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THE QUADRATIC RELATIVE ENTROPY FOR BOUNDED LINEAR OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR[1,2]

ABSTRACT. In this paper we introduce the *quadratic relative entropy*

$$\odot (T|V) := T^{*} \ln \left(\left| VT^{-1} \right|^{2} \right) T$$

for invertible bounded linear operators T, V in the Hilbert space H. Some fundamental inequalities connecting this relative entropy and *quadratic Tsallis relative entropy*

$$\circledcirc_{t} (T|V) := \frac{1}{t} \left(\left| \left| VT^{-1} \right|^{t} T \right|^{2} - |T|^{2} \right), \ t > 0$$

are also provided.

1. INTRODUCTION

Kamei and Fujii [10], [11] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}} \left(\ln \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right) A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [17].

In general, we can define for positive operators A, B

$$S(A|B) := s\text{-} \lim_{\varepsilon \to 0+} S(A + \varepsilon 1_{H}|B)$$

if the strong limit exists, here 1_H is the identity operator.

For the entropy function $\eta(t) = -t \ln t$, the *operator entropy* has the following expression:

$$\eta(A) = -A \ln A = S(A|1_{H}) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy.

Following [12, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities

$$S(A|B) = -A^{1/2} \left(\ln A^{1/2} B^{-1} A^{1/2} \right) A^{1/2} = B^{1/2} \eta \left(B^{-1/2} A B^{-1/2} \right) B^{1/2}; \tag{1.2}$$

(ii) We have the inequalities

$$S(A|B) \leq A (\ln \|B\| - \ln A) \text{ and } S(A|B) \leq B - A; \tag{1.3}$$

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(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \geq S(A|C) + S(B|D);$$

(iv) If $B \leq C$ then

$$S(A|B) \leq S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \leq S(T^*AT|T^*BT).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \geq tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [1], [8], [13], [14], [16] and [18].

Assume that A, B are positive operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. The *weighted operator arithmetic mean* for the pair (A,B) is introduced by

$$A \nabla_\nu B := (1-\nu) A + \nu B.$$

In 1980, Kubo & Ando, [15] introduced the *weighted operator geometric mean* for the pair (A,B) with A positive and invertible and B positive by

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^\nu A^{1/2}.$$

This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

If A, B are positive invertible operators then we can also consider the *weighted operator harmonic mean* defined by (see for instance [15])

$$A!_\nu B := \left((1-\nu) A^{-1} + \nu B^{-1}\right)^{-1}.$$

We have the following fundamental *operator means inequalities*

$$A!_\nu B \leq A \sharp_\nu B \leq A \nabla_\nu B, \ \nu \in [0,1] \tag{1.4}$$

for any A, B positive invertible operators. For $\nu = \frac{1}{2}$, we denote the above means by $A \nabla B$, $A \sharp B$ and $A!B$.

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [7]) by

$$T_t(A|B) := \frac{A \sharp_t B - A}{t}.$$

Consider the scalar function $T_t : (0, \infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}. \tag{1.5}$$

We have

$$T_{-t}(x) = \frac{1 - x^{-t}}{t} = \frac{x^t - 1}{tx^t} = T_t(x) x^{-t}. \tag{1.6}$$

For positive invertible operators A and B and $t>0$ we then have

$$T_t(A|B)=A^{1/2}T_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Also by (1.6) we have

$$T_{-t}(A|B)=A^{1/2}T_{-t}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=T_t(A|B)\left(A\sharp_t B\right)^{-1}A$$

for any positive invertible operators A and B and $t>0$.

In [20], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B)=s\text{-}T_t(A|B)\,, \tag{1.7}$$

for the positive invertible operators A, B.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [10] for $0<t\le 1$. However, it hods for any $t>0$.

Theorem 1 (Fujii-Kamei, 1989, [10])**.** *Let A, B be two positive invertible operators, then for any $t>0$ we have*

$$T_{-t}(A|B)\le S(A|B)\le T_t(A|B)\,. \tag{1.8}$$

In particular, we have for $t=1$ that

$$\left(1_H-AB^{-1}\right)A\le S(A|B)\le B-A,\ [10] \tag{1.9}$$

and for $t=2$ that

$$\frac{1}{2}\left(1_H-\left(AB^{-1}\right)^2\right)A\le S(A|B)\le\frac{1}{2}\left(BA^{-1}B-A\right). \tag{1.10}$$

The case $t=\frac{1}{2}$ is of interest as well, providing the double inequality

$$2\left(1_H-A\left(A\sharp B\right)^{-1}\right)A\le S(A|B)\le 2\left(A\sharp B-A\right)\left(\le B-A\right). \tag{1.11}$$

In the recent paper [5] we extended the concept of weighted operator geometric mean for general bounded linear operators in Hilbert spaces as follows.

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on H. We define the modulus of the operator T by $|T|:=\left(T^*T\right)^{1/2}$. We denote by $\mathcal{B}^{-1}(H)$ the class of all bounded linear invertible operators on H. For $T\in\mathcal{B}^{-1}(H)$ and $V\in\mathcal{B}(H)$ we define the *quadratic weighted operator geometric mean* of (T,V) by

$$T{\textcircled{s}}_\nu V:=\left|\left|VT^{-1}\right|^{\nu}T\right|^2 \tag{1.12}$$

for $\nu\ge 0$. For $V\in\mathcal{B}^{-1}(H)$ we can also extend the definition (1.12) for $\nu<0$. For some fundamental inequalities for this mean, see [5].

By the definition of modulus, we also have

$$T{\textcircled{s}}_\nu V=T^*\left|VT^{-1}\right|^{2\nu}T=T^*\left(\left(T^*\right)^{-1}V^*VT^{-1}\right)^{\nu}T \tag{1.13}$$

for any $T\in\mathcal{B}^{-1}(H)$ and $V\in\mathcal{B}(H)$.

For $\nu=\frac{1}{2}$ we denote

$$T{\textcircled{s}}V:=\left|\left|VT^{-1}\right|^{1/2}T\right|^2=T^*\left|VT^{-1}\right|T=T^*\left(\left(T^*\right)^{-1}V^*VT^{-1}\right)^{1/2}T.$$

It has been shown in [5] that the following representation holds

$$T \circledS_{\nu} V = |T|^2 \sharp_{\nu} |V|^2 \tag{1.14}$$

for T, $V \in \mathcal{B}^{-1}(H)$ and any real ν. We have the following fundamental inequalities extending (1.4):

$$|T|^2 \nabla_{\nu} |V|^2 \geq T \circledS_{\nu} V \geq |T|^2 !_{\nu} |V|^2 \tag{1.15}$$

for T, $V \in \mathcal{B}^{-1}(H)$ and for $\nu \in [0,1]$.

For T, $V \in \mathcal{B}^{-1}(H)$ and $t > 0$ we define the *quadratic Tsallis relative operator entropy* by

$$\begin{aligned} \circledcirc_t (T|V) &:= T^* T_t \left(\left| VT^{-1} \right|^2 \right) T = T^* \frac{\left(\left| VT^{-1} \right|^2 \right)^t - 1}{t} T \\ &= \frac{T \circledS_t V - |T|^2}{t} = \frac{\left| \left| VT^{-1} \right|^t T \right|^2 - |T|^2}{t} \\ &= \frac{|T|^2 \sharp_{\nu} |V|^2 - |T|^2}{t} = T_t \left(|T|^2 \,|\, |V|^2 \right) \end{aligned} \tag{1.16}$$

and the *quadratic relative operator entropy* by

$$\odot (T|V) := T^* \ln \left(\left| VT^{-1} \right|^2 \right) T. \tag{1.17}$$

We also have for $t > 0$ and T, $V \in \mathcal{B}^{-1}(H)$ that

$$\circledcirc_{-t} (T|V) = T^* T_{-t} \left(\left| VT^{-1} \right|^2 \right) T = \circledcirc_t (T|V) \left(T \circledS_t V \right)^{-1} |T|^2 . \tag{1.18}$$

We observe that for $T = A^{1/2} \in \mathcal{B}^{-1}(H)$ and $V = B^{1/2} \in \mathcal{B}^{-1}(H)$ we get the equalities

$$\circledcirc_t \left(A^{1/2} | B^{1/2} \right) = T_t (A|B) \text{ and } \odot \left(A^{1/2} | B^{1/2} \right) = S(A|B),$$

that show the connection between the extended Tsallis and relative entropies with the classical concepts defined for positive operators.

In this paper we establish some fundamental inequalities for the quadratic relative operator entropy in terms of quadratic Tsallis relative operator entropy that extend and generalize the Fujii-Kamei result from (1.8). Applications for invertible positive operators are also provided.

2. Fundamental Inequalities

We have:

Theorem 2. *For any T, $V \in \mathcal{B}^{-1}(H)$ and $t > 0$ we have*

$$\circledcirc_{-t} (T|V) \leq \odot (T|V) \leq \circledcirc_t (T|V). \tag{2.1}$$

In particular,

$$\left(1_H - |T|^2 |V|^{-2} \right) |T|^2 \leq \odot (T|V) \leq |V|^2 - |T|^2 , \tag{2.2}$$

$$2 \left(1_H - |T|^2 (T \circledS V)^{-1} \right) |T|^2 \leq \odot (T|V) \leq 2 \left(T \circledS V - |T|^2 \right) \tag{2.3}$$

and

$$\frac{1}{2}\left(1_H-\left(|T|^2|V|^{-2}\right)^2\right)|T|^2\le \odot(T|V)\le \frac{1}{2}\left(\left(|V|^2|T|^{-2}\right)^2-1_H\right)|T|^2. \tag{2.4}$$

Proof. Consider the convex function $f(t)=-\ln t$, $t>0$. By the gradient inequality for f, namely

$$f'(b)(b-a)\ge f(b)-f(a)\ge f'(a)(b-a)$$

we have

$$\frac{a-b}{b}\ge \ln a-\ln b\ge \frac{a-b}{a}$$

for any a, $b>0$.

If we take in this inequality $b=1$ and $a=x^t$ with $t>0$, then we get

$$\frac{x^t-1}{t}\ge \ln x\ge \frac{1-x^{-t}}{t}=\frac{x^{-t}-1}{-t}$$

namely

$$T_{-t}(x)\le \ln x\le T_t(x)$$

for any t, $x>0$.

If we use the continuous functional calculus for the positive invertible operator X, then we have

$$T_{-t}(X)\le \ln X\le T_t(X). \tag{2.5}$$

For any T, $V\in\mathcal{B}^{-1}(H)$ we have that $X=\left|VT^{-1}\right|^2\in\mathcal{B}^{-1}(H)$. If we take in (2.5) $X=\left|VT^{-1}\right|^2$ then we get

$$T_{-t}\left(\left|VT^{-1}\right|^2\right)\le \ln\left|VT^{-1}\right|^2\le T_t\left(\left|VT^{-1}\right|^2\right), \tag{2.6}$$

for any $t>0$.

It is well know that, if $P\ge 0$ then by multiplying at left with T^* and at right with T where $T\in\mathcal{B}(H)$ we have that $T^*PT\ge 0$. If A, B are selfadjoint operators with $A\ge B$ then for any $T\in\mathcal{B}(H)$ we have $T^*AT\ge T^*BT$.

Therefore, by (2.6) we get

$$T^*T_{-t}\left(\left|VT^{-1}\right|^2\right)T\le T^*\ln\left(\left|VT^{-1}\right|^2\right)T\le T^*T_t\left(\left|VT^{-1}\right|^2\right)T,$$

which proves the desired result (2.1).

For $t=1$ we have

$$\circledcirc_1(T|V)=T^*\left((T^*)^{-1}V^*VT^{-1}-1_H\right)T=|V|^2-|T|^2$$

and

$$\begin{aligned}\circledcirc_{-1}(T|V)&=T^*\left(1-\left(\left|VT^{-1}\right|^2\right)^{-1}\right)T=T^*\left(1-\left((T^*)^{-1}V^*VT^{-1}\right)^{-1}\right)T\\&=T^*\left(1-TV^{-1}(V^*)^{-1}T^*\right)T=|T|^2-|T|^2|V|^{-2}|T|^2\end{aligned}$$

which by (2.1) gives (2.2).

For $t=1/2$ we have

$$\circledcirc_{1/2}(T|V)=2\left(T\circledS V-|T|^2\right)$$

and

$$\circledcirc_{-1/2}(T|V) = \circledcirc_{1/2}(T|V)(T\circledS V)^{-1}|T|^2 = 2\left(T\circledS V - |T|^2\right)(T\circledS V)^{-1}|T|^2$$
$$= 2\left(1_H - |T|^2 (T\circledS V)^{-1}\right)|T|^2,$$

which by (2.1) gives (2.3).

For $t = 2$ we have

$$\circledcirc_2(T|V) = \frac{T^*\left((T^*)^{-1}V^*VT^{-1}\right)^2 T - |T|^2}{2}$$
$$= \frac{V^*VT^{-1}(T^*)^{-1}V^*V - |T|^2}{2} = \frac{|V|^2|T|^{-2}|V|^2 - |T|^2}{2}$$
$$= \frac{|V|^2|T|^{-2}|V|^2|T|^{-2}|T|^2 - |T|^2}{2} = \frac{1}{2}\left(\left(|V|^2|T|^{-2}\right)^2 - 1_H\right)|T|^2$$

and

$$\circledcirc_{-2}(T|V) = \circledcirc_2(T|V)(T\circledS_2 V)^{-1}|T|^2$$
$$= \frac{1}{2}\left(1_H - |T|^2\left(|V|^2|T|^{-2}|V|^2\right)^{-1}\right)|T|^2$$
$$= \frac{1}{2}\left(1 - |T|^2|V|^{-2}|T|^2|V|^{-2}\right)|T|^2 = \frac{1}{2}\left(1_H - \left(|T|^2|V|^{-2}\right)^2\right)|T|^2,$$

which by (2.1) gives (2.4). □

Corollary 1. *For any T, $V \in \mathcal{B}^{-1}(H)$ we have the strong limit*

$$s\text{-}\lim_{t\to 0}\circledcirc_t(T|V) = \odot(T|V). \tag{2.7}$$

Proof follows by the double inequality (2.1).

Corollary 2. *For any T, $V \in \mathcal{B}^{-1}(H)$ we have the representation*

$$\odot(T|V) = S\left(|T|^2\,|\,|V|^2\right). \tag{2.8}$$

Proof. We have for any T, $V \in \mathcal{B}^{-1}(H)$ that

$$\begin{aligned}\odot(T|V) &= s\text{-}\lim_{t\to 0}\circledcirc_t(T|V) \text{ by (2.7)}\\ &= s\text{-}\lim_{t\to 0}T_t\left(|T|^2\,|\,|V|^2\right) \text{ by (1.16)}\\ &= S\left(|T|^2\,|\,|V|^2\right) \text{ by Uhlmann's identity (1.7)}\end{aligned}$$

and the representation (2.8). □

For the invertible operator $T \in \mathcal{B}^{-1}(H)$ we define the *quadratic operator entropy* by

$$\odot(T) := \odot(T|1_H) = T^*\ln\left(\left|T^{-1}\right|^2\right)T = T^*\ln\left(|T^*|^{-2}\right)T = -2T^*\ln(|T^*|)T$$

and *quadratic Tsallis operator entropy* by

$$\circledcirc_t(T) := T^* T_t\left(\left|T^{-1}\right|^2\right) T = T^* \frac{\left(\left|T^{-1}\right|^2\right)^t - 1}{t} T = \frac{T \circledS_t 1_H - |T|^2}{t}$$
$$= \frac{T^* |T^*|^{-2t} T - |T|^2}{t} = \frac{\left| |T^*|^{-t} T \right|^2 - |T|^2}{t}$$

for $t > 0$.

Corollary 3. *For any $T \in \mathcal{B}^{-1}(H)$ and $t > 0$ we have*

$$\circledcirc_{-t}(T) \leq \odot(T) \leq \circledcirc_t(T). \tag{2.9}$$

In particular,

$$\left(1_H - |T|^2\right)|T|^2 \leq \odot(T) \leq 1_H - |T|^2, \tag{2.10}$$

$$2\left(1_H - T^*|T|\left(T^*\right)^{-1}\right)|T|^2 \leq \odot(T) \leq 2\left(T^*|T|^{-1} T - |T|^2\right) \tag{2.11}$$

and

$$\frac{1}{2}\left(1_H - |T|^4\right)|T|^2 \leq \odot(T) \leq \frac{1}{2}\left(|T|^{-4} - 1_H\right)|T|^2. \tag{2.12}$$

We observe that for $T = A^{1/2} \in \mathcal{B}^{-1}(H)$ and $V = B^{1/2} \in \mathcal{B}^{-1}(H)$ we get from from Theorem 2 Fujii and Kamei result from (1.8) as well as the particular inequalities from (1.9)-(1.11).

3. Refinements and Reverses

The following theorem is well known in the literature as *Taylor's theorem with the integral remainder.*

Theorem 3. *Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$ and let m be a positive integer. If $f : I \longrightarrow \mathbb{R}$ is such that $f^{(m)}$ is absolutely continuous on I, then for each $x \in I$*

$$f(x) = T_m(f; a, x) + R_m(f; a, x), \tag{3.1}$$

where $T_m(f; a, x)$ is Taylor's polynomial, i.e.,

$$T_m(f; a, x) := \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

Note that $f^{(0)} := f$ and $0! := 1$, and the remainder is given by

$$R_m(f; a, x) := \frac{1}{m!} \int_a^x (x-t)^m f^{(m+1)}(t)\, dt.$$

We need the following result [3]:

Lemma 1. *For any a, $b > 0$ we have for $n \geq 1$ that*

$$\frac{(b-a)^{2n}}{2n \max^{2n}\{a, b\}} \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(b-a)^k}{k a^k} - \ln b + \ln a \leq \frac{(b-a)^{2n}}{2n \min^{2n}\{a, b\}} \tag{3.2}$$

and

$$\frac{(b-a)^{2n}}{2n\max^{2n}\{a,b\}} \leq \ln b - \ln a - \sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \leq \frac{(b-a)^{2n}}{2n\min^{2n}\{a,b\}}. \tag{3.3}$$

Proof. For the sake of completeness, we give here a simple proof.

The following identity holds, see for instance [6] where further applications in Information Theory were provided:

$$\ln b - \ln a + \sum_{k=1}^{m}\frac{(-1)^k (b-a)^k}{ka^k} = (-1)^m \int_a^b \frac{(b-t)^m}{t^{m+1}}dt. \tag{3.4}$$

for any a, $b > 0$ we have for $m \geq 1$

For $m = 2n-1$ with $n \geq 1$, then from (3.2) we have

$$\ln b - \ln a + \sum_{k=1}^{2n-1}\frac{(-1)^k (b-a)^k}{ka^k} = -\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt, \tag{3.5}$$

for any a, $b > 0$, giving that

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt = \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}(b-a)^k}{ka^k} - \ln b + \ln a. \tag{3.6}$$

If $b > a > 0$, then

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt \geq \frac{1}{b^{2n}}\int_a^b (b-t)^{2n-1}dt = \frac{(b-a)^{2n}}{2nb^{2n}} \tag{3.7}$$

and

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt \leq \frac{1}{a^{2n}}\int_a^b (b-t)^{2n-1}dt = \frac{(b-a)^{2n}}{2na^{2n}}. \tag{3.8}$$

If $a > b > 0$, then

$$\int_a^b \frac{(b-t)^{2n-1}}{t^{2n}}dt = -\int_b^a \frac{(b-t)^{2n-1}}{t^{2n}}dt = \int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt.$$

Therefore

$$\int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt \geq \frac{1}{a^{2n}}\int_b^a (t-b)^{2n-1}dt = \frac{(a-b)^{2n}}{2na^{2n}} \tag{3.9}$$

and

$$\int_b^a \frac{(t-b)^{2n-1}}{t^{2n}}dt \leq \frac{1}{b^{2n}}\int_b^a (t-b)^{2n-1}dt = \frac{(a-b)^{2n}}{2nb^{2n}}. \tag{3.10}$$

By making use of (3.7)-(3.10), we deduce the desired result (3.2).

Now, if we replace a with b in (3.2) we get (3.3). □

Corollary 4. *For any a, $b > 0$ we have for $n \geq 1$ that*

$$\sum_{k=1}^{2n-1}\frac{(b-a)^k}{kb^k} \leq \ln b - \ln a \leq \sum_{k=1}^{2n-1}(-1)^{k-1}\frac{(b-a)^k}{ka^k}. \tag{3.11}$$

We have:

Theorem 4. *For any* T, $V \in \mathcal{B}^{-1}(H)$, $n \geq 1$ *and* $t > 0$ *we have*

$$\begin{aligned} &\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} \left(\circledcirc_{-t} (T|V) |T|^{-2} \right)^k |T|^2 \\ &\leq \odot (T|V) \\ &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} \left(\circledcirc_{t} (T|V) |T|^{-2} \right)^k |T|^2 . \end{aligned} \tag{3.12}$$

Proof. If we take in (3.11) $a = 1$ and $b = x^t$ with x, $t > 0$, then we get

$$\sum_{k=1}^{2n-1} \frac{(x^t - 1)^k}{k (x^t)^k} \leq \ln x^t \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(x^t - 1)^k}{k},$$

which can be written as

$$\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} (T_{-t}(x))^k \leq \ln x \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} (T_t(x))^k \tag{3.13}$$

for x, $t > 0$

If we use the continuous functional calculus for the positive invertible operator X we get

$$\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} (T_{-t}(X))^k \leq \ln X \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} (T_t(X))^k \tag{3.14}$$

for $t > 0$ and $n \geq 1$.

For any T, $V \in \mathcal{B}^{-1}(H)$ we have that $X = \left|VT^{-1}\right|^2 \in \mathcal{B}^{-1}(H)$. If we take in (3.14) $X = \left|VT^{-1}\right|^2$, then we get

$$\begin{aligned} \sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} \left(T_{-t} \left(\left|VT^{-1}\right|^2 \right) \right)^k &\leq \ln \left(\left|VT^{-1}\right|^2 \right) \\ &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} \left(T_t \left(\left|VT^{-1}\right|^2 \right) \right)^k \end{aligned} \tag{3.15}$$

for $t > 0$ and $n \geq 1$.

By multiplying the inequality (3.15) at left with T^* and at right with T we get

$$\begin{aligned} &\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} T^* \left(T_{-t} \left(\left|VT^{-1}\right|^2 \right) \right)^k T \\ &\leq T^* \ln \left(\left|VT^{-1}\right|^2 \right) T \\ &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} T^* \left(T_t \left(\left|VT^{-1}\right|^2 \right) \right)^k T \end{aligned} \tag{3.16}$$

for $t > 0$ and $n \geq 1$.

For $k \geq 2$ we have

$$
\begin{aligned}
& T^{*}\left(T_{t}\left(\left|VT^{-1}\right|^{2}\right)\right)^{k} T \\
& =T^{*}\left(\left(T^{*}\right)^{-1} T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1}\right)^{k} T \\
& =T^{*}\left(\left(T^{*}\right)^{-1} T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1}\right) \ldots\left(\left(T^{*}\right)^{-1} T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1}\right) T \\
& =T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1} \ldots\left(T^{*}\right)^{-1} T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1} T \\
& =T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1}\left(T^{*}\right)^{-1} \ldots T^{*} T_{t}\left(\left|VT^{-1}\right|^{2}\right) TT^{-1}\left(T^{*}\right)^{-1} T^{*} T \\
& =\circledcirc_{t}(T|V)|T|^{-2} \ldots \circledcirc_{t}(T|V)|T|^{-2}|T|^{2}=\left(\circledcirc_{t}(T|V)|T|^{-2}\right)^{k}|T|^{2},
\end{aligned}
$$

$t>0$. This equality remains true for $k=1$ as well.

We also have

$$
T^{*}\left(T_{-t}\left(\left|VT^{-1}\right|^{2}\right)\right)^{k} T=\left(\circledcirc_{-t}(T|V)|T|^{-2}\right)^{k}|T|^{2}
$$

for $t>0$ and $k \geq 1$.

Using the inequality (3.16) we deduce the desired result (3.12). □

For $n=1$ we recapture the inequality (2.1).

Corollary 5. *For any $T, V \in \mathcal{B}^{-1}(H)$, $n \geq 1$ we have*

$$
\begin{aligned}
\sum_{k=1}^{2n-1} \frac{1}{k}\left(1_{H}-|T|^{2}|V|^{-2}\right)^{k}|T|^{2} & \leq \odot(T|V) \\
& \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}\left(|V|^{2}|T|^{-2}-1_{H}\right)^{k}|T|^{2} .
\end{aligned} \tag{3.17}
$$

If we take in (3.12) $T=A^{1/2} \in \mathcal{B}^{-1}(H)$ and $V=B^{1/2} \in \mathcal{B}^{-1}(H)$, then we get

$$
\begin{aligned}
\sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1}\left(T_{-t}(A|B) A^{-1}\right)^{k} A & \leq S(A|B) \\
& \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1}\left(T_{t}(A|B) A^{-1}\right)^{k} A
\end{aligned} \tag{3.18}
$$

for $t>0$, while from (3.17) we get

$$
\sum_{k=1}^{2n-1} \frac{1}{k}\left(1_{H}-AB^{-1}\right)^{k} A \leq S(A|B) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k}\left(BA^{-1}-1_{H}\right)^{k} A, \tag{3.19}
$$

where $n \geq 1$.

4. Further Upper and Lower Bounds

We need the following result that has been obtained in [4]:

Lemma 2. *For any $a, b>0$ we have that*

$$
\frac{1}{2b \max\{a, b\}}(b-a)^{2} \leq \ln b-\ln a-\frac{b-a}{b} \leq \frac{1}{2b \min\{a, b\}}(b-a)^{2} \tag{4.1}
$$

and

$$\frac{1}{2a\max\{a,b\}}(b-a)^2 \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2a\min\{a,b\}}(b-a)^2. \tag{4.2}$$

If $n \ge 1$, then for any a, $b > 0$ we have that

$$\begin{aligned}&\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,b\max^{2n+1}\{a,b\}} \\ &\le \ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{2n+1}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} \\ &\le \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,b\min^{2n+1}\{a,b\}}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,a\max^{2n+1}\{a,b\}} \\ &\le \frac{b-a}{a} - \frac{1}{a}\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\frac{(b-a)^k}{b^{k-1}} - \ln b + \ln a \\ &\le \frac{(b-a)^{2n+2}}{(2n+1)(2n+2)\,a\min^{2n+1}\{a,b\}}.\end{aligned} \tag{4.4}$$

Proof. For the sake of completeness we give here a simple proof.

For any a, $b > 0$ we claim that

$$\ln b - \ln a - \frac{b-a}{b} = \frac{1}{b}\int_a^b \frac{b-t}{t}dt, \tag{4.5}$$

and for any $m \ge 2$ and any a, $b > 0$

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b}\sum_{k=2}^{m}\frac{(-1)^{k-1}}{k(k-1)}\frac{(b-a)^k}{a^{k-1}} = \frac{(-1)^{m-1}}{mb}\int_a^b \frac{(b-t)^m}{t^m}dt. \tag{4.6}$$

Consider the function $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x) = x\ln x$, then

$$f'(x) = \ln x + 1 \text{ and } f''(x) = \frac{1}{x}$$

and, in general, for $m \ge 2$ we have

$$f^{(m)}(x) = \frac{(-1)^m (m-2)!}{x^{m-1}},$$

where $0! := 1$.

If we use Taylor's representation (3.1) for $m = 1$, then we have

$$f(x) = f(a) + (x-a)f'(a) + \int_a^x (x-t)f''(t)\,dt$$

for any x, $a \in I$.

If we write this equality for $f(x) = x\ln x$ and $x = b$, we get

$$b\ln b = b\ln a + b - a + \int_a^b \frac{b-t}{t}dt$$

for any a, $b > 0$ that is equivalent to (4.5).

If we use Taylor's representation (3.1) for $m \geq 1$, then we have

$$f(x) = f(a) + (x-a) f'(a) + \sum_{k=2}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{m!} \int_a^x (x-t)^m f^{(m+1)}(t)\, dt$$

for any $x,\ a \in I$.

If we write this equality for $f(x) = x \ln x$ and $x = b$ we get

$$b \ln b = a \ln a + (b-a)(\ln a + 1) + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} + \frac{(-1)^{m-1}}{m} \int_a^b \frac{(b-t)^m}{t^m} dt,$$

namely

$$b \ln b = b \ln a + b - a + \sum_{k=2}^{m} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} + \frac{(-1)^{m-1}}{m} \int_a^b \frac{(b-t)^m}{t^m} dt$$

for any $a,\ b > 0$ that is equivalent to (4.6).

Now, let $b > a > 0$, then

$$\frac{1}{a} \int_a^b (b-t)\, dt \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{b} \int_a^b (b-t)\, dt$$

giving that

$$\frac{1}{2a}(b-a)^2 \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{2b}(b-a)^2. \tag{4.7}$$

Let $a > b > 0$, then

$$\frac{1}{b} \int_b^a (t-b)\, dt \geq \int_a^b \frac{b-t}{t} dt = \int_b^a \frac{t-b}{t} dt \geq \frac{1}{a} \int_b^a (t-b)\, dt$$

giving that

$$\frac{1}{2b}(b-a)^2 \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{2a}(b-a)^2. \tag{4.8}$$

Therefore, by (4.1) and (4.2) we get

$$\frac{1}{2\min\{a,b\}}(b-a)^2 \geq \int_a^b \frac{b-t}{t} dt \geq \frac{1}{2\max\{a,b\}}(b-a)^2,$$

for any $a,\ b > 0$.

By utilising the equality (4.5) we get the desired result (4.1).

Let $m = 2n + 1$ with $n \geq 1$. Then by (4.6) we have

$$\ln b - \ln a - \frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} = \frac{1}{(2n+1)b} \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}} dt. \tag{4.9}$$

Let $b > a > 0$, then

$$\frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}} dt \geq \frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)}. \tag{4.10}$$

If $a > b > 0$, then

$$\int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}} dt = \int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}} dt$$

and

$$\frac{(b-a)^{2n+2}}{b^{2n+1}(2n+2)} \geq \int_b^a \frac{(t-b)^{2n+1}}{t^{2n+1}} dt \geq \frac{(b-a)^{2n+2}}{a^{2n+1}(2n+2)}. \tag{4.11}$$

Using (4.10) and (4.11) we get

$$\frac{(b-a)^{2n+2}}{\min^{2n+1}\{a,b\}(2n+2)} \geq \int_a^b \frac{(b-t)^{2n+1}}{t^{2n+1}} dt \geq \frac{(b-a)^{2n+2}}{\max^{2n+1}\{a,b\}(2n+2)} \tag{4.12}$$

for any a, $b > 0$.

Finally, on utilising the representation (4.9) and the inequality (4.12) we get the desired result (4.3).

The inequality (4.4) follows from (4.3) by replacing a with b. $\square$

Corollary 6. *If $n \geq 1$, then for any a, $b > 0$ we have that*

$$\frac{b-a}{b} + \frac{1}{b} \sum_{k=2}^{2n+1} \frac{(-1)^k}{k(k-1)} \frac{(b-a)^k}{a^{k-1}} \leq \ln b - \ln a \leq \frac{b-a}{a} - \frac{1}{a} \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(b-a)^k}{b^{k-1}}. \tag{4.13}$$

We also have the following upper and lower bounds:

Theorem 5. *For any T, $V \in \mathcal{B}^{-1}(H)$, $n \geq 1$ and $t > 0$ we have*

$$\begin{aligned} &\circledcirc_{-t}(T|V) + \sum_{k=2}^{2n+1} \frac{(-1)^k}{k(k-1)} t^{k-1} \left(\circledcirc_t (T|V) |T|^{-2}\right)^k T \circledS_{-t} V \\ &\leq \odot (T|V) \\ &\leq \circledcirc_t (T|V) - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} t^{k-1} \left(\circledcirc_{-t} (T|V) |T|^{-2}\right)^k T \circledS_t V. \end{aligned} \tag{4.14}$$

Proof. If we take in (4.13) $a=1$ and $b=x^t$ with $x,\, t>0$, then we get

$$\begin{aligned}&\frac{x^t-1}{x^t}+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}\left(x^t-1\right)^k x^{-t}\\&\le \ln x^t\\&\le x^t-1-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(\frac{x^t-1}{x^t}\right)^k x^t,\end{aligned}$$

namely

$$\begin{aligned}&\frac{x^t-1}{tx^t}+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(\frac{x^t-1}{t}\right)^k x^{-t}\\&\le \ln x\\&\le \frac{x^t-1}{t}-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(\frac{x^t-1}{tx^t}\right)^k x^t,\end{aligned}$$

that can be written as

$$\begin{aligned}&T_{-t}(x)+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}(T(x))^k x^{-t}\\&\le \ln x\\&\le T(x)-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}(T_{-t}(x))^k x^t,\end{aligned}$$

for $x,\, t>0$ and $n\ge 1$.

If we use the continuous functional calculus for the positive invertible operator X we get

$$\begin{aligned}&T_{-t}(X)+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}(T_t(X))^k X^{-t}\\&\le \ln X\\&\le T_t(X)-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}(T_{-t}(X))^k X^t,\end{aligned}$$

for $t>0$ and $n\ge 1$.

Now, if we take $X=\left|VT^{-1}\right|^2\in\mathcal{B}^{-1}(H)$ where $T,\, V\in\mathcal{B}^{-1}(H)$, then we get

$$\begin{aligned}(4.15)\quad &T_{-t}\left(\left|VT^{-1}\right|^2\right)+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}\left(T_t\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^{-t}\\&\le \ln\left(\left|VT^{-1}\right|^2\right)\\&\le T_t\left(\left|VT^{-1}\right|^2\right)-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}\left(T_{-t}\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^t,\end{aligned}$$

for $t>0$ and $n\ge 1$.

By multiplying the inequality (4.15) at left with T^* and at right with T we get

$$
\begin{aligned}
(4.16)\qquad & T^*T_{-t}\left(\left|VT^{-1}\right|^2\right)T \\
&+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}t^{k-1}T^*\left(T_t\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&\le T^*\ln\left(\left|VT^{-1}\right|^2\right)T \\
&\le T^*T_t\left(\left|VT^{-1}\right|^2\right)T \\
&-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}t^{k-1}T^*\left(T_{-t}\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^{t}T,
\end{aligned}
$$

for $t>0$ and $n\ge 1$.

For $k\ge 2$ we have

$$
\begin{aligned}
& T^*\left(T_t\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&= T^*\left(T_t\left(\left|VT^{-1}\right|^2\right)\right)^k TT^{-1}\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&= \left(\circledcirc_t(T|V)\,|T|^{-2}\right)^k|T|^2\,T^{-1}\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&= \left(\circledcirc_t(T|V)\,|T|^{-2}\right)^k T^*TT^{-1}\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&= \left(\circledcirc_t(T|V)\,|T|^{-2}\right)^k T^*\left(\left|VT^{-1}\right|^2\right)^{-t}T \\
&= \left(\circledcirc_t(T|V)\,|T|^{-2}\right)^k T\circledS_{-t}V
\end{aligned}
$$

and

$$
T^*\left(T_{-t}\left(\left|VT^{-1}\right|^2\right)\right)^k\left(\left|VT^{-1}\right|^2\right)^{t}T=\left(\circledcirc_{-t}(T|V)\,|T|^{-2}\right)^k T\circledS_{t}V
$$

for $t>0$.

Using (4.16) we get the desired result (4.14). □

Corollary 7. *For any $T,\,V\in\mathcal{B}^{-1}(H)$, $n\ge 1$ we have*

$$
\begin{aligned}
(4.17)\qquad & |T|^2-|T|^2\,|V|^{-2}\,|T|^2+\sum_{k=2}^{2n+1}\frac{(-1)^k}{k(k-1)}\left(|V|^2\,|T|^{-2}-1_H\right)^k|T|^2\,|V|^{-2}\,|T|^2 \\
&\le \odot(T|V) \\
&\le |V|^2-|T|^2-\sum_{k=2}^{2n+1}\frac{1}{k(k-1)}\left(1-|T|^2\,|V|^{-2}\right)^k|V|^2.
\end{aligned}
$$

If we take in (4.14) $T = A^{1/2} \in \mathcal{B}^{-1}(H)$ and $V = B^{1/2} \in \mathcal{B}^{-1}(H)$, then we get

$$\begin{aligned} & T_{-t}(A|B) + \sum_{k=2}^{2n+1} \frac{(-1)^k}{k(k-1)} t^{k-1} \left(T_t(A|B) AH - 1\right)^k A\sharp_{-t}B \\ & \leq S(A|B) \\ & \leq T_t(A|B) - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} t^{k-1} \left(T_{-t}(A|B) A^{-1}\right)^k A\sharp_t B, \end{aligned} \tag{4.18}$$

for $t > 0$, while from (4.17) we get

$$\begin{aligned} & A - AB^{-1}A + \sum_{k=2}^{2n+1} \frac{(-1)^k}{k(k-1)} \left(BA^{-1} - 1_H\right)^k AB^{-1}A \\ & \leq S(A|B) \\ & \leq B - A - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \left(1 - AB^{-1}\right)^k B, \end{aligned} \tag{4.19}$$

where $n \geq 1$.

5. Inequalities Under Boundedness Conditions

If we take $a = 1$ and $b = y > 0$ in (3.2) and (3.3), then we get

$$\frac{(y-1)^{2n}}{2n \max^{2n}\{y,1\}} \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(y-1)^k}{k} - \ln y \leq \frac{(y-1)^{2n}}{2n \min^{2n}\{y,1\}} \tag{5.1}$$

and

$$\frac{(y-1)^{2n}}{2n \max^{2n}\{y,1\}} \leq \ln y - \sum_{k=1}^{2n-1} \frac{(y-1)^k}{ky^k} \leq \frac{(y-1)^{2n}}{2n \min^{2n}\{y,1\}}. \tag{5.2}$$

Since for $y > 0$ we have

$$\frac{(y-1)^{2n}}{\max^{2n}\{1,y\}} = \left(1 - \frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n},$$

and

$$\frac{(y-1)^{2n}}{\min^{2n}\{1,y\}} = \left(\frac{\max\{1,y\}}{\min\{1,y\}} - 1\right)^{2n},$$

then by (5.1) and (5.2) we have

$$\begin{aligned} \frac{1}{2n}\left(1 - \frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n} & \leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(y-1)^k}{k} - \ln y \\ & \leq \frac{1}{2n}\left(\frac{\max\{1,y\}}{\min\{1,y\}} - 1\right)^{2n} \end{aligned} \tag{5.3}$$

and

$$\frac{1}{2n}\left(1 - \frac{\min\{1,y\}}{\max\{1,y\}}\right)^{2n} \leq \ln y - \sum_{k=1}^{2n-1} \frac{(y-1)^k}{ky^k} \leq \frac{1}{2n}\left(\frac{\max\{1,y\}}{\min\{1,y\}} - 1\right)^{2n}. \tag{5.4}$$

Assume that v, $V > 0$ with $v < V$. If $y \in [v, V] \subset (0, \infty)$, then by analyzing all possible locations of the interval $[v, V]$ and 1 we have

$$\min\{1, v\} \leq \min\{1, y\} \leq \min\{1, V\}$$

and

$$\max\{1, v\} \leq \max\{1, y\} \leq \max\{1, V\}.$$

We also have

$$0 \leq 1 - \frac{\min\{1, V\}}{\max\{1, v\}} \leq 1 - \frac{\min\{1, y\}}{\max\{1, y\}} \tag{5.5}$$

and

$$0 \leq \frac{\max\{1, y\}}{\min\{1, y\}} - 1 \leq \frac{\max\{1, V\}}{\min\{1, v\}} - 1. \tag{5.6}$$

From (5.3) and (5.4) we get the following global lower and upper bounds

$$\begin{aligned}\frac{1}{2n}\left(1 - \frac{\min\{1, V\}}{\max\{1, v\}}\right)^{2n} &\leq \sum_{k=1}^{2n-1} (-1)^{k-1} \frac{(y-1)^k}{k} - \ln y \\ &\leq \frac{1}{2n}\left(\frac{\max\{1, V\}}{\min\{1, v\}} - 1\right)^{2n}\end{aligned} \tag{5.7}$$

and

$$\frac{1}{2n}\left(1 - \frac{\min\{1, V\}}{\max\{1, v\}}\right)^{2n} \leq \ln y - \sum_{k=1}^{2n-1} \frac{(y-1)^k}{ky^k} \leq \frac{1}{2n}\left(\frac{\max\{1, V\}}{\min\{1, v\}} - 1\right)^{2n} \tag{5.8}$$

for any $y \in [v, V] \subset (0, \infty)$ and $n \geq 1$.

Lemma 3. *Let T, $V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$. Then the following statements are equivalent:*

(i) The inequality

$$m\|Tx\| \leq \|Vx\| \leq M\|Tx\| \tag{5.9}$$

holds for any $x \in H$;

(ii) We have the operator inequality

$$m1_H \leq \left|VT^{-1}\right| \leq M1_H. \tag{5.10}$$

Proof. The inequality (5.9) is equivalent to

$$m^2\|Tx\|^2 \leq \|Vx\|^2 \leq M^2\|Tx\|^2$$

for any $x \in H$, namely

$$m^2\langle T^*Tx, x\rangle \leq \langle V^*Vx, x\rangle \leq M^2\langle T^*Tx, x\rangle$$

for any $x \in H$, which can be written in the operator order as

$$m^2T^*T \leq V^*V \leq M^2T^*T.$$

Since $T \in \mathcal{B}^{-1}(H)$, then this inequality is equivalent to

$$m^21_H \leq \left(T^{-1}\right)^* V^*VT^{-1} \leq M^21_H,$$

namely

$$m^21_H \leq \left|VT^{-1}\right|^2 \leq M^21_H,$$

which in its turn is equivalent to (5.10). □

We have:

Theorem 6. *Let T, $V \in \mathcal{B}^{-1}(H)$ and $0 < m < M < \infty$. Assume that the pair of operators (T, V) satisfies either the condition (5.9) or, equivalently, the condition (5.10). Then for any $t > 0$ and $n \geq 1$ we have*

$$\begin{aligned}
&\frac{1}{2n} t^{2n-1} T_t^{2n} \left(\frac{\min\{1, M^2\}}{\max\{1, m^2\}} \right) |T|^2 \\
&\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} t^{k-1} \left(\circledcirc_t (T|V) |T|^{-2} \right)^k |T|^2 - \odot (T|V) \\
&\leq \frac{1}{2n} t^{2n-1} T_t^{2n} \left(\frac{\max\{1, M^2\}}{\min\{1, m^2\}} \right) |T|^2
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
&\frac{1}{2n} t^{2n-1} T_t^{2n} \left(\frac{\min\{1, M^2\}}{\max\{1, m^2\}} \right) |T|^2 \\
&\leq \odot (T|V) - \sum_{k=1}^{2n-1} \frac{1}{k} t^{k-1} \left(\circledcirc_{-t} (T|V) |T|^{-2} \right)^k |T|^2 \\
&\leq \frac{1}{2n} t^{2n-1} T_t^{2n} \left(\frac{\max\{1, M^2\}}{\min\{1, m^2\}} \right) |T|^2 ,
\end{aligned} \tag{5.12}$$

where T_t is defined by (1.5).

Proof. Let $x \in [m, M] \subset (0, \infty)$ and for $t > 0$ put $y = x^t \in [m^t, M^t]$. Then by (5.7) and (5.8) for $v = m^t$ and $V = M^t$ we have

$$\begin{aligned}
\frac{1}{2n} \left(1 - \frac{\min\{1, M^t\}}{\max\{1, m^t\}} \right)^{2n} &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \left(x^t - 1 \right)^k - \ln x^t \\
&\leq \frac{1}{2n} \left(\frac{\max\{1, M^t\}}{\min\{1, m^t\}} - 1 \right)^{2n},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2n} \left(1 - \frac{\min\{1, M^t\}}{\max\{1, m^t\}} \right)^{2n} &\leq \ln x^t - \sum_{k=1}^{2n-1} \frac{1}{k} \left(\frac{x^t - 1}{x^t} \right)^k \\
&\leq \frac{1}{2n} \left(\frac{\max\{1, M^t\}}{\min\{1, m^t\}} - 1 \right)^{2n},
\end{aligned}$$

for any $n \geq 1$.

These inequalities may be written as

$$\begin{aligned}
\frac{1}{2n} \left(1 - \left(\frac{\min\{1, M\}}{\max\{1, m\}} \right)^t \right)^{2n} &\leq \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k} \left(x^t - 1 \right)^k - \ln x^t \\
&\leq \frac{1}{2n} \left(\left(\frac{\max\{1, M\}}{\min\{1, m\}} \right)^t - 1 \right)^{2n},
\end{aligned}$$

and

$$\frac{1}{2n}\left(1-\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right)^{t}\right)^{2n} \le \ln x^{t}-\sum_{k=1}^{2n-1}\frac{1}{k}\left(\frac{x^{t}-1}{x^{t}}\right)^{k} \le \frac{1}{2n}\left(\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right)^{t}-1\right)^{2n}$$

or, in terms of $T_{\pm t}$ as

$$\frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right) \le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t(x)\right)^{k}-\ln x \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right), \tag{5.13}$$

and

$$\frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M\}}{\max\{1,m\}}\right) \le \ln x-\sum_{k=1}^{2n-1}\frac{1}{k}t^{k-1}\left(T_{-t}(x)\right)^{k} \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M\}}{\min\{1,m\}}\right) \tag{5.14}$$

for $x\in[m,M]\subset(0,\infty)$, $n\ge 1$ and for $t>0$.

Let $T,V\in\mathcal{B}^{-1}(H)$ and consider $X=\left|VT^{-1}\right|^{2}$. Then by (5.9) we have $m^{2}1_H\le \left|VT^{-1}\right|^{2}\le M^{2}1_H$ and by making use of (5.13) and the functional calculus for selfadjoint operators we have

$$\frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\min\{1,M^{2}\}}{\max\{1,m^{2}\}}\right) \le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}t^{k-1}\left(T_t\left(\left|VT^{-1}\right|^{2}\right)\right)^{k}-\ln\left|VT^{-1}\right|^{2} \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\{1,M^{2}\}}{\min\{1,m^{2}\}}\right), \tag{5.15}$$

for $n\ge 1$ and for $t>0$.

On making use of a similar argument to the one in the proof of Theorem 4 we obtain from (5.13) the desired result (5.11).

The inequality (5.12) follows by (5.14) in a similar way. □

Remark 1. *If we take in (5.11) and (5.12) $n=1$, then we get*

$$\frac{1}{2}tT_t^{2}\left(\frac{\min\{1,M^{2}\}}{\max\{1,m^{2}\}}\right)|T|^{2} \le \circledcirc_t(T|V)-\odot(T|V) \le \frac{1}{2}tT_t^{2}\left(\frac{\max\{1,M^{2}\}}{\min\{1,m^{2}\}}\right)|T|^{2} \tag{5.16}$$

and

$$\frac{1}{2}tT_t^2\left(\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\right)|T|^2 \le \odot(T|V) - \circledcirc_{-t}(T|V) \le \frac{1}{2n}t^{2n-1}T_t^{2n}\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}\right)|T|^2, \tag{5.17}$$

for any $t>0$. These provides refinements of the inequalities in (2.1).

If we take $t=1$ in (5.11) and (5.12), then we get

$$\begin{aligned}&\frac{1}{2n}\left(1-\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\right)^{2n}|T|^2\\ &\le \sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}\left(|V|^2|T|^{-2}-1_H\right)^k|T|^2-\odot(T|V)\\ &\le \frac{1}{2n}\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1\right)^{2n}|T|^2\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}&\frac{1}{2n}\left(1-\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\right)^{2n}|T|^2\\ &\le \odot(T|V)-\sum_{k=1}^{2n-1}\frac{1}{k}\left(1_H-|T|^2|V|^{-2}\right)^k|T|^2\\ &\le \frac{1}{2n}\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1\right)^{2n}|T|^2\end{aligned} \tag{5.19}$$

for $n\ge 1$.

If we take $a=1$ and $b=y>0$ in Lemma 2, then by using (5.5) and (5.6) we get

$$\frac{1}{2}\frac{1}{\min\{1,V\}}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^2 \le \ln y-\frac{y-1}{y} \le \frac{1}{2}\frac{1}{\max\{1,v\}}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^2 \tag{5.20}$$

and

$$\frac{1}{2}\max\{1,v\}\left(1-\frac{\min\{1,V\}}{\max\{1,v\}}\right)^2 \le y-1-\ln y \le \frac{1}{2}\min\{1,V\}\left(\frac{\max\{1,V\}}{\min\{1,v\}}-1\right)^2 \tag{5.21}$$

for any $y\in[v,V]$.

If $n \geq 1$, then for any $y \in [v, V]$ we also have that

$$\frac{1}{(2n+1)(2n+2)} \frac{1}{\min\{1, V\}} \left(1 - \frac{\min\{1, V\}}{\max\{1, v\}}\right)^{2n+2} \tag{5.22}$$
$$\leq \ln y - \frac{y-1}{y} + \frac{1}{y} \sum_{k=2}^{2n+1} \frac{(-1)^{k-1}}{k(k-1)} (y-1)^k$$
$$\leq \frac{1}{(2n+1)(2n+2)} \frac{1}{\max\{1, v\}} \left(\frac{\max\{1, V\}}{\min\{1, v\}} - 1\right)^{2n+2}$$

and

$$\frac{1}{(2n+1)(2n+2)} \max\{1, v\} \left(1 - \frac{\min\{1, V\}}{\max\{1, v\}}\right)^{2n+2} \tag{5.23}$$
$$\leq y - 1 - \sum_{k=2}^{2n+1} \frac{1}{k(k-1)} \frac{(y-1)^k}{y^{k-1}} - \ln y$$
$$\leq \frac{1}{(2n+1)(2n+2)} \min\{1, V\} \left(\frac{\max\{1, V\}}{\min\{1, v\}} - 1\right)^{2n+2}.$$

These inequalities can be used to obtain the operator versions similar to the ones from Theorem 6. The details are omitted.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa