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REVERSE INEQUALITIES OF HÖLDER TYPE FOR QUADRATIC WEIGHTED GEOMETRIC MEAN OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. In this paper we consider the *quadratic weighted geometric mean*

$$T\circledS_\nu V := \left|\left|VT^{-1}\right|^\nu T\right|^2$$

for bounded linear operators T, V in the Hilbert space H with T invertible and $\nu \in [0,1]$. If T, V are invertible operators and $0 < m < M < \infty$ are such that $m\|Tx\| \le \|Vx\| \le M\|Tx\|$ for any $x \in H$, then we show among others that

$$\|Vx\|^{1/p}\|Tx\|^{1/q}$$
$$\le \min\left\{\mathcal{S}^{1/2}\left(\left(\frac{M}{m}\right)^2\right), \mathcal{K}^{\frac{1}{2}\max\{1/p,1/q\}}\left(\left(\frac{M}{m}\right)^2\right)\right\}\left\langle T\circledS_{1/p}Vx, x\right\rangle^{1/2}$$

for any $x \in H$, where p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, $\mathcal{S}$ is *Specht's ratio* and $\mathcal{K}$ is *Kantorovich's constant.* Applications for the classical weighted operator geometric mean introduced by Kubo and Ando in 1980 are also provided.

1. Introduction

Assume that A, B are positive operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. The *weighted operator arithmetic mean* for the pair (A, B) is defined by

$$A\nabla_\nu B := (1-\nu)A + \nu B.$$

In 1980, Kubo & Ando, [18] introduced the *weighted operator geometric mean* for the pair (A, B) with A positive and invertible and B positive by

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}.$$

If A, B are positive invertible operators then we can also consider the *weighted operator harmonic mean* defined by (see for instance [18])

$$A!_\nu B := \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}.$$

We have the following fundamental operator means inequalities, or Young's inequalities

$$A!_\nu B \le A\sharp_\nu B \le A\nabla_\nu B,\ \nu\in[0,1] \tag{1.1}$$

for any A, B positive invertible operators. For $\nu = \frac{1}{2}$, we denote the above means by $A\nabla B$, $A\sharp B$ and $A!B$.

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For some new reverses and refinements of Young's inequality see [4]-[5], [16]-[17], [19] and [24].

We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T,V) by [7]

$$T \circledS_\nu V := \left| \left| VT^{-1} \right|^\nu T \right|^2 \tag{1.2}$$

for $\nu \geq 0$. For $V \in \mathcal{B}^{-1}(H)$ we can also extend the definition (1.2) for $\nu < 0$.

By the definition of operator modulus, i.e. we recall that $|U| := \sqrt{U^*U}$ where $\in \mathcal{B}(H)$, we also have

$$T \circledS_\nu V = T^* \left| VT^{-1} \right|^{2\nu} T = T^* \left((T^*)^{-1} V^* V T^{-1} \right)^\nu T \tag{1.3}$$

for any $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. For $\nu = \frac{1}{2}$ we denote

$$T \circledS V := \left| \left| VT^{-1} \right|^{1/2} T \right|^2 = T^* \left| VT^{-1} \right| T = T^* \left((T^*)^{-1} V^* V T^{-1} \right)^{1/2} T.$$

It has been shown in [7] that the following representation holds

$$T \circledS_\nu V = |T|^2 \sharp_\nu |V|^2 \tag{1.4}$$

for T, $V \in \mathcal{B}^{-1}(H)$ and any real ν.

We have the following fundamental inequalities extending (1.1):

$$|T|^2 \nabla_\nu |V|^2 \geq T \circledS_\nu V \geq |T|^2 !_\nu |V|^2 \tag{1.5}$$

for $T, V \in \mathcal{B}^{-1}(H)$ and for $\nu \in [0,1]$. In particular, we have

$$|T|^2 \nabla |V|^2 \geq T \circledS V \geq |T|^2 ! |V|^2 \tag{1.6}$$

for T, $V \in \mathcal{B}^{-1}(H)$.

We have the following identities [8] as well

$$(T \circledS_\nu V)^{-1} = (T^*)^{-1} \circledS_\nu (V^*)^{-1} \text{ and } T \circledS_{1-t} V = V \circledS_t T \tag{1.7}$$

for any T, $V \in \mathcal{B}^{-1}(H)$ and $\nu \in [0,1]$.

We also obtained the following Hölder type inequality:

Theorem 1 (Dragomir, 2016, [9])**.** *Let $T \in \mathcal{B}^{-1}(H)$ and $V \in \mathcal{B}(H)$. Then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have that*

$$\left\langle T \circledS_{1/p} V x, x \right\rangle \leq \left\langle |V|^2 x, x \right\rangle^{1/p} \left\langle |T|^2 x, x \right\rangle^{1/q} \tag{1.8}$$

for any $x \in H$.

In particular, we have

$$\langle T \circledS V x, x \rangle \leq \left\langle |V|^2 x, x \right\rangle^{1/2} \left\langle |T|^2 x, x \right\rangle^{1/2} \tag{1.9}$$

for any $x \in H$.

Motivated by the above results, we establish in this paper several multiplicative reverses of Hölder type inequality (1.8) under appropriate boundedness assumptions for the operators involved. These are done via some recent reverses of scalar Young's inequality obtained in [24], [19], [4] and [5]. Applications for the classical weighted operator geometric mean introduced by Kubo and Ando in 1980 are also provided

2. The Main Results

We need the following simple fact:

Lemma 1. *Let T, $V \in \mathcal{B}^{-1}(H)$ and the numbers m, M with $0 < m < M < \infty$. The following statements are equivalent:*
(i) The inequality

$$m\|Tx\| \leq \|Vx\| \leq M\|Tx\| \tag{2.1}$$

holds for any $x \in H$;
(ii) We have the operator inequality

$$m1_H \leq \left|VT^{-1}\right| \leq M1_H. \tag{2.2}$$

Proof. The inequality (2.1) is equivalent to

$$m^2\|Tx\|^2 \leq \|Vx\|^2 \leq M^2\|Tx\|^2$$

for any $x \in H$, namely

$$m^2\langle T^*Tx, x\rangle \leq \langle V^*Vx, x\rangle \leq M^2\langle T^*Tx, x\rangle$$

for any $x \in H$, which can be written in the operator order as

$$m^2T^*T \leq V^*V \leq M^2T^*T. \tag{2.3}$$

Since $T \in \mathcal{B}^{-1}(H)$, then the inequality (2.3) is equivalent to

$$m^2 1_H \leq \left(T^{-1}\right)^* V^*VT^{-1} \leq M^2 1_H,$$

namely

$$m^2 1_H \leq \left|VT^{-1}\right|^2 \leq M^2 1_H,$$

which in its turn is equivalent to (2.2). □

We recall that *Specht's ratio* [23] is defined by

$$\mathcal{S}(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1,\infty), \\ \\ 1 & \text{if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1}\mathcal{S}(h) = 1$, $\mathcal{S}(h) = \mathcal{S}\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function $\mathcal{S}$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

Tominaga [24] had proved a reverse Young inequality with the Specht's ratio [23] as follows:

$$\left(a^{1-\nu}b^{\nu} \leq\right)(1-\nu)a + \nu b \leq \mathcal{S}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{2.4}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

We have the following result for operators:

Theorem 2. *Let T, $V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$ such that either the condition (2.1) or, equivalently (2.2) is true. Then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have that*

$$\left\langle |V|^2 x, x\right\rangle^{1/p}\left\langle |T|^2 x, x\right\rangle^{1/q} \leq \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right)\left\langle T\circledS_{1/p}Vx, x\right\rangle \tag{2.5}$$

for any $x \in H$.

In particular, we have

$$\left\langle |V|^2 x, x \right\rangle^{1/2} \left\langle |T|^2 x, x \right\rangle^{1/2} \leq \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right) \langle T \circledS V x, x\rangle \tag{2.6}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$. Let a, $b \in [k, K] \subset (0, \infty)$, then $\frac{k}{K} \leq \frac{a}{b} \leq \frac{K}{k}$ with $\frac{k}{K} < 1 < \frac{K}{k}$. If $\frac{a}{b} \in \left[\frac{k}{K}, 1\right)$ then $\mathcal{S}\left(\frac{a}{b}\right) \leq \mathcal{S}\left(\frac{k}{K}\right) = \mathcal{S}\left(\frac{K}{k}\right)$. If $\frac{a}{b} \in \left(1, \frac{K}{k}\right]$ then also $\mathcal{S}\left(\frac{a}{b}\right) \leq \mathcal{S}\left(\frac{K}{k}\right)$. Therefore for any a, $b \in [k, K]$ we have by (2.4) that

$$(1-\nu) a + \nu b \leq \mathcal{S}\left(\frac{K}{k}\right) a^{1-\nu} b^{\nu}. \tag{2.7}$$

Using the functional calculus for the operator A with $k1_H \leq A \leq K1_H$, we have from (2.7) that

$$(1-\nu) aI + \nu A \leq \mathcal{S}\left(\frac{K}{k}\right) a^{1-\nu} A^{\nu}$$

for any $a \in [k, K]$ and $\nu \in (0,1)$, that is equivalent to

$$(1-\nu) a + \nu \langle Au, u\rangle \leq \mathcal{S}\left(\frac{K}{k}\right) a^{1-\nu} \langle A^{\nu} u, u\rangle \tag{2.8}$$

for any $u \in H$ with $\|u\| = 1$.

If $u \in H$ with $\|u\| = 1$ then $\langle Au, u\rangle \in [k, K]$ and by taking $a = \langle Au, u\rangle$ in (2.8) we get

$$\langle Au, u\rangle \leq \mathcal{S}\left(\frac{K}{k}\right) \langle Au, u\rangle^{1-\nu} \langle A^{\nu} u, u\rangle$$

and by dividing with $\langle Au, u\rangle^{1-\nu} > 0$ we deduce

$$\langle Au, u\rangle^{\nu} \leq \mathcal{S}\left(\frac{K}{k}\right) \langle A^{\nu} u, u\rangle \tag{2.9}$$

for any $u \in H$ with $\|u\| = 1$.

Let $y \in H$ with $y \neq 0$ and take $u = \frac{y}{\|y\|}$ in (2.9) to get

$$\langle Ay, y\rangle^{\nu} \langle y, y\rangle^{1-\nu} \leq \mathcal{S}\left(\frac{K}{k}\right) \langle A^{\nu} y, y\rangle, \tag{2.10}$$

for any $y \in H$.

Since $m^2 1_H \leq \left|VT^{-1}\right|^2 \leq M^2 1_H$, then by taking $A = \left|VT^{-1}\right|^2$, $k = m^2$ and $K = M^2$ in (2.10), we get

$$\left\langle \left|VT^{-1}\right|^2 y, y \right\rangle^{\nu} \langle y, y\rangle^{1-\nu} \leq \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right) \left\langle \left(\left|VT^{-1}\right|^2\right)^{\nu} y, y \right\rangle, \tag{2.11}$$

for any $y \in H$.

If we take $y = Tx$ with $x \in H$, then we get

$$\left\langle \left|VT^{-1}\right|^2 Tx, Tx \right\rangle^{\nu} \langle Tx, Tx\rangle^{1-\nu} \leq \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right) \left\langle \left(\left|VT^{-1}\right|^2\right)^{\nu} Tx, Tx \right\rangle,$$

that is equivalent to

$$\left\langle |V|^2 x, x \right\rangle^{\nu} \left\langle |T|^2 x, x \right\rangle^{1-\nu} \leq \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right) \left\langle T^* \left(\left|VT^{-1}\right|^2\right)^{\nu} Tx, x \right\rangle,$$

and by taking $\nu = \frac{1}{p} \in (0,1)$, with the desired inequality (2.5). $\square$

Remark 1. *Let T, $V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$ such that either the condition (2.1) or, equivalently (2.2) is true. The inequalities (2.5) and (2.6) may be written in an equivalent form for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ as*

$$\|Vx\|^{1/p} \|Tx\|^{1/q} \le \mathcal{S}^{1/2}\left(\left(\frac{M}{m}\right)^2\right) \langle T \circledS_{1/p} Vx, x\rangle^{1/2} \tag{2.12}$$

for any $x \in H$.

In particular, we have

$$\|Vx\| \|Tx\| \le \mathcal{S}\left(\left(\frac{M}{m}\right)^2\right) \langle T \circledS Vx, x\rangle \tag{2.13}$$

for any $x \in H$.

Corollary 1. *Let A, B be two invertible positive operators and k, $K > 0$ such that*

$$kA \le B \le KA. \tag{2.14}$$

Then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have that

$$\langle Bx, x\rangle^{1/p} \langle Ax, x\rangle^{1/q} \le \mathcal{S}\left(\frac{K}{k}\right) \langle A \sharp_{1/p} Bx, x\rangle \tag{2.15}$$

for any $x \in H$.

In particular, we have

$$\langle Bx, x\rangle^{1/2} \langle Ax, x\rangle^{1/2} \le \mathcal{S}\left(\frac{K}{k}\right) \langle A \sharp Bx, x\rangle \tag{2.16}$$

for any $x \in H$.

Proof. By multiplying both sides of (2.14) by $A^{-1/2}$ we have $k1_H \le A^{-1/2}BA^{-1/2} \le K1_H$. If we take $T = A^{1/2}$ and $V = B^{1/2}$ then

$$\left|VT^{-1}\right|^2 = (T^*)^{-1} V^*VT^{-1} = A^{-1/2}BA^{-1/2},$$

which satisfy the condition (2.2) with $m = \sqrt{k}$ and $M = \sqrt{K}$.

By applying Theorem 2 for this selection we get the desired results (2.15) and (2.16). $\square$

We consider the *Kantorovich's constant* defined by

$$\mathcal{K}(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{2.17}$$

The function $\mathcal{K}$ is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $\mathcal{K}(h) \ge 1$ for any $h > 0$ and $\mathcal{K}(h) = \mathcal{K}\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$(1-\nu)a + \nu b \le \mathcal{K}^{\max\{1-\nu,\nu\}}\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{2.18}$$

where a, $b > 0$, $\nu \in [0,1]$.

This inequality has been obtained by Liao et al. in [19].

Theorem 3. *With the assumptions of Theorem 2 we have we have that*

$$\left\langle |V|^2 x, x\right\rangle^{1/p} \left\langle |T|^2 x, x\right\rangle^{1/q} \le \mathcal{K}^{\max\{1/p,1/q\}}\left(\left(\frac{M}{m}\right)^2\right)\left\langle T \circledS_{1/p} V x, x\right\rangle \tag{2.19}$$

for any $x \in H$.

In particular, we have

$$\left\langle |V|^2 x, x\right\rangle \left\langle |T|^2 x, x\right\rangle \le \mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)\left\langle T \circledS V x, x\right\rangle^2 \tag{2.20}$$

for any $x \in H$.

Proof. Assume that $\nu \in (0,1)$ and put $R = \max\{1-\nu, \nu\}$. Let $a, b \in [k, K] \subset (0, \infty)$, then $\frac{k}{K} \le \frac{a}{b} \le \frac{K}{k}$ with $\frac{k}{K} < 1 < \frac{K}{k}$. If $\frac{a}{b} \in \left[\frac{k}{K}, 1\right)$ then $\mathcal{K}^R\left(\frac{a}{b}\right) \le \mathcal{K}^R\left(\frac{k}{K}\right) = \mathcal{K}^R\left(\frac{K}{k}\right)$. If $\frac{a}{b} \in \left(1, \frac{K}{k}\right]$ then also $\mathcal{K}^R\left(\frac{a}{b}\right) \le \mathcal{K}^R\left(\frac{K}{k}\right)$. Therefore for any $a, b \in [k, K]$ we have by (2.18) that

$$(1-\nu)\, a + \nu b \le \mathcal{K}^R\left(\frac{K}{k}\right) a^{1-\nu} b^{\nu}.$$

Now, on making use of a similar argument to the one from Theorem 2, we get (2.19). We omit the details. □

Remark 2. *Let $T, V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$ such that either the condition (2.1) or, equivalently (2.2) is true. The inequalities (2.19) and (2.20) may be written in an equivalent form for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ as*

$$\|Vx\|^{1/p} \|Tx\|^{1/q} \le \mathcal{K}^{\frac{1}{2}\max\{1/p,1/q\}}\left(\left(\frac{M}{m}\right)^2\right)\left\langle T \circledS_{1/p} V x, x\right\rangle^{1/2} \tag{2.21}$$

for any $x \in H$.

In particular, we have

$$\|Vx\|^2 \|Tx\|^2 \le \mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)\left\langle T \circledS V x, x\right\rangle^2 \tag{2.22}$$

for any $x \in H$.

Corollary 2. *With the assumptions of Corollary 1 we have that*

$$\langle Bx, x\rangle^{1/p} \langle Ax, x\rangle^{1/q} \le \mathcal{K}^{\max\{1/p,1/q\}}\left(\frac{K}{k}\right)\left\langle A \sharp_{1/p} Bx, x\right\rangle \tag{2.23}$$

for any $x \in H$.

In particular, we have

$$\langle Bx, x\rangle \langle Ax, x\rangle \le \mathcal{K}\left(\frac{K}{k}\right)\langle A \sharp Bx, x\rangle^2 \tag{2.24}$$

for any $x \in H$.

3. Related Results

In the recent paper [4] we obtained the following exponential reverse of Young's inequality

$$\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu\,(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{3.1}$$

where $a,\ b>0,\ \nu\in[0,1]$.

By the use of this inequality we can state the following result as well:

Theorem 4. *With the assumptions of Theorem 2 we have that*

$$\begin{aligned}&\left\langle |V|^2\,x,x\right\rangle^{1/p}\left\langle |T|^2\,x,x\right\rangle^{1/q}\\ &\le \exp\left(\frac{4}{pq}\left[\mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)-1\right]\right)\left\langle T\circledS_{1/p}Vx,x\right\rangle\end{aligned} \tag{3.2}$$

for any $x\in H$.

In particular, we have

$$\left\langle |V|^2\,x,x\right\rangle\left\langle |T|^2\,x,x\right\rangle \le \exp\left[2\mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left\langle T\circledS Vx,x\right\rangle^2 \tag{3.3}$$

for any $x\in H$.

Proof. For any $a,\ b\in[k,K]\subset(0,\infty)$ we have from (3.1) that

$$(1-\nu)\,a+\nu b \le a^{1-\nu}b^{\nu}\exp\left[4\nu\,(1-\nu)\left(\mathcal{K}\left(\frac{K}{k}\right)-1\right)\right],$$

where $a,\ b>0,\ \nu\in[0,1]$.

On making use of a similar argument to the one in the proof of Theorem 2 and we omit the details. □

We observe that the inequality (3.2) can be written in an equivalent form as

$$\|Vx\|^{1/p}\,\|Tx\|^{1/q} \le \exp\left(\frac{2}{pq}\left[\mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)-1\right]\right)\left\langle T\circledS_{1/p}Vx,x\right\rangle^{1/2}, \tag{3.4}$$

for any $x\in H$, while (3.3) as

$$\|Vx\|\,\|Tx\| \le \exp\left[\mathcal{K}\left(\left(\frac{M}{m}\right)^2\right)-1\right]\left\langle T\circledS Vx,x\right\rangle \tag{3.5}$$

for any $x\in H$.

Corollary 3. *With the assumptions of Corollary 1 we have that*

$$\langle Bx,x\rangle^{1/p}\,\langle Ax,x\rangle^{1/q} \le \exp\left(\frac{4}{pq}\left[\mathcal{K}\left(\frac{K}{k}\right)-1\right]\right)\left\langle A\sharp_{1/p}Bx,x\right\rangle \tag{3.6}$$

for any $x\in H$.

In particular, we have

$$\langle Bx,x\rangle^{1/2}\,\langle Ax,x\rangle^{1/2} \le \exp\left(\mathcal{K}\left(\frac{K}{k}\right)-1\right)\langle A\sharp Bx,x\rangle \tag{3.7}$$

for any $x\in H$.

In [5] we also proved the following inequality

$$(1-\nu)a+\nu b\leq a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right], \tag{3.8}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

A similar result has been obtained independently in [1] for $a<b$.

By the use of this inequality we can state the following result as well:

Theorem 5. *With the assumptions of Theorem 2 we have that*

$$\left\langle |V|^2 x,x\right\rangle^{1/p}\left\langle |T|^2 x,x\right\rangle^{1/q}\leq\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\langle T\circledS_{1/p}Vx,x\rangle \tag{3.9}$$

for any $x\in H$.

In particular, we have

$$\left\langle |V|^2 x,x\right\rangle\left\langle |T|^2 x,x\right\rangle\leq\exp\left[\frac{1}{4}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\langle T\circledS Vx,x\rangle^2 \tag{3.10}$$

for any $x\in H$.

Proof. If $a,\ b\in[k,K]\subset(0,\infty)$ and since

$$0<\frac{\max\{a,b\}}{\min\{a,b\}}-1\leq\frac{K}{k}-1,$$

hence

$$\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\leq\left(\frac{K}{k}-1\right)^2.$$

By using (3.8) we have

$$(1-\nu)a+\nu b\leq a^{1-\nu}b^{\nu}\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{K}{k}-1\right)^2\right],$$

for any $a,\ b\in[k,K]\subset(0,\infty)$ and $\nu\in[0,1]$.

On making use of a similar argument to the one in the proof of Theorem 2 we deduce the desired result (3.9) and we omit the details. □

We observe that the inequality (3.9) can be written in an equivalent form as

$$\|Vx\|^{1/p}\|Tx\|^{1/q}\leq\exp\left[\frac{1}{pq}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\left\langle T\circledS_{1/p}Vx,x\right\rangle^{1/2}, \tag{3.11}$$

for any $x\in H$, while (3.10) as

$$\|Vx\|\|Tx\|\leq\exp\left[\frac{1}{2}\left(\left(\frac{M}{m}\right)^2-1\right)^2\right]\langle T\circledS Vx,x\rangle \tag{3.12}$$

for any $x\in H$.

Corollary 4. *With the assumptions of Corollary 1 we have that*

$$\langle Bx,x\rangle^{1/p}\langle Ax,x\rangle^{1/q}\leq\exp\left[\frac{1}{2pq}\left(\frac{K}{k}-1\right)^{2}\right]\left\langle A\sharp_{1/p}Bx,x\right\rangle \tag{3.13}$$

for any $x\in H$.

In particular, we have

$$\langle Bx,x\rangle^{1/2}\langle Ax,x\rangle^{1/2}\leq\exp\left[\frac{1}{8}\left(\frac{K}{k}-1\right)^{2}\right]\langle A\sharp Bx,x\rangle \tag{3.14}$$

for any $x\in H$.

If from the inequalities (2.5), (2.19), (3.2) and (3.9) we consider the bounds

$$\mathcal{S}\left(\left(\frac{M}{m}\right)^{2}\right),\ \mathcal{K}^{\max\{1/p,1/q\}}\left(\left(\frac{M}{m}\right)^{2}\right),\ \exp\left(\frac{4}{pq}\left[\mathcal{K}\left(\left(\frac{M}{m}\right)^{2}\right)-1\right]\right)$$

and

$$\exp\left[\frac{1}{2pq}\left(\left(\frac{M}{m}\right)^{2}-1\right)^{2}\right],$$

where $0<m<M<\infty$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by denoting $\frac{1}{p}=\nu$ and $x=\left(\frac{M}{m}\right)^{2}>1$ we should consider the functions of two variables

$$B_{1}(\nu,x):=\mathcal{S}(x),\ B_{2}(\nu,x):=\mathcal{K}^{\max\{\nu,1-\nu\}}(x),$$

$$B_{3}(\nu,x):=\exp\left(4\nu(1-\nu)\left[\mathcal{K}(x)-1\right]\right)$$

and

$$B_{4}(\nu,x):=\exp\left[\frac{1}{2}\nu(1-\nu)(x-1)^{2}\right]$$

in order to study which of the inequalities (2.5), (2.19), (3.2) and (3.9) is best and when. Some numerical experiments have been conducted in the recent work [10] showing that, in general, these bounds do not compare, meaning that some time one is better then the other for different selection of the parameters (ν,x). The details are left to the interested reader.

References

[1] H. Alzer, C. M. da Fonseca and A. Kovačec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63** (2015), Issue 3, 622-635.

[2] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *J. Inequal. & Appl.*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821. Preprint RGMIA *Res. Rep. Coll.*, **11** (2008), Supliment. Art. 15. [http://rgmia.org/papers/v11e/RevJensenOp.pdf].

[3] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type,* Springer Briefs in Mathematics, Springer, 2012.

[4] S. S. Dragomir, A note on Young's inequality, *Transylv. J. Math. Mech*. **8** (2016), No. 1, 45-49. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 126. [http://rgmia.org/papers/v18/v18a126.pdf].

[5] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 2016 *to appear*, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 131. [http://rgmia.org/papers/v18/v18a131.pdf].

[6] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 143. [http://rgmia.org/papers/v18/v18a143.pdf].
[7] S. S. Dragomir, The quadratic weighted geometric mean for bounded linear operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 145. [http://rgmia.org/papers/v19/v18a145.pdf]
[8] S. S. Dragomir, Some properties of quadratic weighted geometric mean of bounded linear operators in Hilbert spaces via Kato's inequality, Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 147. [http://rgmia.org/papers/v19/v18a147.pdf]
[9] S. S. Dragomir, Some inequalities of Hölder type for quadratic weighted geometric mean of bounded linear operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 151. [http://rgmia.org/papers/v19/v18a151.pdf]
[10] S. S. Dragomir and A. McAndrew, A note on numerical comparison of some multiplicative bounds related to weighted arithmetic and geometric means, Preprint *RGMIA Res. Rep. Coll.* **19** (2016), Art. 102. [http://rgmia.org/papers/v19/v19a102.pdf].
[11] M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality. *Nihonkai Math. J.* **5** (1994), no. 1, 61–67.
[12] M. Fuji, S. Izumino, R. Nakamoto and Y. Seo, Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities, *Nihonkai Math. J.*, **8** (1997), 117-122.
[13] T. Furuta, Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications. *Proc. Japan Acad. Ser. A Math. Sci.* **73** (1997), no. 3, 38–41.
[14] T. Furuta, Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities. *J. Inequal. Appl.* **2** (1998), no. 2, 137–148.
[15] T. Furuta, The Hölder-McCarthy and the Young inequalities are equivalent for Hilbert space operators. *Amer. Math. Monthly* **108** (2001), no. 1, 68–69.
[16] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.
[17] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.
[18] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **246** (1979/80), no. 3, 205–224.
[19] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[20] C.-S. Lin and Y. J. Cho, On Hölder-McCarthy-type inequalities with powers. *J. Korean Math. Soc.* **39** (2002), no. 3, 351–361.
[21] C.-S. Lin and Y. J. Cho, On Kantorovich inequality and Hölder-McCarthy inequalities. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **11** (2004), no. 4, 481–490.
[22] C. A. McCarthy, c_p, *Israel J. Math.* **5** (1967), 249–271.
[23] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.*, **74** (1960), pp. 91-98.
[24] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

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