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Inequalities of Čebyšev Type for Lipschitian Functions in Banach Algebras

Marius V. Boldea

Abstract. In these paper we give some Čebyšev type norm inequalities for two Lipschitian functions on Banach algebras. Some examples for power function, exponential and the resolvent functions are also provided.

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1 Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further

$$\|ab\| \leq \|a\| \, \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*.

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.
Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.
For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;

(ii) $\{a \in \mathcal{B}\colon\ \|1 - a\| < 1\} \subset$Inv$\mathcal{B}$;

(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;

(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ.
The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} :\ \lambda - a \in \mathrm{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to$Inv$\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} :\ |\lambda| \leq \|a\|\}$. The *spectral radius* of a is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.
If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \leq \nu(a)\nu(b) \text{ and } \nu(a + b) \leq \nu(a) + \nu(b).$$

Let f be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j\ (|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.
It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}[\exp(bx) - \exp(ax)].$$

Moreover, if x and y are commuting and $y-x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left((1-s)\,x+sy\right)ds &= \int_0^1 \exp\left(s\,(y-x)\right)\exp\left(x\right)ds\\ &= \left(\int_0^1 \exp\left(s\,(y-x)\right)ds\right)\exp\left(x\right)\\ &= (y-x)^{-1}\left[\exp\left(y-x\right)-I\right]\exp\left(x\right)\\ &= (y-x)^{-1}\left[\exp\left(y\right)-\exp\left(x\right)\right].\end{aligned}$$

Inequalities for functions of operators in Hilbert spaces may be found in the papers [15], [14] and in the recent monographs [35], [36] and the references therein.
Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.
Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n|\,\lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map

$$x \mapsto \widetilde{f}(x) : B(0,R) \to \mathcal{B}, \widetilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \leq \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.
The following result has been obtained in [38].

Theorem 1.1. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R > 0$. If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| \leq 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \tag{1.1}$$
$$\leq \|x - 1\| \|y - 1\| \left\{ f_A(|\lambda|) \left[|\lambda| f'_A(|\lambda|) + |\lambda|^2 f''_A(|\lambda|) \right] - \left[|\lambda| f'_A(|\lambda|) \right]^2 \right\}.$$

For other similar results see [37] and [38].
Motivated by the above results, in this paper we consider the similar problem to provide upper bounds for the Čebyšev quantity

$$\left\| P_n \sum_{i=1}^{n} p_i F(x_i) G(y_i) - \sum_{i=1}^{n} p_i F(x_i) \sum_{i=1}^{n} p_i G(y_i) \right\|,$$

where F, G are some functions defined on a Banach algebra $\mathcal{B}$, $p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$ and $x_i, y_i \in \mathcal{C}$, $i = 1, 2, ..., n$ where $\mathcal{C}$ is an appropriate subset of $\mathcal{B}$.
For various results on Čebyšev's inequality in different settings, see [1]-[15], [17]-[36] and [39]-[57].

2 Some New Inequalities

We say that the function $F : \mathcal{C} \subset \mathcal{B} \to \mathcal{B}$ is a Lipschitian with constant $L > 0$ on the subset $\mathcal{C}$ of the Banach algebra $(\mathcal{B}, \|.\|)$ if

$$\|F(x)-F(y)\| \le L\|x-y\|$$

for any $x, y \in \mathcal{C}$.

Theorem 2.1. *Let $F, G : \mathcal{C} \subset \mathcal{B} \to \mathcal{B}$ be Lipschitian functions with constants $L > 0$ and $K > 0$. If $x_i, y_i \in \mathcal{C}$, $i = 1, 2, ..., n$ and $p_i \ge 0$, $i = 1, 2, ..., n$ with $P_n := \sum_{i=1}^n p_i > 0$, then we have:*

$$\left\| P_n \sum_{i=1}^n p_i F(x_i) G(y_i) - \sum_{i=1}^n p_i F(x_i) \sum_{i=1}^n p_i G(y_i) \right\|$$

$$\le LK \sum_{1 \le i < j \le n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| \tag{2.1}$$

where Δx_l is the forward difference, namely: $\Delta x_l = x_{l+1} - x_l, l = 1, ..., n-1$.

Proof. Since F and G are Lipschitian with the constants L and K we have:

$$\|F(x_i) - F(x_j)\| \le L\|x_i - x_j\|$$

and

$$\|G(y_i) - G(y_j)\| \le K\|y_i - y_j\|$$

for any $i, j \in \{1, ..., n\}$. If we multiply these inequalities we get

$$\|F(x_i) - F(x_j)\| \|G(y_i) - G(y_j)\| \le LK \|x_i - x_j\| \|y_i - y_j\|.$$

Using the property of Banach algebra $\mathcal{B}$, $\|uv\| \le \|u\| \|v\|$, that for any $u, v \in \mathcal{B}$ we have,

$$\|(F(x_i) - F(x_j))(G(y_i) - G(y_j))\| \le LK \|x_i - x_j\| \|y_i - y_j\|$$

for any $i, j \in \{1, ..., n\}$, that is equivalent to:

$$\|F(x_i)G(y_i) - F(x_j)G(y_i) - F(x_i)G(y_j) + F(x_j)G(y_j)\|$$

$$\leq LK \|x_i - x_j\| \|y_i - y_j\| \tag{2.2}$$

for any $i, j \in \{1, ..., n\}$.
If we multiply (2.2) by $p_i p_j > 0$ sum over i and j from 1 to n and use the generalised triangle inequality, we get:

$$\left\| \sum_{i,j=1}^{n} p_i p_j \left(F(x_i)G(y_i) - F(x_j)G(y_i) - F(x_i)G(y_j) + F(x_j)G(y_j) \right) \right\| \tag{2.3}$$

$$\leq \sum_{i,j=1}^{n} p_i p_j \|F(x_i)G(y_i) - F(x_j)G(y_i) - F(x_i)G(y_j) + F(x_j)G(y_j)\|$$

$$\leq LK \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| .$$

Now, observe that

$$\sum_{i,j=1}^{n} p_i p_j \left(F(x_i)G(y_i) - F(x_j)G(y_i) - F(x_i)G(y_j) + F(x_j)G(y_j) \right) \tag{2.4}$$

$$= P_n \sum_{i=1}^{n} p_i F(x_i)G(y_i) - \sum_{j=1}^{n} p_j F(x_j) \sum_{i=1}^{n} p_i G(y_i)$$

$$- \sum_{i=1}^{n} p_i F(x_i) \sum_{j=1}^{n} p_j G(y_j) + P_n \sum_{j=1}^{n} p_j F(x_j)G(y_j)$$

$$= 2 \left(P_n \sum_{i=1}^{n} p_i F(x_i)G(y_i) - \sum_{i=1}^{n} p_i F(x_i) \sum_{i=1}^{n} p_i G(y_i) \right).$$

We also have, by the symmetry in the sum:

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| = 2 \sum_{1 \le i < j \le n} p_i p_j \|x_j - x_i\| \|y_j - y_i\| . \quad (2.5)$$

Moreover, we have $x_j - x_i = \sum_{l=1}^{j-1} \Delta x_l$ and $y_j - y_i = \sum_{s=1}^{j-1} \Delta y_s$ where $\Delta x_l := x_{l+1} - x_l$, $l = i, ..., j-1$ and by the generalized triangle inequality we have:

$$\|x_j - x_i\| \le \sum_{l=i}^{j-1} \|\Delta x_l\|$$

and

$$\|y_j - y_i\| \le \sum_{s=i}^{j-1} \|\Delta y_s\| . \quad (2.6)$$

By the inequality (2.5) and by (2.6) we then have:

$$\sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| \le 2 \sum_{1 \le i < j \le n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \quad (2.7)$$

Now, by making use of (2.3), (2.4) and (2.7) we get the desired result (2.1). □

Corollary 2.2. *With the assumptions of Theorem 1 we have:*

$$\left\| P_n \sum_{i=1}^{n} p_i F(x_i) G(y_i) - \sum_{i=1}^{n} p_i F(x_i) \sum_{i=1}^{n} p_i G(y_i) \right\|$$

$$\le \frac{1}{2} LK \left(P_n^2 - \sum_{i=1}^{n} p_i^2 \right) \sum_{k=1}^{j-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\| . \quad (2.8)$$

Proof. It is obvious that for all $1 \le i < j \le n$ we have that:

$$\sum_{l=i}^{j-1} \|\Delta x_l\| \le \sum_{k=1}^{n-1} \|\Delta x_k\|$$

and

$$\sum_{s=i}^{j-1} \|\Delta y_s\| \le \sum_{k=1}^{n-1} \|\Delta y_k\| .$$

Then

$$\sum_{1\le i<j\le n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| \le \sum_{1\le i<j\le n} p_i p_j \sum_{k=1}^{n-1} \|\Delta x_k\| \sum_{k=1}^{n-1} \|\Delta y_k\| .$$

Now, let observe that:

$$\begin{aligned}\sum_{1\le i<j\le n} p_i p_j &= \frac{1}{2}\left[\sum_{i,j=1}^{n} p_i p_j - \sum_{i=j} p_i p_j\right] \\ &= \frac{1}{2}\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i^2\right] \\ &= \frac{1}{2}\left(P_n^2 - \sum_{i=1}^{n} p_i^2\right) .\end{aligned}$$

Using (2.1) we deduce (2.8). □

Corollary 2.3. *With the assumptions of Theorem 1 we have:*

$$\left\| P_n \sum_{i=1}^{n} p_i F(x_i) G(y_i) - \sum_{i=1}^{n} p_i F(x_i) \sum_{i=1}^{n} p_i G(y_i) \right\|$$

$$\le LK \sum_{1\le i<j\le n} (j-i)\, p_i p_j \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q\right)^{\frac{1}{q}} \tag{2.9}$$

when $p,\ q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Hölder's discrete inequality, we can state that

$$\sum_{l=i}^{j-1} \|\Delta x_l\| \leq (j-i)^{\frac{1}{q}} \left(\sum_{l=i}^{j-1} \|\Delta x_l\|^p \right)^{\frac{1}{p}}$$

and

$$\sum_{s=i}^{j-1} \|\Delta y_s\| \leq (j-i)^{\frac{1}{p}} \left(\sum_{s=i}^{j-1} \|\Delta y_s\|^q \right)^{\frac{1}{q}}$$

where $j-1 \geq i, p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and then, by multiplication we have

$$\sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\|$$

$$\leq \sum_{1 \leq i < j \leq n} p_i p_j \left(j-i\right)^{\frac{1}{p}+\frac{1}{q}} \left(\sum_{l=i}^{j-1} \|\Delta x_l\|^p \right)^{\frac{1}{p}} \left(\sum_{s=i}^{j-1} \|\Delta y_s\|^q \right)^{\frac{1}{q}}$$

$$= \sum_{1 \leq i < j \leq n} p_i p_j \left(j-i\right) \left(\sum_{l=i}^{j-1} \|\Delta x_l\|^p \right)^{\frac{1}{p}} \left(\sum_{s=i}^{j-1} \|\Delta y_s\|^q \right)^{\frac{1}{q}}$$

$$\leq \sum_{1 \leq i < j \leq n} p_i p_j \left(j-i\right) \left(\sum_{k=1}^{n-1} \|\Delta x_k\|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n-1} \|\Delta y_k\|^q \right)^{\frac{1}{q}}$$

and the corollary is proved. □

Corollary 2.4. *With the assumptions of Theorem 1 we have:*

$$\left\| P_n \sum_{i=1}^{n} p_i F(x_i) G(y_i) - \sum_{i=1}^{n} p_i F(x_i) \sum_{i=1}^{n} p_i G(y_i) \right\|$$

$$\leq LK \left[P_n \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max_{1 \leq k \leq n-1} \|\Delta x_k\| \max_{1 \leq k \leq n-1} \|\Delta y_k\| . \quad (2.10)$$

Proof. We have

$$\sum_{l=i}^{j-1}\|\Delta x_l\| \leq (j-i)\max_{l=\overline{1,n-1}}\|\Delta x_l\|$$

and

$$\sum_{s=i}^{j-1}\|\Delta y_s\| \leq (j-i)\max_{s=\overline{1,n-1}}\|\Delta y_s\|.$$

Then

$$\sum_{1\leq i<j\leq n} p_i p_j \sum_{l=i}^{j-1}\|\Delta x_l\| \sum_{s=i}^{j-1}\|\Delta y_s\|$$

$$\leq \sum_{1\leq i<j\leq n} p_i p_j (j-i)^2 \max_{l=\overline{1,n-1}}\|\Delta x_l\| \max_{s=\overline{1,n-1}}\|\Delta y_s\|$$

$$= \left[\sum_{i=1}^{n} p_i \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i\right)^2\right] \max_{l=\overline{1,n-1}}\|\Delta x_l\| \max_{s=\overline{1,n-1}}\|\Delta y_s\|$$

and the corollary is proved. □

3 Some Applications

In [16] S. S. Dragomir proved the following result:

Lemma 3.1. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,r)\subset\mathbb{C}$, $R>0$ and let $f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n$. For any $x,y\in\mathcal{B}$ with $\|x\|,\|y\|\leq M<R$ we have:*

$$\|f(y)-f(x)\| \leq f_a^{'}(M)\|y-x\|.$$

If we take $f(z) = z^p$, $p \in \mathbb{N}$, $p \geq 1$ and assume that $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq M$ then we have

$$\|y^p - x^p\| \leq pM^{p-1} \|y - x\| .$$

If we take $f(z) = \exp(\mu z)$, with $\mu \in \mathbb{R}$, $\mu \neq 0$ and assume that $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq M$ then we have

$$\|\exp(\mu y) - \exp(\mu x)\| \leq |\mu| \exp(\mu M) \|y - x\| .$$

If we write the inequality (2.1) for $F(z) = z^p, G(z) = z^q$, with $p, q \in \mathbb{N}$, $p, q \geq 1$ then we have for $\|x_i\|, \|y_i\| \leq M$, $i \in \{1, 2, ..., n\}$ that

$$\left\| P_n \sum_{i=1}^{n} p_i x_i^p y_i^q - \sum_{i=1}^{n} p_i x_i^p \sum_{i=1}^{n} p_i y_i^q \right\|$$

$$\leq pqM^{p+q-2} \sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \qquad (3.1)$$

If $p = q$ and x_i commutes with y_i for each $i \in \{1, 2, ..., n\}$, then we get from (3.1) that:

$$\left\| P_n \sum_{i=1}^{n} p_i (x_i y_i)^p - \sum_{i=1}^{n} p_i x_i^p \sum_{i=1}^{n} p_i y_i^p \right\|$$

$$\leq p^2 M^{2p-2} \sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \qquad (3.2)$$

In particular, for $x_i = y_i$ and $p = q$, $i \in \{1, 2, ..., n\}$, we get

$$\left\| P_n \sum_{i=1}^{n} p_i (x_i)^{2p} - \left(\sum_{i=1}^{n} p_i x_i^p \right)^2 \right\|$$

$$\leq p^2 M^{2p-2} \sum_{1 \leq i < j \leq n} p_i p_j \left(\sum_{l=i}^{j-1} \|\Delta x_l\| \right)^2 . \tag{3.3}$$

Now, if we write the inequality (2.1) for the functions $F(z) = \exp(\mu z)$ and $G(z) = \exp(\nu z)$, with $\mu, \nu \in \mathbb{R}$, $\mu, \nu \neq 0$ we get for $\|x_i\|, \|y_i\| \leq M$, for $i \in \{1, 2, ..., n\}$ that

$$\left\| P_n \sum_{i=1}^{n} p_i \exp(\mu x_i) \exp(\nu y_i) - \sum_{i=1}^{n} p_i \exp(\mu x_i) \sum_{i=1}^{n} p_i \exp(\nu y_i) \right\|$$

$$\leq |\mu\nu| \exp[(\mu + \nu) M] \sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \tag{3.4}$$

If x_i commutes with y_i for each $i \in \{1, 2, ..., n\}$, then from (3.4) we get:

$$\left\| P_n \sum_{i=1}^{n} p_i \exp(\mu x_i + \nu y_i) - \sum_{i=1}^{n} p_i \exp(\mu x_i) \sum_{i=1}^{n} p_i \exp(\nu y_i) \right\|$$

$$\leq |\mu\nu| \exp[(\mu + \nu) M] \sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \tag{3.5}$$

In particular, for $x_i = y_i$ and $\mu = \nu$, $i \in \{1, 2, ..., n\}$, we get:

$$\left\| P_n \sum_{i=1}^{n} p_i \exp(2\mu x_i) - \left(\sum_{i=1}^{n} p_i \exp(\mu x_i) \right)^2 \right\|$$

$$\leq |\mu|^2 \exp(2\mu M) \sum_{1 \leq i < j \leq n} p_i p_j \left(\sum_{l=i}^{j-1} \|\Delta x_l\| \right)^2 . \tag{3.6}$$

Consider now the function $F_p(z) = (1 - z^p)^{-1} = \sum_{n=0}^{\infty} z^{np}$ for $p \geq 1$ a natural number. Then $F_p^{'}(z) = -(1 - z^p)^{-2}\left(-pz^{p-1}\right) = p(1 - z^p)^{-2} z^{p-1}$.
If $M < 1$, then $F_p^{'}(M) = p(1 - M^p)^{-2} M^{p-1}$.
We have then $F_p(x) = (1 - x^p)^{-1}$, $x \in \mathcal{B}$, $\|x\| < 1$, $p \geq 1$ is Lipschitzian with the constant $p(1 - M^p)^{-2} M^{p-1}$ and the function $G_q(x) = (1 - x^q)^{-1}$, $x \in \mathcal{B}$, $\|x\| < 1$, $q \geq 1$ is Lipschitzian with the constant $q(1 - M^q)^{-2} M^{q-1}$. Therefore, by using Theorem 2.1 for $x_i, y_i \in \mathcal{B}$ with $\|x_i\|, \|y_i\| \leq M < 1$, $i \in \{1, 2, ..., n\}$, we get for $p_i \geq 0$, $i \in \{1, 2, ..., n\}$ that

$$\left\| P_n \sum_{i=1}^{n} p_i \left(1 - x_i^p\right)^{-1} \left(1 - y_i^q\right)^{-1} - \sum_{i=1}^{n} p_i \left(1 - x_i^p\right)^{-1} \sum_{i=1}^{n} p_i \left(1 - y_i^q\right)^{-1} \right\|$$

$$\leq pq(1 - M^p)^{-2}(1 - M^q)^{-2} M^{p+q-2} \sum_{1 \leq i < j \leq n} p_i p_j \sum_{l=i}^{j-1} \|\Delta x_l\| \sum_{s=i}^{j-1} \|\Delta y_s\| . \tag{3.7}$$

In particular for $p = q$ and $x_i = y_i$ for each $i \in \{1, 2, ..., n\}$, we get from (3.1) that:

$$\left\| P_n \sum_{i=1}^{n} p_i \left(1 - x_i^p\right)^{-2} - \left(\sum_{i=1}^{n} p_i \left(1 - x_i^p\right)^{-1} \right)^2 \right\|$$

$$\leq p^2 (1 - M^p)^{-4} M^{2p-2} \sum_{1 \leq i < j \leq n} p_i p_j \left(\sum_{l=i}^{j-1} \|\Delta x_l\| \right)^2 . \tag{3.8}$$

Finally, we notice that if one uses the upper bounds from Corollaries 2.2 - 2.4 that one can get further upper bounds in the examples outlined above. The details are omitted.

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Marius V. Boldea

Department Mathematics and Statistics
Banat University of Agricultural Sciences and Veterinary Medicine Timişoara
119 Calea Aradului
Timişoara
Timiş, România

E-mail: marius.boldea@usab-tm.ro