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SOME INEQUALITIES OF HÖLDER TYPE FOR QUADRATIC WEIGHTED GEOMETRIC MEAN OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. In this paper we consider the *quadratic weighted geometric mean*

$$T\circledS_\nu V := \left|\left|VT^{-1}\right|^\nu T\right|^2$$

for bounded linear operators T, V in the Hilbert space H with T invertible and $\nu \in [0,1]$. Using the celebrated McCarthy's inequality we show that

$$\left\langle T\circledS_{1/p}Vx, x\right\rangle \le \left\langle |V|^2 x, x\right\rangle^{1/p} \left\langle |T|^2 x, x\right\rangle^{1/q}$$

for any $x \in H$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We also provide some norm inequalities such as

$$\left\| \sum_{j=1}^n p_j T_j \circledS_{1/p} V_j \right\| \le \left\| \sum_{j=1}^n p_j |V_j|^2 \right\|^{1/p} \left\| \sum_{j=1}^n p_j |T_j|^2 \right\|^{1/q}$$

for any n-tuples of invertible operators $(T_1, \dots, T_n)$, $(V_1, \dots, V_n)$ and any n-tuple of positive weights. $(p_1, ..., p_n)$.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$, namely $\langle Ax, x\rangle \ge 0$ for any $x \in H$. We write this as $A \ge 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [18] proved that

$$\langle Ax, x\rangle^p \le \langle A^p x, x\rangle, \ p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x\rangle \le \langle Ax, x\rangle^p, \ p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

For various related inequalities, see [1], [2], [8]-[12] and [16]-[17].

Assume that A, B are positive operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The *weighted operator arithmetic mean* for the pair (A, B) is defined by

$$A\nabla_\nu B := (1-\nu) A + \nu B.$$

In 1980, Kubo & Ando, [15] introduced the *weighted operator geometric mean* for the pair (A, B) with A positive and invertible and B positive by

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}.$$

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If A, B are positive invertible operators then we can also consider the *weighted operator harmonic mean* defined by (see for instance [15])

$$A!_{\nu}B := \left((1-\nu) A^{-1} + \nu B^{-1} \right)^{-1}.$$

We have the following fundamental operator means inequalities, or Young's inequalities

$$A!_{\nu}B \leq A\sharp_{\nu}B \leq A\nabla_{\nu}B,\ \ \nu\in[0,1] \tag{1.3}$$

for any A, B positive invertible operators. For $\nu = \frac{1}{2}$, we denote the above means by $A\nabla B$, $A\sharp B$ and $A!B$.

For some new reverses and refinements of Young's inequality see [3]-[4], [13]-[14], [19] and [21].

We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T, V) by [6]

$$T \circledS_{\nu} V := \left| \left| VT^{-1} \right|^{\nu} T \right|^{2} \tag{1.4}$$

for $\nu \geq 0$. For $V \in \mathcal{B}^{-1}(H)$ we can also extend the definition (1.4) for $\nu < 0$.

By the definition of modulus, we also have

$$T \circledS_{\nu} V = T^{*} \left| VT^{-1} \right|^{2\nu} T = T^{*} \left((T^{*})^{-1} V^{*}VT^{-1} \right)^{\nu} T \tag{1.5}$$

for any $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. For $\nu = \frac{1}{2}$ we denote

$$T \circledS V := \left| \left| VT^{-1} \right|^{1/2} T \right|^{2} = T^{*} \left| VT^{-1} \right| T = T^{*} \left((T^{*})^{-1} V^{*}VT^{-1} \right)^{1/2} T.$$

It has been shown in [6] that the following representation holds

$$T \circledS_{\nu} V = |T|^{2} \sharp_{\nu} |V|^{2} \tag{1.6}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and any real ν.

We have the following fundamental inequalities extending (1.3):

$$|T|^{2} \nabla_{\nu} |V|^{2} \geq T \circledS_{\nu} V \geq |T|^{2} !_{\nu} |V|^{2} \tag{1.7}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and for $\nu \in [0,1]$. In particular, we have

$$|T|^{2} \nabla |V|^{2} \geq T \circledS V \geq |T|^{2} ! |V|^{2} \tag{1.8}$$

for T, $V \in \mathcal{B}^{-1}(H)$.

We have the following identities [7] as well

$$(T \circledS_{\nu} V)^{-1} = (T^{*})^{-1} \circledS_{\nu} (V^{*})^{-1} \text{ and } T \circledS_{1-t} V = V \circledS_{t} T \tag{1.9}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$.

In this paper we establish some Hölder type inequalities for the quadratic weighted operator geometric mean for both a pair and two n-tuples of invertible operators on the Hilbert space H. Refinements, reverses and norm inequalities with applications to the usual weighted operator geometric mean are also given.

2. Some Hölder's Type Inequalities

We have the following result:

Theorem 1. *Let* $T \in \mathcal{B}^{-1}(H)$ *and* $V \in \mathcal{B}(H)$. *Then for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have that*

$$\left\langle T \circledS_{1/p} V x, x \right\rangle \leq \left\langle |V|^2 x, x \right\rangle^{1/p} \left\langle |T|^2 x, x \right\rangle^{1/q} \tag{2.1}$$

for any $x \in H$.

In particular, we have

$$\langle T \circledS V x, x \rangle \leq \left\langle |V|^2 x, x \right\rangle^{1/2} \left\langle |T|^2 x, x \right\rangle^{1/2} \tag{2.2}$$

for any $x \in H$.

Proof. By the inequality (1.2) we have

$$\langle A^{\nu} u, u \rangle \leq \langle A u, u \rangle^{\nu},\ \nu \in (0,1) \tag{2.3}$$

for any $u \in H$ with $\|u\| = 1$.

If we take in (2.3) $u = y/\|y\|$ for $y \neq 0$, then we have

$$\langle A^{\nu} y, y \rangle \leq \langle A y, y \rangle^{\nu} \langle y, y \rangle^{1-\nu} \tag{2.4}$$

for any $y \in H$.

Let $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. If we take in (2.4) $A = \left|VT^{-1}\right|^2 = (T^*)^{-1} V^* V T^{-1}$ and $\nu = \frac{1}{p} \in (0,1)$, then we have

$$\left\langle \left|VT^{-1}\right|^{2/p} y, y \right\rangle \leq \left\langle (T^*)^{-1} V^* V T^{-1} y, y \right\rangle^{1/p} \langle y, y \rangle^{1/q} \tag{2.5}$$

for any $y \in H$.

Now, if we take in (2.5) $y = Tx$, $x \in H$, then we get

$$\left\langle \left|VT^{-1}\right|^{2/p} Tx, Tx \right\rangle \leq \left\langle (T^*)^{-1} V^* V T^{-1} Tx, Tx \right\rangle^{1/p} \langle Tx, Tx \rangle^{1/q}$$

that is equivalent to

$$\left\langle T^* \left|VT^{-1}\right|^{2/p} Tx, x \right\rangle \leq \langle V^* V x, x \rangle^{1/p} \langle T^* T x, x \rangle^{1/q}$$

for any $x \in H$, which proves (2.1). □

If we assume that A, B are positive invertible operators, then by taking $T = A^{1/2}$ and $V = B^{1/2}$ in (2.1) and (2.2) we get for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\left\langle A \sharp_{1/p} B x, x \right\rangle \leq \langle A x, x \rangle^{1/p} \langle B x, x \rangle^{1/q} \tag{2.6}$$

for any $x \in H$.

In particular, we have

$$\langle A \sharp B x, x \rangle \leq \langle A x, x \rangle^{1/2} \langle B x, x \rangle^{1/2} \tag{2.7}$$

for any $x \in H$.

Remark 1. *If we use the inequality (1.7) for $\nu = \frac{1}{p}$ and take the inner product, then we get*

$$\left\langle T \circledS_{1/p} V x, x \right\rangle \leq \frac{1}{p} \left\langle |V|^2 x, x \right\rangle + \frac{1}{q} \left\langle |T|^2 x, x \right\rangle \tag{2.8}$$

for any $x \in H$.

By the elementary inequality

$$a^{1/p} b^{1/q} \leq \frac{1}{p} a + \frac{1}{q} b,$$

that holds for $a,\ b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p,\ q > 1$, we have

$$\left\langle |V|^2 x, x \right\rangle^{1/p} \left\langle |T|^2 x, x \right\rangle^{1/q} \leq \frac{1}{p} \left\langle |V|^2 x, x \right\rangle + \frac{1}{q} \left\langle |T|^2 x, x \right\rangle$$

and by (2.1) we deduce

$$\begin{aligned} \left\langle T \circledS_{1/p} V x, x \right\rangle &\leq \left\langle |V|^2 x, x \right\rangle^{1/p} \left\langle |T|^2 x, x \right\rangle^{1/q} \\ &\leq \frac{1}{p} \left\langle |V|^2 x, x \right\rangle + \frac{1}{q} \left\langle |T|^2 x, x \right\rangle \end{aligned} \tag{2.9}$$

for any $x \in H$, which provides a refinement of (2.8).

Since

$$\left\langle |V|^2 x, x \right\rangle = \|Vx\|^2 \text{ and } \left\langle |T|^2 x, x \right\rangle = \|Tx\|^2$$

then (2.1) is equivalent to

$$\left\langle T \circledS_{1/p} V x, x \right\rangle^{1/2} \leq \|Vx\|^{1/p} \|Tx\|^{1/q}, \tag{2.10}$$

for any $x \in H$, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, while (2.2) is equivalent to

$$\left\langle T \circledS V x, x \right\rangle \leq \|Vx\| \, \|Tx\| \tag{2.11}$$

for any $x \in H$.

Since

$$\left\langle T \circledS_{1/p} V x, x \right\rangle = \left\langle \left| \left| VT^{-1} \right|^{1/p} T \right|^2 x, x \right\rangle = \left\| \left| VT^{-1} \right|^{1/p} Tx \right\|^2, \ x \in H$$

then the inequality (2.10) can also be written as

$$\left\| \left| VT^{-1} \right|^{1/p} Tx \right\| \leq \|Vx\|^{1/p} \|Tx\|^{1/q}, \ x \in H \tag{2.12}$$

while the inequality (2.11), as

$$\left\| \left| VT^{-1} \right|^{1/2} Tx \right\|^2 \leq \|Vx\| \, \|Tx\|, \ x \in H. \tag{2.13}$$

Corollary 1. *With the assumptions of Theorem 1 we have the norm inequalities*

$$\left\| T \circledS_{1/p} V \right\|^{1/2} \leq \|V\|^{1/p} \|T\|^{1/q} \text{ and } \|T \circledS V\| \leq \|V\| \, \|T\|. \tag{2.14}$$

Equivalently, we have

$$\left\| \left| VT^{-1} \right|^{1/p} T \right\| \leq \|V\|^{1/p} \|T\|^{1/q} \text{ and } \left\| \left| VT^{-1} \right|^{1/2} T \right\|^2 \leq \|V\| \, \|T\|. \tag{2.15}$$

The proof follows by (2.10) and (2.11) on taking the supremum over $x \in H$, $\|x\| = 1$.

If we assume that A, B are positive invertible operators, then by taking $T = A^{1/2}$ and $V = B^{1/2}$ in (2.14) we get for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\left\|A\sharp_{1/p}B\right\| \leq \|A\|^{1/p} \|B\|^{1/q} \text{ and } \|A\sharp B\|^2 \leq \|A\| \|B\| . \tag{2.16}$$

Further on, consider the Cartesian product $\mathcal{B}^{-1,(n)}(H) := \mathcal{B}^{-1}(H) \times \cdots \times \mathcal{B}^{-1}(H)$, where $\mathcal{B}^{-1}(H)$ denotes the class of all bounded linear invertible operators on H.

Corollary 2. *Let $(T_1, \ldots, T_n)$, $(V_1, \ldots, V_n) \in \mathcal{B}^{-1,(n)}(H)$ and $(p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ be an n-tuple of nonnegative weights not all of them equal to zero. Then we have*

$$\left\langle \sum_{j=1}^{n} p_j T_j \circledS_{1/p} V_j x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x \right\rangle^{1/q} \tag{2.17}$$

for any $x \in H$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and, in particular

$$\left\langle \sum_{j=1}^{n} p_j T_j \circledS V_j x, x \right\rangle \leq \left\langle \sum_{j=1}^{n} p_j |V_j|^2 x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x \right\rangle^{1/2} \tag{2.18}$$

for any $x \in H$.

Proof. From inequality (2.1) we have

$$\left\langle T_j \circledS_{1/p} V_j x, x \right\rangle \leq \left\langle |V_j|^2 x, x \right\rangle^{1/p} \left\langle |T_j|^2 x, x \right\rangle^{1/q} \tag{2.19}$$

for any $j \in \{1, ..., n\}$ and $x \in H$.

If we multiply by $p_j \geq 0$, $j \in \{1, ..., n\}$ and sum over j from 1 to n, then we get

$$\left\langle \sum_{j=1}^{n} p_j T_j \circledS_{1/p} V_j x, x \right\rangle \leq \sum_{j=1}^{n} p_j \left\langle |V_j|^2 x, x \right\rangle^{1/p} \left\langle |T_j|^2 x, x \right\rangle^{1/q} \tag{2.20}$$

for any $x \in H$.

Now, on making use of the weighted Hölder discrete inequality

$$\sum_{j=1}^{n} p_j a_j b_j \leq \left(\sum_{j=1}^{n} p_j a_j^p \right)^{1/p} \left(\sum_{j=1}^{n} p_j b_j^q \right)^{1/q}, \ p, \ q > 1, \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1,...,a_n)$, $(b_1,...,b_n) \in \mathbb{R}_+^n$ and choose $a_j = \left\langle |V_j|^2 x, x \right\rangle^{1/p}$ while $b_j = \left\langle |T_j|^2 x, x \right\rangle^{1/q}$, then we get

$$\begin{aligned}
(2.21)\qquad & \sum_{j=1}^n p_j \left\langle |V_j|^2 x, x \right\rangle^{1/p} \left\langle |T_j|^2 x, x \right\rangle^{1/q} \\
& \le \left(\sum_{j=1}^n p_j \left(\left\langle |V_j|^2 x, x \right\rangle^{1/p} \right)^p \right)^{1/p} \left(\sum_{j=1}^n p_j \left(\left\langle |T_j|^2 x, x \right\rangle^{1/q} \right)^q \right)^{1/q} \\
& = \left(\sum_{j=1}^n p_j \left\langle |V_j|^2 x, x \right\rangle \right)^{1/p} \left(\sum_{j=1}^n p_j \left\langle |T_j|^2 x, x \right\rangle \right)^{1/q} \\
& = \left\langle \sum_{j=1}^n p_j |V_j|^2 x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^n p_j |T_j|^2 x, x \right\rangle^{1/q}
\end{aligned}$$

for any $x \in H$.

On using (2.20) and (2.21) we deduce (2.17). □

Corollary 3. *With the assumptions of Theorem 2 we have the norm inequalities*

$$\left\| \sum_{j=1}^n p_j T_j \circledS_{1/p} V_j \right\| \le \left\| \sum_{j=1}^n p_j |V_j|^2 \right\|^{1/p} \left\| \sum_{j=1}^n p_j |T_j|^2 \right\|^{1/q} \tag{2.22}$$

and

$$\left\| \sum_{j=1}^n p_j T_j \circledS V_j \right\| \le \left\| \sum_{j=1}^n p_j |V_j|^2 \right\|^{1/2} \left\| \sum_{j=1}^n p_j |T_j|^2 \right\|^{1/2} \tag{2.23}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. *If $(A_1, \dots, A_n)$, $(B_1, \dots, B_n)$ are positive invertible operators, then by taking $T_j = A_j^{1/2}$, $V_j = B_j^{1/2}$ in (2.17) and (2.18) we get*

$$\left\langle \sum_{j=1}^n p_j A_j \sharp_{1/p} B_j x, x \right\rangle \le \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle^{1/p} \left\langle \sum_{j=1}^n p_j B_j x, x \right\rangle^{1/q} \tag{2.24}$$

and, in particular

$$\left\langle \sum_{j=1}^n p_j A_j \sharp B_j x, x \right\rangle \le \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^n p_j B_j x, x \right\rangle^{1/2} \tag{2.25}$$

for any $x \in H$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

By the use of (2.22) and (2.23) we get the norm inequalities

$$\left\| \sum_{j=1}^n p_j A_j \sharp_{1/p} B_j \right\| \le \left\| \sum_{j=1}^n p_j A_j \right\|^{1/p} \left\| \sum_{j=1}^n p_j B_j \right\|^{1/q} \tag{2.26}$$

and, in particular

$$\left\|\sum_{j=1}^{n} p_j A_j \sharp B_j\right\| \leq \left\|\sum_{j=1}^{n} p_j A_j\right\|^{1/2} \left\|\sum_{j=1}^{n} p_j B_j\right\|^{1/2}, \tag{2.27}$$

where $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

3. Refinements and Reverses

We need the following lemma that is of interest in itself, [5]:

Lemma 1. *Let* A *be a positive operator, namely* $\langle Ay, y\rangle > 0$ *for any* $y \in H$, $y \neq 0$ *and* $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. *Then for any* $y \in H$ *with* $\|y\| = 1$, *we have*

$$\begin{aligned} &2r \langle Ay, y\rangle^{\nu - \frac{1}{2}} \left(\langle Ay, y\rangle^{1/2} - \left\langle A^{1/2} y, y\right\rangle\right) \\ &\leq \langle Ay, y\rangle^{\nu} - \langle A^{\nu} y, y\rangle \\ &\leq 2R \langle Ay, y\rangle^{\nu - \frac{1}{2}} \left(\langle Ay, y\rangle^{1/2} - \left\langle A^{1/2} y, y\right\rangle\right), \end{aligned} \tag{3.1}$$

where $r := \min\{1-\nu, \nu\}$ *and* $R := \max\{1-\nu, \nu\}$.

Proof. We use the following double inequality obtained by Kittaneh and Manasrah [13], [14] that provide a refinement and an additive reverse for Young's inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{3.2}$$

where $a,\ b \geq 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

This is equivalent to

$$r\left(a - 2\sqrt{a}\sqrt{b} + b\right) \leq (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \leq R\left(a - 2\sqrt{a}\sqrt{b} + b\right) \tag{3.3}$$

for any $a,\ b \geq 0$, $\nu \in [0,1]$.

Fix $a \geq 0$ and by using the continuous functional calculus, we have by replacing b with the operator $A \geq 0$ that

$$\begin{aligned} r\left(aI - 2\sqrt{a} A^{1/2} + A\right) &\leq (1-\nu) aI + \nu A - a^{1-\nu} A^{\nu} \\ &\leq R\left(aI - 2\sqrt{a} A^{1/2} + A\right). \end{aligned} \tag{3.4}$$

Therefore, by (3.4) we have

$$\begin{aligned} r\left(a - 2\sqrt{a}\left\langle A^{1/2} y, y\right\rangle + \langle Ay, y\rangle\right) &\leq (1-\nu) a + \nu \langle Ay, y\rangle - a^{1-\nu} \langle A^{\nu} y, y\rangle \\ &\leq R\left(a - 2\sqrt{a}\left\langle A^{1/2} y, y\right\rangle + \langle Ay, y\rangle\right) \end{aligned} \tag{3.5}$$

for any $y \in H$ with $\|y\| = 1$ and any $a \geq 0$.

Now, if we take $a = \langle Ay, y\rangle$ with $y \in H$ and $\|y\| = 1$, then by (3.5) we get

$$\begin{aligned} &2r \langle Ay, y\rangle^{1/2} \left(\langle Ay, y\rangle^{1/2} - \left\langle A^{1/2} y, y\right\rangle\right) \\ &\leq \langle Ay, y\rangle^{1-\nu} \left(\langle Ay, y\rangle^{\nu} - \langle A^{\nu} y, y\rangle\right) \\ &\leq 2R \langle Ay, y\rangle^{1/2} \left(\langle Ay, y\rangle^{1/2} - \left\langle A^{1/2} y, y\right\rangle\right) \end{aligned}$$

and, by dividing with $\langle Ay, y\rangle^{1-\nu} > 0$, we get the desired result (3.1). □

Theorem 2. *Let T, $V \in \mathcal{B}^{-1}(H)$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have that*

$$
\begin{aligned}
(3.6)\qquad & 2\min\left\{\frac{1}{p},\frac{1}{q}\right\}\left\langle |V|^2 x,x\right\rangle^{\frac{2-p}{2p}}\left\langle |T|^2 x,x\right\rangle^{\frac{2-q}{2q}} \\
& \times\left(\left\langle |V|^2 x,x\right\rangle^{1/2}\left\langle |T|^2 x,x\right\rangle^{1/2} - \langle T \circledS V x,x\rangle\right) \\
& \le \left\langle |V|^2 x,x\right\rangle^{1/p}\left\langle |T|^2 x,x\right\rangle^{1/q} - \langle T \circledS_{1/p} V x,x\rangle \\
& \le 2\max\left\{\frac{1}{p},\frac{1}{q}\right\}\left\langle |V|^2 x,x\right\rangle^{\frac{2-p}{2p}}\left\langle |T|^2 x,x\right\rangle^{\frac{2-q}{2q}} \\
& \times\left(\left\langle |V|^2 x,x\right\rangle^{1/2}\left\langle |T|^2 x,x\right\rangle^{1/2} - \langle T \circledS V x,x\rangle\right)
\end{aligned}
$$

for any $x \in H$, $x \neq 0$,

Proof. If we take $y = u/\|u\|$ for $u \in H$, $u \neq 0$ in (3.1), then we get

$$
\begin{aligned}
& 2r\frac{\langle Au,u\rangle^{\nu-\frac{1}{2}}}{\|u\|^{2\nu-1}}\left(\frac{\langle Ay,y\rangle^{1/2}}{\|u\|} - \frac{\langle A^{1/2}u,u\rangle}{\|u\|^2}\right) \\
& \le \frac{\langle Au,u\rangle^{\nu}}{\|u\|^{2\nu}} - \frac{\langle A^{\nu}u,u\rangle}{\|u\|^2} \\
& \le 2R\frac{\langle Au,u\rangle^{\nu-\frac{1}{2}}}{\|u\|^{2\nu-1}}\left(\frac{\langle Ay,y\rangle^{1/2}}{\|u\|} - \frac{\langle A^{1/2}u,u\rangle}{\|u\|^2}\right),
\end{aligned}
$$

that is equivalent to

$$
\begin{aligned}
& 2r\frac{\langle Au,u\rangle^{\nu-\frac{1}{2}}}{\|u\|^{2\nu-1}}\left(\frac{\langle Au,u\rangle^{1/2}\|u\| - \langle A^{1/2}u,u\rangle}{\|u\|^2}\right) \\
& \le \frac{\langle Au,u\rangle^{\nu}\|u\|^{2(1-\nu)} - \langle A^{\nu}u,u\rangle}{\|u\|^2} \\
& \le 2R\frac{\langle Au,u\rangle^{\nu-\frac{1}{2}}}{\|u\|^{2\nu-1}}\left(\frac{\langle Au,u\rangle^{1/2}\|u\| - \langle A^{1/2}u,u\rangle}{\|u\|^2}\right),
\end{aligned}
$$

or to

$$
\begin{aligned}
(3.7)\qquad & 2r\langle Au,u\rangle^{\nu-\frac{1}{2}}\langle u,u\rangle^{\frac{1}{2}-\nu}\left(\langle Au,u\rangle^{1/2}\langle u,u\rangle^{1/2} - \left\langle A^{1/2}u,u\right\rangle\right) \\
& \le \langle Au,u\rangle^{\nu}\langle u,u\rangle^{1-\nu} - \langle A^{\nu}u,u\rangle \\
& \le 2R\langle Au,u\rangle^{\nu-\frac{1}{2}}\|u\|^{1-2\nu}\left(\langle Au,u\rangle^{1/2}\langle u,u\rangle^{1/2} - \left\langle A^{1/2}u,u\right\rangle\right),
\end{aligned}
$$

where $\nu \in (0,1) \setminus \left\{\frac{1}{2}\right\}$.

Let $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. If we take in (3.7) $A = \left|VT^{-1}\right|^2 = (T^*)^{-1} V^*VT^{-1}$, then we have

$$
\begin{aligned}
(3.8)\quad & 2r\left\langle (T^*)^{-1} V^*VT^{-1}u, u\right\rangle^{\nu-\frac{1}{2}} \langle u,u\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\left\langle (T^*)^{-1} V^*VT^{-1}u, u\right\rangle^{1/2} \langle u,u\rangle^{1/2} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} u, u\right\rangle\right) \\
& \leq \left\langle (T^*)^{-1} V^*VT^{-1}u, u\right\rangle^{\nu} \langle u,u\rangle^{1-\nu} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{\nu} u, u\right\rangle \\
& \leq 2R\left\langle (T^*)^{-1} V^*VT^{-1}u, u\right\rangle^{\nu-\frac{1}{2}} \langle u,u\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\left\langle (T^*)^{-1} V^*VT^{-1}u, u\right\rangle^{1/2} \langle u,u\rangle^{1/2} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} u, u\right\rangle\right),
\end{aligned}
$$

for any $u \in H$, $u \neq 0$.

If we take in (3.8) $u = Tx$, $x \in H$, $x \neq 0$, then we get

$$
\begin{aligned}
& 2r\left\langle (T^*)^{-1} V^*VT^{-1}Tx, Tx\right\rangle^{\nu-\frac{1}{2}} \langle Tx,Tx\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\left\langle (T^*)^{-1} V^*VT^{-1}Tx, Tx\right\rangle^{1/2} \langle Tx,Tx\rangle^{1/2} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} Tx, Tx\right\rangle\right) \\
& \leq \left\langle (T^*)^{-1} V^*VT^{-1}Tx, Tx\right\rangle^{\nu} \langle Tx,Tx\rangle^{1-\nu} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{\nu} Tx, Tx\right\rangle \\
& \leq 2R\left\langle (T^*)^{-1} V^*VT^{-1}Tx, Tx\right\rangle^{\nu-\frac{1}{2}} \langle Tx,Tx\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\left\langle (T^*)^{-1} V^*VT^{-1}Tx, Tx\right\rangle^{1/2} \langle Tx,Tx\rangle^{1/2} - \left\langle \left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} Tx, Tx\right\rangle\right),
\end{aligned}
$$

namely

$$
\begin{aligned}
& 2r\langle V^*Vx, x\rangle^{\nu-\frac{1}{2}} \langle T^*Tx, x\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\langle V^*Vx, x\rangle^{1/2} \langle T^*Tx, x\rangle^{1/2} - \left\langle T^*\left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} Tx, x\right\rangle\right) \\
& \leq \langle V^*Vx, x\rangle^{\nu} \langle T^*Tx, x\rangle^{1-\nu} - \left\langle T^*\left((T^*)^{-1} V^*VT^{-1}\right)^{\nu} Tx, x\right\rangle \\
& \leq 2R\langle V^*Vx, x\rangle^{\nu-\frac{1}{2}} \langle T^*Tx, x\rangle^{\frac{1}{2}-\nu} \\
& \times \left(\langle V^*Vx, x\rangle^{1/2} \langle T^*Tx, x\rangle^{1/2} - \left\langle T^*\left((T^*)^{-1} V^*VT^{-1}\right)^{1/2} Tx, x\right\rangle\right),
\end{aligned}
$$

which, for $\nu = \frac{1}{p} \in (0,1)$, is equivalent to the desired result (3.6). □

Remark 3. *The inequality (3.6) can be written as*

$$
\begin{aligned}
(3.9)\quad & 2\min\left\{\frac{1}{p}, \frac{1}{q}\right\} \|Vx\|^{\frac{2-p}{p}} \|Tx\|^{\frac{2-q}{q}} \left(\|Vx\| \|Tx\| - \langle T \circledS Vx, x\rangle\right) \\
& \leq \|Vx\|^{2/p} \|Tx\|^{2/q} - \langle T \circledS_{1/p} Vx, x\rangle \\
& \leq 2\max\left\{\frac{1}{p}, \frac{1}{q}\right\} \|Vx\|^{\frac{2-p}{p}} \|Tx\|^{\frac{2-q}{q}} \left(\|Vx\| \|Tx\| - \langle T \circledS Vx, x\rangle\right)
\end{aligned}
$$

for any $x \in H$, $x \neq 0$.

If we divide (3.9) by $\|Vx\|^{2/p}\|Tx\|^{2/q}$, $x\in H$, $x\neq 0$, *then we get*

$$2\min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(1-\frac{\langle T \circledS Vx,x\rangle}{\|Vx\|\|Tx\|}\right)\leq 1-\frac{\left\langle T\circledS_{1/p}Vx,x\right\rangle}{\|Vx\|^{2/p}\|Tx\|^{2/q}} \leq 2\max\left\{\frac{1}{p},\frac{1}{q}\right\}\left(1-\frac{\langle T \circledS Vx,x\rangle}{\|Vx\|\|Tx\|}\right) \tag{3.10}$$

where T, $V\in\mathcal{B}^{-1}(H)$ *and* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Finally, if we assume that A, B are positive invertible operators, then by taking $T=A^{1/2}$ and $V=B^{1/2}$ in (3.10) we get for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned} &2\min\left\{\frac{1}{p},\frac{1}{q}\right\}\left(1-\frac{\langle A\sharp Bx,x\rangle}{\left\|A^{1/2}x\right\|\left\|B^{1/2}x\right\|}\right)\\ &\leq 1-\frac{\left\langle A\sharp_{1/p}Bx,x\right\rangle}{\left\|A^{1/2}x\right\|^{2/p}\left\|B^{1/2}x\right\|^{2/q}}\\ &\leq 2\max\left\{\frac{1}{p},\frac{1}{q}\right\}\left(1-\frac{\langle A\sharp Bx,x\rangle}{\left\|A^{1/2}x\right\|\left\|B^{1/2}x\right\|}\right)\end{aligned} \tag{3.11}$$

for any $x\in H$, $x\neq 0$.

References

[1] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *J. Inequal. & Appl.*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821. Preprint RGMIA *Res. Rep. Coll.*, **11** (2008), Supliment. Art. 15. [http://rgmia.org/papers/v11e/RevJensenOp.pdf].

[2] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type,* Springer Briefs in Mathematics, Springer, 2012.

[3] S. S. Dragomir, A note on Young's inequality, *Transylv. J. Math. Mech*. **8** (2016), No. 1, 45-49. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[4] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 2016 to appear (Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 131. [http://rgmia.org/papers/v18/v18a131.pdf].

[5] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 143. [http://rgmia.org/papers/v18/v18a143.pdf].

[6] S. S. Dragomir, The quadratic weighted geometric mean for bounded linear operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 145. [http://rgmia.org/papers/v19/v19a145.pdf].

[7] S. S. Dragomir, Some properties of quadratic weighted geometric mean of bounded linear operators in Hilbert spaces via Kato's inequality, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 147. [http://rgmia.org/papers/v19/v19a147.pdf].

[8] M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality. *Nihonkai Math. J.* **5** (1994), no. 1, 61–67.

[9] M. Fuji, S. Izumino, R. Nakamoto and Y. Seo, Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.*, **8** (1997), 117-122.

[10] T. Furuta, Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications. *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), no. 3, 38–41.

[11] T. Furuta, Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities. *J. Inequal. Appl.* **2** (1998), no. 2, 137–148.

[12] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators. *Amer. Math. Monthly* **108** (2001), no. 1, 68–69.
[13] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.
[14] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.
[15] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **246** (1979/80), no. 3, 205–224.
[16] C.-S. Lin and Y. J. Cho, On Hölder-McCarthy-type inequalities with powers. *J. Korean Math. Soc.* **39** (2002), no. 3, 351–361.
[17] C.-S. Lin and Y. J. Cho, On Kantorovich inequality and Hölder-McCarthy inequalities. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **11** (2004), no. 4, 481–490.
[18] C. A. McCarthy, c_p, *Israel J. Math.* **5** (1967), 249–271.
[19] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[20] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[21] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

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