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QUADRATIC RELATIVE OPERATOR ENTROPY IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we introduce the *quadratic relative operator entropy*

$$\odot(x|y) := x^* \ln\left(\left|yx^{-1}\right|^2\right)x$$

for invertible elements x, y in a Hermitian unital Banach $*$-algebra. We show that

$$\odot(x|y) = S\left(|x|^2 \,|\, |y|^2\right),$$

where $S(\cdot|\cdot)$ is the relative operator entropy defined for positive invertible elements c, d by

$$S(c|d) := c^{1/2}\left(\ln\left(c^{-1/2}dc^{-1/2}\right)\right)c^{1/2}.$$

Several upper and lower bounds in terms of the corresponding Tsallis' type relative entropies are also provided.

1. Introduction

We recall some definitions and fundamental facts that will be used in the sequel.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [10] (see also [1, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [9], Tanahashi and Uchiyama [11] proved the following fundamental properties (see also [5]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;

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(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [9] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with *continuous involution*, namely if a, $b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz. \tag{1.1}$$

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [5], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [11, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

We define the following means for $\nu \in [0,1]$, see also [5] for different notations:

$$a \nabla_\nu b := (1-\nu) a + \nu b, \ a, \ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a, b),

$$a !_\nu b := \left((1-\nu) a^{-1} + \nu b^{-1}\right)^{-1}, \ a, \ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a, b) and

$$a \sharp_\nu b := a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^\upsilon a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a, b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a \nabla b$, $a!b$ and $a \sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [5], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a$, $b \in A$, then $a!b = b!a$ and $a \sharp b = b \sharp a$;

(xiii) If $0 < a$, $b \in A$ and $c \in \text{Inv}(A)$, then

$$c^* (a!b) c = (c^*ac)! (c^*bc) \text{ and } c^* (a\sharp b) c = (c^*ac) \sharp (c^*bc);$$

(xiv) If $0 < a$, $b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right) \nabla_\nu \left(b^{-1}\right) \text{ and } \left(a^{-1}\right) \sharp_\nu \left(b^{-1}\right) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [5] the following inequality between the weighted means introduced above:

$$a\nabla_\nu b \geq a\sharp_\nu b \geq a!_\nu b \tag{HGA}$$

for any $0 < a$, $b \in A$ and $\nu \in [0,1]$.

In [11], Tanahashi and Uchiyama obtained the following identity of interest:

Lemma 1. *If $0 < c$, d and λ is a real number, then*

$$(dcd)^\lambda = dc^{1/2} \left(c^{1/2}d^2c^{1/2}\right)^{\lambda-1} c^{1/2}d. \tag{1.2}$$

As a consequence of this equality we proved the following fact [4]:

Proposition 1. *For any $0 < a$, $b \in A$ we have*

$$b\sharp_{1-\nu} a = a\sharp_\nu b \tag{1.3}$$

for any real number ν.

Following [4] we can introduce the *quadratic weighted mean* of (x, y) with x, $y \in \text{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$, as the positive element denoted by $x\circledS_\nu y$ and defined by

$$x\circledS_\nu y := x^* \left((x^*)^{-1} y^*yx^{-1}\right)^\nu x = x^* \left|yx^{-1}\right|^{2\nu} x = \left|\left|yx^{-1}\right|^\nu x\right|^2. \tag{S}$$

When $\nu = 1/2$, we denote $x\circledS_{1/2}y$ by $x\circledS y$ and we have

$$x\circledS y = x^* \left((x^*)^{-1} y^*yx^{-1}\right)^{1/2} x = x^* \left|yx^{-1}\right| x = \left|\left|yx^{-1}\right|^{1/2} x\right|^2.$$

If we take in (S) $x = a^{1/2}$ and $y = b^{1/2}$ with a, $b > 0$ then we get

$$a^{1/2}\circledS_\nu b^{1/2} = a\sharp_\nu b$$

for any $\nu \in \mathbb{R}$ that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let x, $y \in \text{Inv}(A)$. If we take in the definition of "$\sharp_\nu$" the elements $a = |x|^2 > 0$ and $b = |y|^2 > 0$ we also have for real ν

$$|x|^2 \sharp_\nu |y|^2 = |x| \left(|x|^{-1} |y|^2 |x|^{-1}\right)^\upsilon |x| = |x| \left||y| |x|^{-1}\right|^{2\upsilon} |x| = \left|\left||y| |x|^{-1}\right|^\upsilon |x|\right|^2.$$

By utilizing the following lemma [4] that provides a slight generalization of Lemma 1.

Lemma 2. *If $0 < c$, $d \in \text{Inv}(A)$ and λ is a real number, then*

$$(dcd^*)^\lambda = dc^{1/2} \left(c^{1/2} |d|^2 c^{1/2}\right)^{\lambda-1} c^{1/2}d^*, \tag{1.4}$$

we proved the following representation result [4]:

Theorem 1. *If x, $y \in \operatorname{Inv}(A)$ and λ is a real number, then*

$$x \circledS_\nu y = |x|^2 \sharp_\nu |y|^2 . \tag{1.5}$$

In this paper we introduce the *quadratic relative operator entropy* for invertible elements x, y in a Hermitian unital Banach $*$-algebra. We show that it can be expressed in terms of the *relative operator entropy* defined for positive invertible elements c, d. Several upper and lower bounds in terms of the corresponding Tsallis' type relative entropies are also provided.

2. Relative Entropy

Consider the scalar function $T_t : (0,\infty) \to \mathbb{R}$ defined for $t \neq 0$ by

$$T_t(x) := \frac{x^t - 1}{t}. \tag{2.1}$$

We have

$$T_{-t}(x) = \frac{1 - x^{-t}}{t} = \frac{x^t - 1}{t x^t} = T_t(x) x^{-t}. \tag{2.2}$$

For $t > 0$ and the elements $0 < c$, $d \in \operatorname{Inv}(A)$ we define the *Tsallis relative operator entropy* by

$$T_t(c|d) := \frac{c \sharp_t d - c}{t}.$$

We have for $t > 0$ and the elements $0 < c$, $d \in \operatorname{Inv}(A)$ that

$$T_t(c|d) = c^{1/2} T_t\left(c^{-1/2} d c^{-1/2}\right) c^{1/2}$$

and

$$T_{-t}(c|d) = c^{1/2} T_{-t}\left(c^{-1/2} d c^{-1/2}\right) c^{1/2} = T_t(c|d) (c \sharp_t d)^{-1} c.$$

For x, $y \in \operatorname{Inv}(A)$ and $t > 0$ we define the *quadratic Tsallis relative operator entropy* by

$$\begin{aligned} \circledcirc_t(x|y) &:= x^* T_t\left(\left|yx^{-1}\right|^2\right) x = x^* \frac{\left(\left|yx^{-1}\right|^2\right)^t - 1}{t} x \\ &= \frac{x \circledS_t y - |x|^2}{t} = \frac{\left|\left|yx^{-1}\right|^t x\right|^2 - |x|^2}{t}. \end{aligned} \tag{2.3}$$

By the representation (1.5) we have

$$\circledcirc_t(x|y) = \frac{|x|^2 \sharp_\nu |y|^2 - |x|^2}{t} = T_t\left(|x|^2 \mid |y|^2\right) \tag{2.4}$$

for any x, $y \in \operatorname{Inv}(A)$ and $t > 0$.

We also have for $t > 0$ and x, $y \in \operatorname{Inv}(A)$ that

$$\circledcirc_{-t}(x|y) = x^* T_{-t}\left(\left|yx^{-1}\right|^2\right) x = \circledcirc_t(x|y) (x \circledS_t y)^{-1} |x|^2 . \tag{2.5}$$

We observe that for $x = c^{1/2}$ and $y = d^{1/2}$ with $0 < c$, $d \in \operatorname{Inv}(A)$, we get the equality

$$\circledcirc_t\left(c^{1/2} | d^{1/2}\right) = T_t(c|d) . \tag{2.6}$$

Consider the function $f(z) := \operatorname{Ln} z$, where $\operatorname{Ln} z := \ln|z| + i \operatorname{Arg} z$ is the principal of the complex logarithm function. Then $f(z)$ is analytic in the right half open plane

$\{\operatorname{Re} z > 0\}$ of the complex plane and by using the analytic functional calculus (1.1) we can introduce the relative operator entropy

$$S(c|d) := c^{1/2}\left(\ln\left(c^{-1/2}dc^{-1/2}\right)\right)c^{1/2}, \tag{2.7}$$

for $0 < c$, $d \in \operatorname{Inv}(A)$ and the *quadratic relative operator entropy* by

$$\odot(x|y) := x^*\left(\ln\left(\left|yx^{-1}\right|^2\right)\right)x \tag{2.8}$$

for x, $y \in \operatorname{Inv}(A)$.

If we take in (2.8) $x = c^{1/2}$ and $y = d^{1/2}$ with $0 < c$, $d \in \operatorname{Inv}(A)$ then we get

$$\odot\left(c^{1/2}|d^{1/2}\right) = S(c|d). \tag{2.9}$$

Now, assume that $f(z)$ is analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0,\infty)$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0,\infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

Lemma 3. *Let $f(z)$ and $g(z)$ be analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0,\infty)$ assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

We have:

Theorem 2. *For any x, $y \in \operatorname{Inv}(A)$ and $t > 0$ we have the double inequality*

$$\circledcirc_t(x|y) \geq \odot(x|y) \geq \circledcirc_{-t}(x|y). \tag{2.10}$$

In particular,

$$|y|^2 - |x|^2 \geq \odot(x|y) \geq \left(1 - |x|^2|y|^{-2}\right)|x|^2, \tag{2.11}$$

$$2\left(x \circledS y - |x|^2\right) \geq \odot(x|y) \geq 2\left(1 - |x|^2(x \circledS y)^{-1}\right)|x|^2 \tag{2.12}$$

and

$$\frac{1}{2}\left(\left(|y|^2|x|^{-2}\right)^2 - 1\right)|x|^2 \geq \odot(x|y) \geq \frac{1}{2}\left(1 - \left(|x|^2|y|^{-2}\right)^2\right)|x|^2. \tag{2.13}$$

Proof. Consider the convex function $f(t) = -\ln t$, $t > 0$. By the gradient inequality for f, namely

$$f'(b)(b-a) \geq f(b) - f(a) \geq f'(a)(b-a)$$

we have

$$\frac{a-b}{b} \geq \ln a - \ln b \geq \frac{a-b}{a}$$

for any a, $b > 0$.

If we take in this inequality $b = 1$ and $a = z^t$ with $t > 0$, then we get

$$\frac{z^t - 1}{t} \geq \ln z \geq \frac{1 - z^{-t}}{t} = \frac{z^{-t} - 1}{-t}$$

namely

$$T_t(z) \geq \ln z \geq T_{-t}(z) \tag{2.14}$$

for any $t,\ z > 0$.

Consider the functions $f(z) := T_t(z)$, $g(z) := \operatorname{Ln} z$ and $h(z) = T_{-t}(z)$ where $\operatorname{Ln} z$ is the principal of the logarithmic function. Then $f(z)$, $g(z)$ and $h(z)$ are analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ of the complex plane and by (2.14) we have $f(z) \geq g(z) \geq h(z)$ for any $z > 0$.

If we use Lemma 3 for the positive invertible element u, then we have

$$T_t(u) \geq \ln u \geq T_{-t}(u). \tag{2.15}$$

For any $x,\ y \in \operatorname{Inv}(A)$ we have that $u = \left|yx^{-1}\right|^2 \in \operatorname{Inv}(A)$. If we take in (2.15) $u = \left|yx^{-1}\right|^2$, then we get

$$T_t\left(\left|yx^{-1}\right|^2\right) \geq \ln\left|yx^{-1}\right|^2 \geq T_{-t}\left(\left|yx^{-1}\right|^2\right), \tag{2.16}$$

for any $t > 0$.

By multiplying (2.16) at left with x^* and at right with x we get

$$x^*T_t\left(\left|yx^{-1}\right|^2\right)x \geq x^*\left(\ln\left|yx^{-1}\right|^2\right)x \geq x^*T_{-t}\left(\left|yx^{-1}\right|^2\right)x,$$

which proves the desired result (2.10).

For $t = 1$ we have

$$\circledcirc_1(x|y) = x^*\left((x^*)^{-1}y^*yx^{-1} - 1\right)x = |y|^2 - |x|^2$$

and

$$\begin{aligned}\circledcirc_{-1}(x|y) &= x^*\left(1 - \left(\left|yx^{-1}\right|^2\right)^{-1}\right)x = x^*\left(1 - \left((x^*)^{-1}y^*yx^{-1}\right)^{-1}\right)x \\ &= x^*\left(1 - xy^{-1}(y^*)^{-1}x^*\right)x = |x|^2 - |x|^2|y|^{-2}|x|^2,\end{aligned}$$

which by (2.10) gives (2.11).

For $t = 1/2$ we have

$$\circledcirc_{1/2}(x|y) = 2\left(x \circledS y - |x|^2\right)$$

and

$$\begin{aligned}\circledcirc_{-1/2}(x|y) &= \circledcirc_{1/2}(x|y)(x \circledS y)^{-1}|x|^2 = 2\left(x \circledS y - |x|^2\right)(x \circledS y)^{-1}|x|^2 \\ &= 2\left(1 - |x|^2(x \circledS y)^{-1}\right)|x|^2,\end{aligned}$$

which by (2.10) gives (2.12).

For $t = 2$ we have

$$\begin{aligned}\circledcirc_2(x|y) &= \frac{x^*\left((x^*)^{-1}y^*yx^{-1}\right)^2x - |x|^2}{2} \\ &= \frac{y^*yx^{-1}(x^*)^{-1}y^*y - |x|^2}{2} = \frac{|y|^2|x|^{-2}|y|^2 - |x|^2}{2} \\ &= \frac{|y|^2|x|^{-2}|y|^2|x|^{-2}|x|^2 - |x|^2}{2} = \frac{1}{2}\left(\left(|y|^2|x|^{-2}\right)^2 - 1\right)|x|^2\end{aligned}$$

and

$$\begin{aligned}
\circledcirc_{-2}(x|y) &= \circledcirc_2(x|y)\left(x \circledS_2 y\right)^{-1}|x|^2 \\
&= \frac{1}{2}\left(1-|x|^2\left(|y|^2|x|^{-2}|y|^2\right)^{-1}\right)|x|^2 \\
&= \frac{1}{2}\left(1-|x|^2|y|^{-2}|x|^2|y|^{-2}\right)|x|^2 = \frac{1}{2}\left(1-\left(|x|^2|y|^{-2}\right)^2\right)|x|^2,
\end{aligned}$$

which by (2.10) gives (2.13). □

Corollary 1. *For any x, $y \in \operatorname{Inv}(A)$ we have the strong limit*

$$s\text{-}\lim_{t\to 0}\circledcirc_t(x|y) = \odot(x|y). \tag{2.17}$$

Proof follows by the double inequality (2.10) on taking the strong limit over $t \to 0+$.

Remark 1. *If we take in Theorem 2 $x = c^{1/2}$, $y = d^{1/2}$ with $0 < c$, $d \in \operatorname{Inv}(A)$, then we get the inequalities*

$$T_t(c|d) \geq S(c|d) \geq T_{-t}(c|d) \tag{2.18}$$

for any $t > 0$. The inequality (2.18) was obtained by Fujii and Kamei in [6] *for positive invertible operators in Hilbert spaces.*

In particular,

$$d - c \geq S(c|d) \geq \left(1 - cd^{-1}\right)c, \tag{2.19}$$

$$2(c\sharp d - c) \geq S(c|d) \geq 2\left(1 - c\left(c\sharp d\right)^{-1}\right)c \tag{2.20}$$

and

$$\frac{1}{2}\left(\left(dc^{-1}\right)^2 - 1\right)c \geq S(c|d) \geq \frac{1}{2}\left(1-\left(cd^{-1}\right)^2\right)c. \tag{2.21}$$

We have the strong limit

$$s\text{-}\lim_{t\to 0} T_t(c|d) = S(c|d) \tag{2.22}$$

for $0 < c$, $d \in \operatorname{Inv}(A)$. The representation (2.23) was obtained for positive invertible operators in Hilbert spaces by Uhlmann in [12].

Corollary 2. *For any x, $y \in \operatorname{Inv}(A)$ we have the representation*

$$\odot(x|y) = S\left(|x|^2 \,|\, |y|^2\right). \tag{2.23}$$

Proof. We have for any x, $y \in \operatorname{Inv}(A)$ that

$$\begin{aligned}
\odot(x|y) &= s\text{-}\lim_{t\to 0}\circledcirc_t(x|y) \text{ by (2.17)} \\
&= s\text{-}\lim_{t\to 0} T_t\left(|x|^2 \,|\, |y|^2\right) \text{ by (2.6)} \\
&= S\left(|x|^2 \,|\, |y|^2\right) \text{ by (2.22)}
\end{aligned}$$

and the representation (2.23) is proved. □

For the invertible element $x \in \operatorname{Inv}(A)$ we define the *quadratic operator entropy* by

$$\odot(x) := \odot(x|1) = x^* \ln\left(\left|x^{-1}\right|^2\right) x = x^* \ln\left(\left|x^*\right|^{-2}\right) x = -2x^* \ln\left(\left|x^*\right|\right) x \tag{2.24}$$

and *quadratic Tsallis operator entropy* by

$$\begin{aligned} \circledcirc_t(x) &:= x^* T_t\left(\left|x^{-1}\right|^2\right) x = x^* \frac{\left(\left|x^{-1}\right|^2\right)^t - 1}{t} x = \frac{x \circledS_t 1 - |x|^2}{t} \\ &= \frac{x^* \left|x^*\right|^{-2t} x - |x|^2}{t} = \frac{\left|\left|x^*\right|^{-t} x\right|^2 - |x|^2}{t} \end{aligned} \tag{2.25}$$

for $t > 0$.

Corollary 3. *For any $x \in \operatorname{Inv}(A)$ and $t > 0$ we have*

$$\circledcirc_{-t}(x) \le \odot(x) \le \circledcirc_t(x). \tag{2.26}$$

In particular,

$$\left(1 - |x|^2\right)|x|^2 \le \odot(x) \le 1 - |x|^2, \tag{2.27}$$

$$2\left(1 - x^* |x| (x^*)^{-1}\right)|x|^2 \le \odot(x) \le 2\left(x^* |x|^{-1} x - |x|^2\right) \tag{2.28}$$

and

$$\frac{1}{2}\left(1 - |x|^4\right)|x|^2 \le \odot(x) \le \frac{1}{2}\left(|x|^{-4} - 1\right)|x|^2. \tag{2.29}$$

For a positive invertible element c we define the *operator entropy* by

$$\eta(c) := S(c|1) = c^{1/2}\left(\ln\left(c^{-1}\right)\right)c^{1/2} = -c \ln c$$

and the *Tsallis operator entropy* by

$$T_t(c) := T_t(c|1) = c^{1/2} T_t\left(c^{-1}\right)c^{1/2} = \frac{c^{1-t} - c}{t},$$

where $t \neq 0$. The operator entropy for bounded linear operators in Hilbert spaces was considered by Nakamura-Umegaki in [8].

We have

$$\eta(c) = \odot\left(c^{1/2}\right) \text{ and } T_t(c) = \circledcirc_t\left(c^{1/2}\right)$$

for $c > 0$ and $t \neq 0$.

By Corollary 3 we get the inequalities

$$T_{-t}(c) \le \eta(c) \le T_t(c) \tag{2.30}$$

for $c > 0$ and $t > 0$. In particular,

$$(1 - c)c \le \eta(c) \le 1 - c, \tag{2.31}$$

$$2\left(1 - c^{1/2}\right)c \le \eta(c) \le 2\left(1 - c^{1/2}\right)c^{1/2} \tag{2.32}$$

and

$$\frac{1}{2}\left(1 - c^2\right)c \le \eta(c) \le \frac{1}{2}\left(c^{-2} - 1\right)c, \tag{2.33}$$

for $c > 0$.

3. Further Bounds

We start with the following scalar double inequality:

Lemma 4. *For any α, $\beta > 0$ we have*

$$\begin{aligned} \frac{1}{2}\left(1-\frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}\right)^2 &= \frac{1}{2}\frac{(\beta-\alpha)^2}{\max^2\{\alpha,\beta\}} \\ &\leq \frac{\beta-\alpha}{\alpha}-\ln\beta+\ln\alpha \\ &\leq \frac{1}{2}\frac{(\beta-\alpha)^2}{\min^2\{\alpha,\beta\}} = \frac{1}{2}\left(\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}-1\right)^2. \end{aligned} \tag{3.1}$$

Proof. Integrating by parts, we have

$$\begin{aligned} \int_\alpha^\beta \frac{\beta-t}{t^2}dt &= \int_\alpha^\beta (t-\beta)\, d\left(\frac{1}{t}\right) = \left.\frac{t-\beta}{t}\right|_\alpha^\beta - \int_\alpha^\beta \frac{1}{t}dt \\ &= \frac{\beta-\alpha}{\alpha}-\ln\beta+\ln\alpha \end{aligned} \tag{3.2}$$

for any α, $\beta > 0$.

If $\beta > \alpha$, then

$$\frac{1}{2}\frac{(\beta-\alpha)^2}{\alpha^2} \geq \int_\alpha^\beta \frac{\beta-t}{t^2}dt \geq \frac{1}{2}\frac{(\beta-\alpha)^2}{\beta^2}. \tag{3.3}$$

If $\alpha > \beta$ then

$$\int_\alpha^\beta \frac{\beta-t}{t^2}dt = -\int_\beta^\alpha \frac{\beta-t}{t^2}dt = \int_\beta^\alpha \frac{t-\beta}{t^2}dt$$

and

$$\frac{1}{2}\frac{(\beta-\alpha)^2}{\beta^2} \geq \int_\beta^\alpha \frac{t-\beta}{t^2}dt \geq \frac{1}{2}\frac{(\beta-\alpha)^2}{\alpha^2}. \tag{3.4}$$

Therefore, by (3.3) and (3.4) we have for any α, $\beta > 0$ that

$$\int_\alpha^\beta \frac{\beta-t}{t^2}dt \geq \frac{1}{2}\frac{(\beta-\alpha)^2}{\max^2\{\alpha,\beta\}} = \frac{1}{2}\left(\frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}-1\right)^2$$

and

$$\int_\alpha^\beta \frac{\beta-t}{t^2}dt \leq \frac{1}{2}\frac{(\beta-\alpha)^2}{\min^2\{\alpha,\beta\}} = \frac{1}{2}\left(\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}}-1\right)^2.$$

By the representation (3.2) we then get the desired result (3.1). □

Remark 2. *If we take in (3.1) $\alpha = 1$ and $\beta = \tau \in (0,\infty)$, then we get*

$$\begin{aligned} \frac{1}{2}\left(1-\frac{\min\{1,\tau\}}{\max\{1,\tau\}}\right)^2 &= \frac{1}{2}\frac{(\tau-1)^2}{\max^2\{1,\tau\}} \\ &\leq \tau-1-\ln\tau \\ &\leq \frac{1}{2}\frac{(\tau-1)^2}{\min^2\{1,\tau\}} = \frac{1}{2}\left(\frac{\max\{1,\tau\}}{\min\{1,\tau\}}-1\right)^2 \end{aligned} \tag{3.5}$$

and if we take $\alpha=\tau$ *and* $\beta=1$, *then we also get*

$$\frac{1}{2}\left(1-\frac{\min\{1,\tau\}}{\max\{1,\tau\}}\right)^2=\frac{1}{2}\frac{(\tau-1)^2}{\max^2\{1,\tau\}} \tag{3.6}$$
$$\leq \ln\tau-\frac{\tau-1}{\tau}$$
$$\leq \frac{1}{2}\frac{(\tau-1)^2}{\min^2\{1,\tau\}}=\frac{1}{2}\left(\frac{\max\{1,\tau\}}{\min\{1,\tau\}}-1\right)^2.$$

If $\tau\in[k,K]\subset(0,\infty)$, then by analyzing all possible locations of the interval $[k,K]$ and 1 we have

$$\min\{1,k\}\leq\min\{1,\tau\}\leq\min\{1,K\}$$

and

$$\max\{1,k\}\leq\max\{1,\tau\}\leq\max\{1,K\}.$$

By (3.5) and (3.6) we get the *local bounds*

$$\frac{1}{2}\frac{(\tau-1)^2}{\max^2\{1,K\}}\leq\tau-1-\ln\tau\leq\frac{1}{2}\frac{(\tau-1)^2}{\min^2\{1,k\}} \tag{3.7}$$

and

$$\frac{1}{2}\frac{(\tau-1)^2}{\max^2\{1,K\}}\leq\ln\tau-\frac{\tau-1}{\tau}\leq\frac{1}{2}\frac{(\tau-1)^2}{\min^2\{1,k\}} \tag{3.8}$$

for any $\tau\in[k,K]$.

Observe also that for $\tau\in[k,K]$ we have

$$1-\frac{\min\{1,\tau\}}{\max\{1,\tau\}}\geq 1-\frac{\min\{1,K\}}{\max\{1,k\}}\geq 0$$

and

$$0\leq\frac{\max\{1,\tau\}}{\min\{1,\tau\}}-1\leq\frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

Now, by (3.5) and (3.6) we get the *global bounds*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq\tau-1-\ln\tau\leq\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{3.9}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\leq\ln\tau-\frac{\tau-1}{\tau}\leq\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{3.10}$$

for any $\tau\in[k,K]$.

We also have:

Lemma 5. *For any* α, $\beta>0$, *the following inequalities are valid*

$$(0\leq)\,\frac{\beta-\alpha}{\alpha}-\ln\beta+\ln\alpha\leq\frac{(\beta-\alpha)^2}{\alpha\beta} \tag{3.11}$$

and

$$(0\leq)\ln\beta-\ln\alpha-\frac{\beta-\alpha}{\beta}\leq\frac{(\beta-\alpha)^2}{\alpha\beta}. \tag{3.12}$$

Proof. If $\beta > \alpha$, then

$$\int_{\alpha}^{\beta} \frac{\beta - t}{t^2} dt \le (\beta - \alpha) \int_{\alpha}^{\beta} \frac{1}{t^2} dt = (\beta - \alpha) \frac{\beta - \alpha}{\alpha\beta} = \frac{(\beta - \alpha)^2}{\alpha\beta}.$$

If $\alpha > \beta$, then

$$\int_{\alpha}^{\beta} \frac{\beta - t}{t^2} dt = \int_{\beta}^{\alpha} \frac{t - \beta}{t^2} dt \le (\alpha - \beta) \int_{\beta}^{\alpha} \frac{1}{t^2} dt = (\alpha - \beta) \frac{\alpha - \beta}{\alpha\beta} = \frac{(\beta - \alpha)^2}{\alpha\beta}.$$

Therefore,

$$\int_{\alpha}^{\beta} \frac{\beta - t}{t^2} dt \le \frac{(\beta - \alpha)^2}{\alpha\beta}$$

for any α, $\beta > 0$ and by the representation (3.2) we get the desired result (3.11). □

It is natural to ask, which of the upper bounds for the quantity

$$\frac{\beta - \alpha}{\alpha} - \ln \beta + \ln \alpha$$

as provided by (3.1) and (3.11) is better?

Consider the difference

$$\Delta(\alpha, \beta) := \frac{1}{2} \frac{(\beta - \alpha)^2}{\min^2\{\alpha, \beta\}} - \frac{(\beta - \alpha)^2}{\alpha\beta}, \ \alpha, \ \beta > 0.$$

We observe that for $\beta > \alpha$ we get

$$\Delta(\alpha, \beta) := \frac{1}{2} \frac{(\beta - \alpha)^2}{\alpha^2} - \frac{(\beta - \alpha)^2}{\alpha\beta} = \frac{(\beta - \alpha)^2}{2\alpha^2\beta} (\beta - 2\alpha).$$

Therefore $\Delta(\alpha, \beta) > 0$ if $\beta > 2\alpha$ and $\Delta(\alpha, \beta) < 0$ if $\alpha < \beta < 2\alpha$, meaning that neither of the upper bounds in (3.1) and (3.11) is always best.

If we take in (3.11) and (3.12) $\alpha = 1$ and $\beta = \tau \in (0, \infty)$, then we get

$$(0 \le) \, \tau - 1 - \ln \tau \le \frac{(\tau - 1)^2}{\tau} \tag{3.13}$$

and

$$(0 \le) \ln \tau - \frac{\tau - 1}{\tau} \le \frac{(\tau - 1)^2}{\tau} \tag{3.14}$$

for any $\tau > 0$.

If $\tau \in [k, K]$, then we have the global upper bounds

$$(0 \le) \, \tau - 1 - \ln \tau \le U(k, K) \tag{3.15}$$

and

$$(0 \le) \ln \tau - \frac{\tau - 1}{\tau} \le U(k, K), \tag{3.16}$$

where

$$U(k, K) := \begin{cases} \frac{(1-k)^2}{k} \text{ if } K < 1, \\ \max\left\{\frac{(1-k)^2}{k}, \frac{(K-1)^2}{K}\right\} \text{ if } k \le 1 \le K, \\ \frac{(K-1)^2}{K} \text{ if } 1 < k. \end{cases} \tag{3.17}$$

Indeed, if we consider the function $f(\tau)=\frac{(\tau-1)^2}{\tau}$, $\tau>0$, then we observe that

$$f'(\tau)=\frac{\tau^2-1}{\tau^2} \text{ and } f''(\tau)=\frac{2}{\tau^3},$$

which shows that f is strictly decreasing on $(0,1)$, strictly increasing on $[1,\infty)$ and strictly convex for $\tau>0$. We also have $f\left(\frac{1}{\tau}\right)=f(\tau)$ for $\tau>0$.

By (3.13) and by the properties of f we then have that for any $\tau\in[k,K]$

$$\begin{aligned}\tau-1-\ln\tau &\le \max_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau} \\ &= \begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},\frac{(K-1)^2}{K}\right\} \text{ if } k\le 1\le K,\\ \frac{(K-1)^2}{K} \text{ if } 1<k.\end{cases} \\ &= U(k,K).\end{aligned} \tag{3.18}$$

Let $\tau=\frac{1}{\tau}$ with $\tau\in[k,K]$. Then $\tau\in\left[\frac{1}{K},\frac{1}{k}\right]$ and we have like in (3.18) that

$$\begin{aligned}\tau-1-\ln\tau &\le \max_{\tau\in[K^{-1},k^{-1}]}\frac{(\tau-1)^2}{\tau} \\ &= \begin{cases}\frac{\left(K^{-1}-1\right)^2}{K^{-1}} \text{ if } k^{-1}<1,\\ \max\left\{\frac{\left(K^{-1}-1\right)^2}{K^{-1}},\frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}}\right\} \text{ if } k\le 1\le K^{-1},\\ \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}} \text{ if } 1<\frac{1}{K^{-1}}.\end{cases} \\ &= U(k,K),\end{aligned}$$

which implies (3.16).

Now, let

$$V(k,K):=\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2=\frac{1}{2}\begin{cases}\left(\frac{1-k}{k}\right)^2 \text{ if } K<1,\\ \left(\frac{K-k}{k}\right)^2 \text{ if } k\le 1\le K,\\ (K-1)^2 \text{ if } 1<k,\end{cases} \tag{3.19}$$

and

$$v(k,K):=\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2=\frac{1}{2}\begin{cases}(1-K)^2 \text{ if } K<1,\\ 0 \text{ if } k\le 1\le K,\\ \left(\frac{k-1}{k}\right)^2 \text{ if } 1<k,\end{cases} \tag{3.20}$$

then by (3.9) and (3.10) we have

$$v(k,K)\le\tau-1-\ln\tau\le V(k,K) \tag{3.21}$$

and

$$v(k,K)\le\ln\tau-\frac{\tau-1}{\tau}\le V(k,K) \tag{3.22}$$

for any $\tau\in[k,K]$.

Therefore, we have the double inequalities of interest:

$$v(k,K)\le\tau-1-\ln\tau\le\min\{V(k,K),U(k,K)\} \tag{3.23}$$

and

$$v(k,K) \leq \ln \tau - \frac{\tau - 1}{\tau} \leq \min\{V(k,K), U(k,K)\} \tag{3.24}$$

for any $\tau \in [k,K]$.

Lemma 6. *Let $x \in [k,K]$ and $t > 0$, then we have*

$$\frac{1}{t} v\left(k^t, K^t\right) \leq \frac{x^t - 1}{t} - \ln x \leq \frac{1}{t} \min\left\{V\left(k^t, K^t\right), U\left(k^t, K^t\right)\right\} \tag{3.25}$$

and

$$\frac{1}{t} v\left(k^t, K^t\right) \leq \ln x - \frac{1 - x^{-t}}{t} \leq \frac{1}{t} \min\left\{V\left(k^t, K^t\right), U\left(k^t, K^t\right)\right\}. \tag{3.26}$$

The proof follows by choosing $\tau = x^t \in [k^t, K^t]$ in the inequalities (3.23) and (3.24).

Assume that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that

$$M \geq \left|yx^{-1}\right| \geq m. \tag{3.27}$$

The inequality (3.27) is equivalent to

$$M^2 \geq \left|yx^{-1}\right|^2 = (x^*)^{-1} |y|^2 x^{-1} \geq m^2.$$

If we multiply at left with x^* and at right with x we get the equivalent relation

$$M^2 |x|^2 \geq |y|^2 \geq m^2 |x|^2. \tag{3.28}$$

We have:

Theorem 3. *Assume that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that either (3.27), or, equivalently (3.28) is true. Then for any $t > 0$ we have*

$$\begin{aligned} \frac{1}{t} v\left(m^{2t}, M^{2t}\right) |x|^2 &\leq \circledcirc_t (x|y) - \odot (x|y) \\ &\leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\} |x|^2 \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \frac{1}{t} v\left(m^{2t}, M^{2t}\right) |x|^2 &\leq \odot (x|y) - \circledcirc_{-t} (x|y) \\ &\leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\} |x|^2, \end{aligned} \tag{3.30}$$

where the functions v, V and U are defined by (3.20), (3.19) and (3.17), respectively.

Proof. From Lemma 6 for $k = m^2$ and $K = M^2$, if $z \in \left[m^2, M^2\right]$ and $t > 0$, then we have

$$\frac{1}{t} v\left(m^{2t}, M^{2t}\right) \leq \frac{z^t - 1}{t} - \ln z \leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\} \tag{3.31}$$

and

$$\frac{1}{t} v\left(m^{2t}, M^{2t}\right) \leq \ln z - \frac{1 - z^{-t}}{t} \leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\}. \tag{3.32}$$

If x, $y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (3.27) is true, then the element $u=\left|yx^{-1}\right|^{2}$ has the spectrum $\sigma(u)\subset\left[m^{2},M^{2}\right]$. Making use of Lemma 3 we have in the order of A that

$$\begin{aligned}\frac{1}{t}v\left(m^{2t},M^{2t}\right)&\leq\frac{\left(\left|yx^{-1}\right|^{2}\right)^{t}-1}{t}-\ln\left(\left|yx^{-1}\right|^{2}\right)\\&\leq\frac{1}{t}\min\left\{V\left(m^{2t},M^{2t}\right),U\left(m^{2t},M^{2t}\right)\right\}\end{aligned}\tag{3.33}$$

and

$$\begin{aligned}\frac{1}{t}v\left(m^{2t},M^{2t}\right)&\leq\ln\left(\left|yx^{-1}\right|^{2}\right)-\frac{1-\left(\left|yx^{-1}\right|^{2}\right)^{-t}}{t}\\&\leq\frac{1}{t}\min\left\{V\left(m^{2t},M^{2t}\right),U\left(m^{2t},M^{2t}\right)\right\}\end{aligned}\tag{3.34}$$

for any $t>0$.

By multiplying (3.33) and (3.34) at left with x^{*} and at right with x we get the desired results. □

If we take $t=1$ in (3.29) and (3.30), then we get

$$\begin{aligned}v\left(m^{2},M^{2}\right)|x|^{2}&\leq|y|^{2}-|x|^{2}-\odot(x|y)\\&\leq\min\left\{V\left(m^{2},M^{2}\right),U\left(m^{2},M^{2}\right)\right\}|x|^{2}\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}v\left(m^{2},M^{2}\right)|x|^{2}&\leq\odot(x|y)-\left(1-|x|^{2}|y|^{-2}\right)|x|^{2}\\&\leq\min\left\{V\left(m^{2},M^{2}\right),U\left(m^{2},M^{2}\right)\right\}|x|^{2},\end{aligned}\tag{3.36}$$

provided that x, $y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (3.27) is valid.

For $t=\frac{1}{2}$, we get from (3.29) and (3.30) that

$$\begin{aligned}2v(m,M)|x|^{2}&\leq2\left(x\circledS y-|x|^{2}\right)-\odot(x|y)\\&\leq2\min\{V(m,M),U(m,M)\}|x|^{2}\end{aligned}\tag{3.37}$$

and

$$\begin{aligned}2v(m,M)|x|^{2}&\leq\odot(x|y)-2\left(1-|x|^{2}(x\circledS y)^{-1}\right)|x|^{2}\\&\leq2\min\{V(m,M),U(m,M)\}|x|^{2}.\end{aligned}\tag{3.38}$$

Recall that a C^{*}*-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^{*}a\|=\|a\|^{2}\text{ for any }a\in A.$$

If a C^{*}-algebra A has a unit 1, then automatically $\|1\|=1$.

It is well know that, if A is a C^{*}-algebra, then (see for instance [7, 2.2.5 Theorem])

$$b\geq a\geq0\text{ implies that }\|b\|\geq\|a\|.$$

Corollary 4. *Let A be a unital C^*-algebra. If $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that either (3.27), or, equivalently (3.28) is true, then we have the norm inequalities*

$$\frac{1}{t} v\left(m^{2t}, M^{2t}\right) \|x\|^2 \leq \|\circledcirc_t (x|y) - \odot (x|y)\| \leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\} \|x\|^2 \tag{3.39}$$

and

$$\frac{1}{t} v\left(m^{2t}, M^{2t}\right) \|x\|^2 \leq \|\odot (x|y) - \circledcirc_{-t} (x|y)\| \leq \frac{1}{t} \min\left\{V\left(m^{2t}, M^{2t}\right), U\left(m^{2t}, M^{2t}\right)\right\} \|x\|^2, \tag{3.40}$$

where the functions v, V and U are defined by (3.20), (3.19) and (3.17), respectively.

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