,q}}\left(\frac{g^q}{A\left(g^q\right)} - \frac{f^p}{A\left(f^p\right)}\right)^2 \\ &\leq \frac{1}{p}\frac{f^p}{A\left(f^p\right)} + \frac{1}{q}\frac{g^q}{A\left(g^q\right)} - \frac{f}{\left[A\left(f^p\right)\right]^{1/p}}\frac{g}{\left[A\left(g^q\right)\right]^{1/q}} \\ &\leq \frac{M_{p,q}}{2qp}\left(\frac{g^q}{A\left(g^q\right)} - \frac{f^p}{A\left(f^p\right)}\right)^2, \end{aligned}$$

which can be written as

$$\begin{aligned} 0 &\leq \frac{1}{2qpM_{p,q}}\left(\frac{g^{2q}}{A^2\left(g^q\right)} - 2\frac{g^q}{A\left(g^q\right)}\frac{f^p}{A\left(f^p\right)} + \frac{f^{2p}}{A^2\left(f^p\right)}\right) \\ &\leq \frac{1}{p}\frac{f^p}{A\left(f^p\right)} + \frac{1}{q}\frac{g^q}{A\left(g^q\right)} - \frac{f}{\left[A\left(f^p\right)\right]^{1/p}}\frac{g}{\left[A\left(g^q\right)\right]^{1/q}} \\ &\leq \frac{M_{p,q}}{2qp}\left(\frac{g^{2q}}{A^2\left(g^q\right)} - 2\frac{g^q}{A\left(g^q\right)}\frac{f^p}{A\left(f^p\right)} + \frac{f^{2p}}{A^2\left(f^p\right)}\right). \end{aligned} \tag{2.4}$$

If we take the functional A in (2.4), then we get

$$\begin{aligned} 0 &\leq \frac{1}{2qpM_{p,q}}\left(\frac{A\left(g^{2q}\right)}{A^2\left(g^q\right)} - 2\frac{A\left(g^q f^p\right)}{A\left(g^q\right)A\left(f^p\right)} + \frac{A\left(f^{2p}\right)}{A^2\left(f^p\right)}\right) \\ &\leq \frac{1}{p}\frac{A\left(f^p\right)}{A\left(f^p\right)} + \frac{1}{q}\frac{A\left(g^q\right)}{A\left(g^q\right)} - \frac{A\left(fg\right)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}} \\ &\leq \frac{M_{p,q}}{2qp}\left(\frac{A\left(g^{2q}\right)}{A^2\left(g^q\right)} - 2\frac{A\left(g^q f^p\right)}{A\left(g^q\right)A\left(f^p\right)} + \frac{A\left(f^{2p}\right)}{A^2\left(f^p\right)}\right), \end{aligned}$$

which is equivalent to the desired result (2.3). □

We have the following refinement and reverse of (CBS)-inequality:

Corollary 1. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and f, $g : E \to \mathbb{R}$ are such that fg, f^2, g^2, $g^2 f^2$, g^4, $f^4 \in L$ and the condition (2.1) is valid for some constants m_1, M_1, m_2, M_2, then by putting*

$$M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\} \tag{2.5}$$

we have

$$0 \leq \frac{1}{8M^2}\left(\frac{A\left(g^4\right)}{A^2\left(g^2\right)} - 2\frac{A\left(g^2 f^2\right)}{A\left(g^2\right)A\left(f^2\right)} + \frac{A\left(f^4\right)}{A^2\left(f^2\right)}\right) \tag{2.6}$$
$$\leq 1 - \frac{A\left(fg\right)}{\left[A\left(f^2\right)\right]^{1/2}\left[A\left(g^2\right)\right]^{1/2}}$$
$$\leq \frac{M^2}{8}\left(\frac{A\left(g^4\right)}{A^2\left(g^2\right)} - 2\frac{A\left(g^2 f^2\right)}{A\left(g^2\right)A\left(f^2\right)} + \frac{A\left(f^4\right)}{A^2\left(f^2\right)}\right).$$

We also have:

Theorem 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f,\ g : E \to \mathbb{R}$ are such that $fg,\ f^p,\ g^q,\ \frac{g^q}{f^p},\ \frac{f^p}{g^q} \in L$ and the condition (2.1) is valid for some constants $m_1,\ M_1,\ m_2,\ M_2$, then we have*

$$0 \leq \frac{1}{2qpM_{p,q}}\left(\frac{A\left(f^p\right)}{A\left(g^q\right)}A\left(\frac{g^q}{f^p}\right) + \frac{A\left(g^q\right)}{A\left(f^p\right)}A\left(\frac{f^p}{g^q}\right) - 2\right) \tag{2.7}$$
$$\leq 1 - \frac{A\left(fg\right)}{\left[A\left(f^p\right)\right]^{1/p}\left[A\left(g^q\right)\right]^{1/q}}$$
$$\leq \frac{M_{p,q}}{2qp}\left(\frac{A\left(f^p\right)}{A\left(g^q\right)}A\left(\frac{g^q}{f^p}\right) + \frac{A\left(g^q\right)}{A\left(f^p\right)}A\left(\frac{f^p}{g^q}\right) - 2\right),$$

where $M_{p,q}$ is defined by (2.2).

Proof. Since $ab = \min\{a,b\}\max\{a,b\}$ for any $a,\ b > 0$, then from (1.3) we have

$$\frac{1}{2}\nu\left(1-\nu\right)\min\{a,b\}\frac{\left(b-a\right)^2}{ab}$$
$$\leq \left(1-\nu\right)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu\left(1-\nu\right)\max\{a,b\}\frac{\left(b-a\right)^2}{ab},$$

where $\nu \in [0,1]$.

This can be written as

$$\frac{1}{2}\nu\left(1-\nu\right)\min\{a,b\}\left(\frac{b}{a} + \frac{a}{b} - 2\right) \tag{2.8}$$
$$\leq \left(1-\nu\right)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu\left(1-\nu\right)\max\{a,b\}\left(\frac{b}{a} + \frac{a}{b} - 2\right),$$

for any $a,\ b > 0$.

Using the inequality (2.8) for $\nu = \frac{1}{q}$, $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$ we get

$$0 \leq \frac{1}{2qpM_{p,q}}\left(\frac{A\left(f^p\right)}{A\left(g^q\right)}\frac{g^q}{f^p} + \frac{A\left(g^q\right)}{A\left(f^p\right)}\frac{f^p}{g^q} - 2\right) \tag{2.9}$$
$$\leq \frac{1}{p}\frac{f^p}{A\left(f^p\right)} + \frac{1}{q}\frac{g^q}{A\left(g^q\right)} - \frac{f}{\left[A\left(f^p\right)\right]^{1/p}}\frac{g}{\left[A\left(g^q\right)\right]^{1/q}}$$
$$\leq \frac{1}{2qp}M_{p,q}\left(\frac{A\left(f^p\right)}{A\left(g^q\right)}\frac{g^q}{f^p} + \frac{A\left(g^q\right)}{A\left(f^p\right)}\frac{f^p}{g^q} - 2\right).$$

If we take the functional A in (2.9), then we get

$$\begin{aligned} 0 &\le \frac{1}{2qpM_{p,q}}\left(\frac{A(f^p)}{A(g^q)}A\left(\frac{g^q}{f^p}\right)+\frac{A(g^q)}{A(f^p)}A\left(\frac{f^p}{g^q}\right)-2\right) \\ &\le \frac{1}{p}\frac{A(f^p)}{A(f^p)}+\frac{1}{q}\frac{A(g^q)}{A(g^q)}-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\ &\le \frac{1}{2qp}M_{p,q}\left(\frac{A(f^p)}{A(g^q)}A\left(\frac{g^q}{f^p}\right)+\frac{A(g^q)}{A(f^p)}A\left(\frac{f^p}{g^q}\right)-2\right), \end{aligned}$$

which is equivalent to (2.7). □

We have the following refinement and reverse of (CBS)-inequality:

Corollary 2. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and f, $g : E \to \mathbb{R}$ are such that fg, f^2, g^2, $\frac{g^2}{f^2}$, $\frac{f^2}{g^2} \in L$ and the condition (2.1) is valid for some constants m_1, M_1, m_2, M_2, then we have*

$$\begin{aligned} 0 &\le \frac{1}{8M^2}\left(\frac{A(f^2)}{A(g^2)}A\left(\frac{g^2}{f^2}\right)+\frac{A(g^2)}{A(f^2)}A\left(\frac{f^2}{g^2}\right)-2\right) \\ &\le 1-\frac{A(fg)}{[A(f^2)]^{1/2}[A(g^2)]^{1/2}} \\ &\le \frac{M^2}{8}\left(\frac{A(f^2)}{A(g^2)}A\left(\frac{g^2}{f^2}\right)+\frac{A(g^2)}{A(f^2)}A\left(\frac{f^2}{g^2}\right)-2\right), \end{aligned} \tag{2.10}$$

where M is defined by (2.5).

If we write the inequality (1.3) for $a = 1$ and $b = x$ we get

$$\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}} \le 1-\nu+\nu x-x^\nu \le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}} \tag{2.11}$$

for any $x > 0$ and for any $\nu \in [0,1]$.

If $x \in [t,T] \subset (0,\infty)$, then $\max\{x,1\} \le \max\{T,1\}$ and $\min\{t,1\} \le \min\{x,1\}$ and by (2.11) we get

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)\frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}} &\le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}} \\ &\le 1-\nu+\nu x-x^\nu \\ &\le \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}} \\ &\le \frac{1}{2}\nu(1-\nu)\frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}} \end{aligned} \tag{2.12}$$

for any $x \in [t,T]$ and for any $\nu \in [0,1]$.

Observe that

$$\min_{x\in[t,T]}(x-1)^2 = \begin{cases} (T-1)^2 \text{ if } T<1, \\ 0 \text{ if } t\le 1\le T, \\ (t-1)^2 \text{ if } 1<t \end{cases}$$

and

$$\max_{x\in[t,T]}(x-1)^2=\begin{cases}(t-1)^2 \text{ if } T<1,\\ \max\left\{(t-1)^2,(T-1)^2\right\} \text{ if } t\le 1\le T,\\ (T-1)^2 \text{ if } 1<t.\end{cases}$$

Then

$$c(t,T):=\frac{\min_{x\in[t,T]}(x-1)^2}{\max\{T,1\}}=\begin{cases}(T-1)^2 \text{ if } T<1,\\ 0 \text{ if } t\le 1\le T,\\ \frac{(t-1)^2}{T} \text{ if } 1<t\end{cases}\tag{2.13}$$

and

$$\begin{aligned}C(t,T)&:=\frac{\max_{x\in[t,T]}(x-1)^2}{\min\{t,1\}}\\&=\begin{cases}\frac{(t-1)^2}{t} \text{ if } T<1,\\ \frac{1}{t}\max\left\{(t-1)^2,(T-1)^2\right\} \text{ if } t\le 1\le T,\\ (T-1)^2 \text{ if } 1<t.\end{cases}\end{aligned}\tag{2.14}$$

Using the inequality (2.12) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(t,T)&\le\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{T,1\}}\\&\le 1-\nu+\nu x-x^\nu\\&\le\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{t,1\}}\le\frac{1}{2}\nu(1-\nu)C(t,T)\end{aligned}\tag{2.15}$$

for any $x\in[t,T]$ and for any $\nu\in[0,1]$.

Now, if $a,\ b>0$ and assume that $\frac{b}{a}\in[t,T]$, then by (2.15) we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(t,T)a&\le\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{T,1\}a}\\&\le(1-\nu)a+\nu b-b^\nu a^{1-\nu}\\&\le\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{t,1\}a}\le\frac{1}{2}\nu(1-\nu)C(t,T)a\end{aligned}\tag{2.16}$$

for any $\nu\in[0,1]$, where $c(t,T)$ and $C(t,T)$ are defined by (2.13) and (2.14), respectively.

Theorem 3. *Let $A:L\to\mathbb{R}$ be a normalised isotonic functional and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $f,\ g:E\to\mathbb{R}$ are such that $fg,\ f^p,\ g^q,\ \frac{g^{2q}}{f^p}\in L$ and (2.1) is valid for some constants $m_1,\ M_1,\ m_2,\ M_2$, then by putting*

$$\tilde{M}_{p,q}:=\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q\tag{2.17}$$

we have

$$\begin{aligned}0&\le\frac{1}{2pq\tilde{M}_{p,q}}\left(\frac{A(f^p)}{A^2(g^q)}A\left(\frac{g^{2q}}{f^p}\right)-1\right)\le 1-\frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}}\\&\le\frac{1}{2pq}\tilde{M}_{p,q}\left(\frac{A(f^p)}{A^2(g^q)}A\left(\frac{g^{2q}}{f^p}\right)-1\right)\le\frac{1}{2pq}\tilde{M}_{p,q}\left(\tilde{M}_{p,q}-1\right)^2.\end{aligned}\tag{2.18}$$

Proof. Observe that, by (2.1) we have

$$m_1^p \le A(f^p) \le M_1^p \text{ and } m_2^q \le A(g^q) \le M_2^q.$$

Also

$$\left(\frac{m_1}{M_1}\right)^p \le \frac{f^p}{A(f^p)} \le \left(\frac{M_1}{m_1}\right)^p \text{ and } \left(\frac{m_2}{M_2}\right)^q \le \frac{g^q}{A(g^q)} \le \left(\frac{M_2}{m_2}\right)^q$$

giving that

$$\frac{\left(\frac{m_2}{M_2}\right)^q}{\left(\frac{M_1}{m_1}\right)^p} \le \frac{\frac{g^q}{A(g^q)}}{\frac{f^p}{A(f^p)}} \le \frac{\left(\frac{M_2}{m_2}\right)^q}{\left(\frac{m_1}{M_1}\right)^p}. \tag{2.19}$$

Since

$$\frac{\left(\frac{M_2}{m_2}\right)^q}{\left(\frac{m_1}{M_1}\right)^p} = \tilde{M}_{p,q} \text{ and } \frac{\left(\frac{m_2}{M_2}\right)^q}{\left(\frac{M_1}{m_1}\right)^p} = \frac{1}{\tilde{M}_{p,q}}$$

then by (2.19) we have

$$\frac{1}{\tilde{M}_{p,q}} \le \frac{\frac{g^q}{A(g^q)}}{\frac{f^p}{A(f^p)}} \le \tilde{M}_{p,q}.$$

Now by (2.16) for $a = \frac{f^p}{A(f^p)}$, $b = \frac{g^q}{A(g^q)}$, $\nu = \frac{1}{p}$, $t = \frac{1}{\tilde{M}_{p,q}}$ and $T = \tilde{M}_{p,q}$ we have

$$\begin{aligned}
0 &\le \frac{1}{2pq\tilde{M}_{p,q}}\left(\frac{A(f^p)}{A^2(g^q)}\frac{g^{2q}}{f^p} - 2\frac{g^q}{A(g^q)} + \frac{f^p}{A(f^p)}\right) \\
&\le \frac{1}{p}\frac{f^p}{A(f^p)} + \frac{1}{q}\frac{g^q}{A(g^q)} - \frac{f}{[A(f^p)]^{1/p}}\frac{g}{[A(g^q)]^{1/q}} \\
&\le \frac{1}{2pq}\tilde{M}_{p,q}\left(\frac{A(f^p)}{A^2(g^q)}\frac{g^{2q}}{f^p} - 2\frac{g^q}{A(g^q)} + \frac{f^p}{A(f^p)}\right) \\
&\le \frac{1}{2pq}\tilde{M}_{p,q}\max\left\{\left(\frac{1}{\tilde{M}_{p,q}} - 1\right)^2, \left(\tilde{M}_{p,q} - 1\right)^2\right\}\frac{f^p}{A(f^p)} \\
&= \frac{1}{2pq}\tilde{M}_{p,q}\left(\tilde{M}_{p,q} - 1\right)^2\frac{f^p}{A(f^p)}.
\end{aligned} \tag{2.20}$$

If we take the functional A in (2.20), then we get

$$\begin{aligned}
0 &\le \frac{1}{2pq\tilde{M}_{p,q}}\left(\frac{A(f^p)}{A^2(g^q)}A\left(\frac{g^{2q}}{f^p}\right) - 2\frac{A(g^q)}{A(g^q)} + \frac{A(f^p)}{A(f^p)}\right) \\
&\le \frac{1}{p}\frac{A(f^p)}{A(f^p)} + \frac{1}{q}\frac{A(g^q)}{A(g^q)} - \frac{A(fg)}{[A(f^p)]^{1/p}[A(g^q)]^{1/q}} \\
&\le \frac{1}{2pq}\tilde{M}_{p,q}\left(\frac{A(f^p)}{A^2(g^q)}A\left(\frac{g^{2q}}{f^p}\right) - 2\frac{A(g^q)}{A(g^q)} + \frac{A(f^p)}{A(f^p)}\right) \\
&\le \frac{1}{2pq}\tilde{M}_{p,q}\left(\tilde{M}_{p,q} - 1\right)^2\frac{A(f^p)}{A(f^p)},
\end{aligned}$$

which is equivalent to the desired result (2.18). □

We have the following refinement and reverse of (CBS)-inequality:

Corollary 3. *Let $A : L \to \mathbb{R}$ be a normalised isotonic functional and $f,\, g : E \to \mathbb{R}$ are such that $fg,\, f^2,\, g^2,\, \frac{g^4}{f^2} \in L$ and the condition (2.1) is valid for some constants $m_1,\, M_1,\, m_2,\, M_2$, then by putting*

$$\tilde{M} := \frac{M_1 M_2}{m_1 m_2} \tag{2.21}$$

we have

$$\begin{aligned} 0 &\le \frac{1}{8\tilde{M}^2}\left(\frac{A\left(f^2\right)}{A^2\left(g^2\right)}A\left(\frac{g^4}{f^2}\right)-1\right) \le 1-\frac{A\left(fg\right)}{\left[A\left(f^2\right)\right]^{1/2}\left[A\left(g^2\right)\right]^{1/2}} \\ &\le \frac{1}{8}\tilde{M}^2\left(\frac{A\left(f^2\right)}{A^2\left(g^2\right)}A\left(\frac{g^4}{f^2}\right)-1\right) \le \frac{1}{8}\tilde{M}^2\left(\tilde{M}^2-1\right)^2. \end{aligned} \tag{2.22}$$

3. Applications for Integrals

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ -algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ -a.e. (almost every) $x \in \Omega$ and $p \ge 1$ consider the Lebesgue space

$$L_w^p(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p w(x)\, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\, d\mu(x)$. The same for other integrals involved below. We assume that $\int_\Omega w d\mu = 1$.

Let $f,\, g$ be μ-measurable functions with the property that

$$0 < m_1 \le f \le M_1 < \infty,\ 0 < m_2 \le g \le M_2 < \infty, \tag{3.1}$$

μ-almost everywhere (a.e.) on Ω for some constants $m_1,\, M_1,\, m_2,\, M_2$, then by (2.3) we have

$$\begin{aligned} 0 &\le \frac{1}{2qpM_{p,q}}\left(\frac{\int_\Omega wg^{2q}d\mu}{\left(\int_\Omega wg^q d\mu\right)^2}-2\frac{\int_\Omega wg^q f^p d\mu}{\int_\Omega wg^q d\mu \int_\Omega wf^p d\mu}+\frac{\int_\Omega wf^{2p}d\mu}{\left(\int_\Omega wf^p d\mu\right)^2}\right) \\ &\le 1-\frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}} \\ &\le \frac{M_{p,q}}{2qp}\left(\frac{\int_\Omega wg^{2q}d\mu}{\left(\int_\Omega wg^q d\mu\right)^2}-2\frac{\int_\Omega wg^q f^p d\mu}{\int_\Omega wg^q d\mu \int_\Omega wf^p d\mu}+\frac{\int_\Omega wf^{2p}d\mu}{\left(\int_\Omega wf^p d\mu\right)^2}\right), \end{aligned} \tag{3.2}$$

where

$$M_{p,q} := \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\} \tag{3.3}$$

and $p,\, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have for

$$M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\} \tag{3.4}$$

that

$$0 \leq \frac{1}{8M^2}\left(\frac{\int_\Omega wg^4 d\mu}{\left(\int_\Omega wg^2 d\mu\right)^2} - 2\frac{\int_\Omega wg^2 f^2 d\mu}{\int_\Omega wg^2 d\mu \int_\Omega wf^2 d\mu} + \frac{\int_\Omega wf^4 d\mu}{\left(\int_\Omega wf^2 d\mu\right)^2}\right) \tag{3.5}$$
$$\leq 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^2 d\mu\right)^{1/2}\left(\int_\Omega wg^2 d\mu\right)^{1/2}}$$
$$\leq \frac{M^2}{8}\left(\frac{\int_\Omega wg^4 d\mu}{\left(\int_\Omega wg^2 d\mu\right)^2} - 2\frac{\int_\Omega wg^2 f^2 d\mu}{\int_\Omega wg^2 d\mu \int_\Omega wf^2 d\mu} + \frac{\int_\Omega wf^4 d\mu}{\left(\int_\Omega wf^2 d\mu\right)^2}\right).$$

Let f, g are μ-measurable functions with the property (3.1), then by (2.7) we have

$$0 \leq \frac{1}{2qpM_{p,q}}\left(\frac{\int_\Omega wf^p d\mu}{\int_\Omega wg^q d\mu}\int_\Omega w\frac{g^q}{f^p}d\mu + \frac{\int_\Omega wg^q d\mu}{\int_\Omega wf^p d\mu}\int_\Omega w\frac{f^p}{g^q}d\mu - 2\right) \tag{3.6}$$
$$\leq 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}$$
$$\leq \frac{M_{p,q}}{2qp}\left(\frac{\int_\Omega wf^p d\mu}{\int_\Omega wg^q d\mu}\int_\Omega w\frac{g^q}{f^p}d\mu + \frac{\int_\Omega wg^q d\mu}{\int_\Omega wf^p d\mu}\int_\Omega w\frac{f^p}{g^q}d\mu - 2\right),$$

where $M_{p,q}$ is defined by (3.3) and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$0 \leq \frac{1}{8M^2}\left(\frac{\int_\Omega wf^2 d\mu}{\int_\Omega wg^2 d\mu}\int_\Omega w\frac{g^2}{f^2}d\mu + \frac{\int_\Omega wg^2 d\mu}{\int_\Omega wf^2 d\mu}\int_\Omega w\frac{f^2}{g^2}d\mu - 2\right) \tag{3.7}$$
$$\leq 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^2 d\mu\right)^{1/2}\left(\int_\Omega wg^2 d\mu\right)^{1/2}}$$
$$\leq \frac{M^2}{8}\left(\frac{\int_\Omega wf^2 d\mu}{\int_\Omega wg^2 d\mu}\int_\Omega w\frac{g^2}{f^2}d\mu + \frac{\int_\Omega wg^2 d\mu}{\int_\Omega wf^2 d\mu}\int_\Omega w\frac{f^2}{g^2}d\mu - 2\right),$$

where M is defined by (3.4).

Let f, g are μ-measurable functions with the property (3.1), then by (2.18) we have

$$0 \leq \frac{1}{2pq\tilde{M}_{p,q}}\left(\frac{\int_\Omega wf^p d\mu}{\left(\int_\Omega wg^q d\mu\right)^2}\int_\Omega w\frac{g^{2q}}{f^p}d\mu - 1\right) \tag{3.8}$$
$$\leq 1 - \frac{\int_\Omega wfg d\mu}{\left(\int_\Omega wf^p d\mu\right)^{1/p}\left(\int_\Omega wg^q d\mu\right)^{1/q}}$$
$$\leq \frac{1}{2pq}\tilde{M}_{p,q}\left(\frac{\int_\Omega wf^p d\mu}{\left(\int_\Omega wg^q d\mu\right)^2}\int_\Omega w\frac{g^{2q}}{f^p}d\mu - 1\right) \leq \frac{1}{2pq}\tilde{M}_{p,q}\left(\tilde{M}_{p,q} - 1\right)^2,$$

where

$$\tilde{M}_{p,q} := \left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q \tag{3.9}$$

and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$0 \leq \frac{1}{8\tilde{M}^2}\left(\frac{\int_\Omega wf^2d\mu}{\left(\int_\Omega wg^2d\mu\right)^2}\int_\Omega w\frac{g^4}{f^2}d\mu - 1\right) \tag{3.10}$$
$$\leq 1 - \frac{\int_\Omega wfgd\mu}{\left(\int_\Omega wf^2d\mu\right)^{1/2}\left(\int_\Omega wg^2d\mu\right)^{1/2}}$$
$$\leq \frac{1}{8}\tilde{M}^2\left(\frac{\int_\Omega wf^2d\mu}{\left(\int_\Omega wg^2d\mu\right)^2}\int_\Omega w\frac{g^4}{f^2}d\mu - 1\right) \leq \frac{1}{8}\tilde{M}^2\left(\tilde{M}^2 - 1\right)^2,$$

where

$$\tilde{M} := \frac{M_1M_2}{m_1m_2}. \tag{3.11}$$

4. Applications for Real Numbers

We consider the n-tuples of positive numbers $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and the probability distribution $p = (p_1, ..., p_n)$, i.e. $p_i \geq 0$ for any $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. If there exist the constants m_1, M_1, m_2, M_2, such that

$$0 < m_1 \leq a_i \leq M_1 < \infty,\ 0 < m_2 \leq b_i \leq M_2 < \infty, \tag{4.1}$$

for any $i \in \{1, ..., n\}$, then by (3.2) and (3.5) for the counting discrete measure, we have

$$0 \leq \frac{1}{2qpM_{p,q}}\left(\frac{\sum_{i=1}^n p_ib_i^{2q}}{\left(\sum_{i=1}^n p_ib_i^q\right)^2} - \frac{2\sum_{i=1}^n p_ib_i^qa_i^p}{\sum_{i=1}^n p_ib_i^q\sum_{i=1}^n p_ia_i^p} + \frac{\sum_{i=1}^n p_ia_i^{2p}}{\left(\sum_{i=1}^n p_ia_i^p\right)^2}\right) \tag{4.2}$$
$$\leq 1 - \frac{\sum_{i=1}^n p_ia_ib_i}{\left(\sum_{i=1}^n p_ia_i^p\right)^{1/p}\left(\sum_{i=1}^n p_ib_i^q\right)^{1/q}}$$
$$\leq \frac{M_{p,q}}{2qp}\left(\frac{\sum_{i=1}^n p_ib_i^{2q}}{\left(\sum_{i=1}^n p_ib_i^q\right)^2} - \frac{2\sum_{i=1}^n p_ib_i^qa_i^p}{\sum_{i=1}^n p_ib_i^q\sum_{i=1}^n p_ia_i^p} + \frac{\sum_{i=1}^n p_ia_i^{2p}}{\left(\sum_{i=1}^n p_ia_i^p\right)^2}\right),$$

where $M_{p,q} = \max\left\{\left(\frac{M_1}{m_1}\right)^p, \left(\frac{M_2}{m_2}\right)^q\right\}$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$0 \leq \frac{1}{8M^2}\left(\frac{\sum_{i=1}^n p_ib_i^4}{\left(\sum_{i=1}^n p_ib_i^2\right)^2} - \frac{2\sum_{i=1}^n p_ib_i^2a_i^2}{\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2} + \frac{\sum_{i=1}^n p_ia_i^4}{\left(\sum_{i=1}^n p_ia_i^2\right)^2}\right) \tag{4.3}$$
$$\leq 1 - \frac{\sum_{i=1}^n p_ia_ib_i}{\left(\sum_{i=1}^n p_ia_i^2\right)^{1/2}\left(\sum_{i=1}^n p_ib_i^2\right)^{1/2}}$$
$$\leq \frac{M^2}{8}\left(\frac{\sum_{i=1}^n p_ib_i^4}{\left(\sum_{i=1}^n p_ib_i^2\right)^2} - \frac{2\sum_{i=1}^n p_ib_i^2a_i^2}{\sum_{i=1}^n p_ib_i^2\sum_{i=1}^n p_ia_i^2} + \frac{\sum_{i=1}^n p_ia_i^4}{\left(\sum_{i=1}^n p_ia_i^2\right)^2}\right),$$

where $M := \max\left\{\frac{M_1}{m_1}, \frac{M_2}{m_2}\right\}$.

From (3.6) and (3.7) we have

$$
\begin{aligned}
(4.4)\qquad 0 &\leq \frac{1}{2qpM_{p,q}}\left(\frac{\sum_{i=1}^n p_i a_i^p}{\sum_{i=1}^n p_i b_i^q}\sum_{i=1}^n p_i\frac{b_i^q}{a_i^p}+\frac{\sum_{i=1}^n p_i b_i^q}{\sum_{i=1}^n p_i a_i^p}\sum_{i=1}^n p_i\frac{a_i^p}{b_i^q}-2\right)\\
&\leq 1-\frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^n p_i b_i^q\right)^{1/q}}\\
&\leq \frac{M_{p,q}}{2qp}\left(\frac{\sum_{i=1}^n p_i a_i^p}{\sum_{i=1}^n p_i b_i^q}\sum_{i=1}^n p_i\frac{b_i^q}{a_i^p}+\frac{\sum_{i=1}^n p_i b_i^q}{\sum_{i=1}^n p_i a_i^p}\sum_{i=1}^n p_i\frac{a_i^p}{b_i^q}-2\right),
\end{aligned}
$$

where $M_{p,q}$ is as above and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$
\begin{aligned}
(4.5)\qquad 0 &\leq \frac{1}{8M^2}\left(\frac{\sum_{i=1}^n p_i a_i^2}{\sum_{i=1}^n p_i b_i^2}\sum_{i=1}^n p_i\frac{b_i^2}{a_i^2}+\frac{\sum_{i=1}^n p_i b_i^2}{\sum_{i=1}^n p_i a_i^2}\sum_{i=1}^n p_i\frac{a_i^2}{b_i^2}-2\right)\\
&\leq 1-\frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}}\\
&\leq \frac{M^2}{8}\left(\frac{\sum_{i=1}^n p_i a_i^2}{\sum_{i=1}^n p_i b_i^2}\sum_{i=1}^n p_i\frac{b_i^2}{a_i^2}+\frac{\sum_{i=1}^n p_i b_i^2}{\sum_{i=1}^n p_i a_i^2}\sum_{i=1}^n p_i\frac{a_i^2}{b_i^2}-2\right),
\end{aligned}
$$

where M is as above.

By putting $\tilde{M}_{p,q}:=\left(\frac{M_1}{m_1}\right)^p\left(\frac{M_2}{m_2}\right)^q$ and using (3.8) we have

$$
\begin{aligned}
(4.6)\qquad 0 &\leq \frac{1}{2pq\tilde{M}_{p,q}}\left(\frac{\sum_{i=1}^n p_i a_i^p}{\left(\sum_{i=1}^n p_i b_i^q\right)^2}\sum_{i=1}^n p_i\frac{b_i^{2q}}{a_i^p}-1\right)\\
&\leq 1-\frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^p\right)^{1/p}\left(\sum_{i=1}^n p b_i^q\right)^{1/q}}\\
&\leq \frac{1}{2pq}\tilde{M}_{p,q}\left(\frac{\sum_{i=1}^n p_i a_i^p}{\left(\sum_{i=1}^n p_i b_i^q\right)^2}\sum_{i=1}^n p_i\frac{b_i^{2q}}{a_i^p}-1\right)\leq \frac{1}{2pq}\tilde{M}_{p,q}\left(\tilde{M}_{p,q}-1\right)^2,
\end{aligned}
$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Finally, if $\tilde{M}:=\frac{M_1M_2}{m_1m_2}$ then by (3.10) we have

$$
\begin{aligned}
(4.7)\qquad 0 &\leq \frac{1}{8\tilde{M}^2}\left(\frac{\sum_{i=1}^n p_i a_i^2}{\left(\sum_{i=1}^n p_i b_i^2\right)^2}\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}-1\right)\\
&\leq 1-\frac{\sum_{i=1}^n p_i a_i b_i}{\left(\sum_{i=1}^n p_i a_i^2\right)^{1/2}\left(\sum_{i=1}^n p_i b_i^2\right)^{1/2}}\\
&\leq \frac{1}{8}\tilde{M}^2\left(\frac{\sum_{i=1}^n p_i a_i^2}{\left(\sum_{i=1}^n p_i b_i^2\right)^2}\sum_{i=1}^n p_i\frac{b_i^4}{a_i^2}-1\right)\leq \frac{1}{8}\tilde{M}^2\left(\tilde{M}^2-1\right)^2.
\end{aligned}
$$

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