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INEQUALITIES FOR WEIGHTED GEOMETRIC MEAN IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS VIA A RESULT OF CARTWRIGHT AND FIELD

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Abstract. Consider the *quadratic weighted geometric mean*

$$x \circledS_\nu y := \left| \left| y x^{-1} \right|^\nu x \right|^2$$

for invertible elements x, y in a *Hermitian unital Banach $*$-algebra* and real number ν. In this paper, by utilizing a result of Cartwright and Field, we obtain various upper and lower bounds for the positive difference

$$(1-\nu)|x|^2 + \nu |y|^2 - x \circledS_\nu y,$$

where $\nu \in [0,1]$, under various assumptions for the elements involved. Applications for the classical *weighted geometric mean*

$$a \sharp_\nu b := a^{1/2} \left(a^{-1/2} b a^{-1/2} \right)^\upsilon a^{1/2}$$

of positive elements a, b that satisfy the condition $0 < ka \le b \le Ka$ for certain numbers $0 < k < K$, are also given.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \ge 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \ge 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \ge b$ means that $a - b \ge 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [14] (see also [2, Theorem 41.5])

$$a^* a \ge 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [13], Tanahashi and Uchiyama [15] proved the following fundamental properties (see also [7]):

(i) If a, $b \in A$, then $a \ge 0$, $b \ge 0$ imply $a + b \ge 0$ and $\alpha \ge 0$ implies $\alpha a \ge 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \ge 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \ge b > 0$ or $a > b \ge 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;

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(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [13] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [15, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

We define the following means for $\nu \in [0, 1]$, see also [7] for different notations:

$$a \nabla_\nu b := (1 - \nu) a + \nu b, \ a, \ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a, b),

$$a !_\nu b := \left((1 - \nu) a^{-1} + \nu b^{-1}\right)^{-1}, \ a, \ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a, b) and

$$a \sharp_\nu b := a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^\upsilon a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a, b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [7], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)\,c = (c^*ac)!\,(c^*bc) \text{ and } c^*(a\sharp b)\,c = (c^*ac)\,\sharp\,(c^*bc);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [7] the following inequality between the weighted means introduced above:

$$a\nabla_\nu b \geq a\sharp_\nu b \geq a!_\nu b \tag{HGA}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

In [15], Tanahashi and Uchiyama obtained the following identity of interest:

Lemma 1. *If $0 < c,\ d$ and λ is a real number, then*

$$(dcd)^\lambda = dc^{1/2}\left(c^{1/2}d^2c^{1/2}\right)^{\lambda-1}c^{1/2}d. \tag{1.1}$$

Using this equality we can prove the following fact [6]:

Proposition 1. *For any $0 < a,\ b \in A$ we have*

$$b\sharp_{1-\nu}a = a\sharp_\nu b \tag{1.2}$$

for any real number ν.

In [6] we introduced the *quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$, as the positive element denoted by $x\circledS_\nu y$ and defined by

$$x\circledS_\nu y := x^*\left(\left(x^*\right)^{-1}y^*yx^{-1}\right)^\nu x = x^*\left|yx^{-1}\right|^{2\nu}x = \left|\left|yx^{-1}\right|^\nu x\right|^2. \tag{S}$$

When $\nu = 1/2$, we denote $x\circledS_{1/2}y$ by $x\circledS y$ and we have

$$x\circledS y = x^*\left(\left(x^*\right)^{-1}y^*yx^{-1}\right)^{1/2}x = x^*\left|yx^{-1}\right|x = \left|\left|yx^{-1}\right|^{1/2}x\right|^2.$$

We can also introduce the $1/2$-*quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$ by

$$x\circledS_\nu^{1/2}y := (x\circledS_\nu y)^{1/2} = \left|\left|yx^{-1}\right|^\nu x\right|. \tag{1/2-S}$$

Correspondingly, when $\nu = 1/2$ we denote $x\circledS^{1/2}y$ and we have

$$x\circledS^{1/2}y = \left|\left|yx^{-1}\right|^{1/2}x\right|.$$

The following equalities hold [6]:

Proposition 2. *For any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$ we have*

$$(x\circledS_\nu y)^{-1} = \left(x^*\right)^{-1}\circledS_\nu\left(y^*\right)^{-1}$$

and

$$\left(x^{-1}\right)\circledS_\nu\left(y^{-1}\right) = \left(x^*\circledS_\nu y^*\right)^{-1}.$$

If we take in (S) $x=a^{1/2}$ and $y=b^{1/2}$ with $a,\ b>0$ then we get

$$a^{1/2}Ⓢ_{\nu}b^{1/2}=a\sharp_{\nu}b$$

for any $\nu\in\mathbb{R}$ that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let $x,\ y\in\operatorname{Inv}(A)$. If we take in the definition of "$\sharp_{\nu}$" the elements $a=|x|^{2}>0$ and $b=|y|^{2}>0$ we also have for real ν

$$|x|^{2}\sharp_{\nu}|y|^{2}=|x|\left(|x|^{-1}|y|^{2}|x|^{-1}\right)^{\upsilon}|x|=|x|\left||y|\,|x|^{-1}\right|^{2\upsilon}|x|=\left|\left||y|\,|x|^{-1}\right|^{\upsilon}|x|\right|^{2}.$$

It is then natural to ask how the positive elements $xⓈ_{\nu}y$ and $|x|^{2}\sharp_{\nu}|y|^{2}$ do compare, when $x,\ y\in\operatorname{Inv}(A)$ and $\nu\in\mathbb{R}$?

In [6] we proved the following lemma that provides a slight generalization of Lemma 1.

Lemma 2. *If $0<c,\ d\in\operatorname{Inv}(A)$ and λ is a real number, then*

$$(dcd^{*})^{\lambda}=dc^{1/2}\left(c^{1/2}|d|^{2}c^{1/2}\right)^{\lambda-1}c^{1/2}d^{*}. \tag{1.3}$$

Remark 1. *The identity (1.3) was proved by. T. Furuta in* [8] *for positive operator c and invertible operator d in the Banach algebra of all bonded linear operators on a Hilbert space by using the polar decomposition of the invertible operator $dc^{1/2}$.*

The following fundamental fact that connects the quadratic weighted geometric mean $Ⓢ_{\nu}$ to the weighted geometric mean $\sharp_{\nu}$ holds [6]:

Theorem 1. *If $x,\ y\in\operatorname{Inv}(A)$ and λ is a real number, then*

$$xⓈ_{\nu}y=|x|^{2}\sharp_{\nu}|y|^{2} \tag{1.4}$$

Now, assume that $f(z)$ is analytic in the right half open plane $\{\operatorname{Re}z>0\}$ and for the interval $I\subset(0,\infty)$ assume that $f(z)\geq0$ for any $z\in I$. If $u\in A$ such that $\sigma(u)\subset I$, then by (SMT) we have

$$\sigma(f(u))=f(\sigma(u))\subset f(I)\subset[0,\infty)$$

meaning that $f(u)\geq0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

Lemma 3. *Let $f(z)$ and $g(z)$ be analytic in the right half open plane $\{\operatorname{Re}z>0\}$ and for the interval $I\subset(0,\infty)$ assume that $f(z)\geq g(z)$ for any $z\in I$. Then for any $u\in A$ with $\sigma(u)\subset I$ we have $f(u)\geq g(u)$ in the order of A.*

We have the following inequalities between means [6]:

Theorem 2. *For any $x,\ y\in\operatorname{Inv}(A)$ and $\nu\in[0,1]$ we have*

$$|x|^{2}\nabla_{\nu}|y|^{2}\geq xⓈ_{\nu}y\geq|x|^{2}!_{\nu}|y|^{2}. \tag{1.5}$$

In particular,

$$|x|^{2}\nabla|y|^{2}\geq xⓈy\geq|x|^{2}!\,|y|^{2}. \tag{1.6}$$

We can define the weighted means for $\nu \in [0,1]$ and the elements $x, y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ by

$$x\nabla_\nu^{1/2} y := \left(|x|^2 \nabla_\nu |y|^2\right)^{1/2} = \left((1-\nu)|x|^2 + \nu |y|^2\right)^{1/2}$$

and

$$x!_\nu^{1/2} y := \left(|x|^2 !_\nu |y|^2\right)^{1/2} = \left((1-\nu)|x|^{-2} + \nu |y|^{-2}\right)^{-1/2}.$$

For $\nu = 1/2$ we consider

$$x\nabla^{1/2} y := \left(|x|^2 \nabla |y|^2\right)^{1/2} = \frac{\sqrt{2}}{2}\left(|x|^2 + |y|^2\right)^{1/2}$$

and

$$x!^{1/2} y := \left(|x|^2 ! |y|^2\right)^{1/2} = \sqrt{2}\left(|x|^{-2} + |y|^{-2}\right)^{-1/2}.$$

Corollary 1. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Then for any $x, y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$x\nabla_\nu^{1/2} y \geq x Ⓢ_\nu^{1/2} y \geq x!_\nu^{1/2} y. \tag{1.7}$$

In particular, we have

$$x\nabla^{1/2} y \geq x Ⓢ^{1/2} y \geq x!^{1/2} y. \tag{1.8}$$

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^* a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [12, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \|b\| \geq \|a\|.$$

Corollary 2. *Let A be a unital C^*-algebra. Then for any $x, y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$(1-\nu)\|x\|^2 + \nu\|y\|^2 \geq \left\|(1-\nu)|x|^2 + \nu|y|^2\right\| \geq \left\|\left|yx^{-1}\right|^\nu x\right\|^2. \tag{1.9}$$

In particular,

$$\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) \geq \frac{1}{2}\left\||x|^2 + |y|^2\right\| \geq \left\|\left|yx^{-1}\right|^{1/2} x\right\|^2. \tag{1.10}$$

Motivated by the above facts, in this paper we obtain various upper and lower bounds for the positive difference

$$(1-\nu)|x|^2 + \nu|y|^2 - x Ⓢ_\nu y,$$

where $\nu \in [0,1]$, under various assumptions for the elements involved. Applications for the classical geometric mean $a\sharp_\nu b := a^{1/2}\left(a^{-1/2} b a^{-1/2}\right)^\upsilon a^{1/2}$ of positive elements a, b that satisfy the condition $0 < ka \leq b \leq Ka$ for certain numbers $0 < k < K$, are also given.

2. Refinements and Reverses

We have the following inequality that provides a refinement and a reverse for the celebrated scalar Young's inequality

$$\frac{1}{2}\nu\left(1-\nu\right)\frac{\left(\beta-\alpha\right)^2}{\max\left\{\alpha,\beta\right\}}\leq\left(1-\nu\right)\alpha+\nu\beta-\alpha^{1-\nu}\beta^{\nu}\leq\frac{1}{2}\nu\left(1-\nu\right)\frac{\left(\beta-\alpha\right)^2}{\min\left\{\alpha,\beta\right\}}\tag{2.1}$$

for any $\alpha,\ \beta>0$ and $\nu\in[0,1]$.

This result was obtained in 1978 by Cartwright and Field [3] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Assume that $x,\ y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that

$$M\geq\left|yx^{-1}\right|\geq m.\tag{2.2}$$

The inequality (2.2) is equivalent to

$$M^2\geq\left|yx^{-1}\right|^2=\left(x^*\right)^{-1}\left|y\right|^2x^{-1}\geq m^2.\tag{2.3}$$

If we multiply at left with x^* and at right with x we get the equivalent relation

$$M^2\left|x\right|^2\geq\left|y\right|^2\geq m^2\left|x\right|^2.\tag{2.4}$$

For $[k,K]\subset(0,\infty)$ we consider the coefficients

$$c\left(k,K\right):=\begin{cases}\left(K-1\right)^2 & \text{if } K<1,\\ 0 \text{ if } k\leq 1\leq K, & \\ \frac{(k-1)^2}{K} & \text{if } 1<k\end{cases}\tag{2.5}$$

and

$$C\left(k,K\right):=\begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \frac{1}{k}\max\left\{\left(k-1\right)^2,\left(K-1\right)^2\right\} \text{ if } k\leq 1\leq K,\\ \left(K-1\right)^2 \text{ if } 1<k.\end{cases}\tag{2.6}$$

We have:

Theorem 3. *Assume that $x,\ y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (2.2) is true. Then we have the inequalities*

$$\begin{aligned}\frac{1}{2}\nu\left(1-\nu\right)c\left(m^2,M^2\right)\left|x\right|^2&\leq\frac{1}{2}\frac{\nu\left(1-\nu\right)}{\max\left\{M^2,1\right\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2\\&\leq\left|x\right|^2\nabla_\nu\left|y\right|^2-x\circledS_\nu y\\&\leq\frac{1}{2}\frac{\nu\left(1-\nu\right)}{\min\left\{m^2,1\right\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2\\&\leq\frac{1}{2}\nu\left(1-\nu\right)C\left(m^2,M^2\right)\left|x\right|^2\end{aligned}\tag{2.7}$$

for any $\nu\in[0,1]$.

In particular, we have

$$\begin{aligned}\frac{1}{8}c\left(m^2,M^2\right)|x|^2 &\leq \frac{1}{8}\frac{1}{\max\{M^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2 \\ &\leq |x|^2\nabla|y|^2 - x \circledS y \\ &\leq \frac{1}{8}\frac{1}{\min\{m^2,1\}}\left|\left(\left|yx^{-1}\right|^2-1\right)x\right|^2 \\ &\leq \frac{1}{8}C\left(m^2,M^2\right)|x|^2 .\end{aligned} \tag{2.8}$$

Proof. If we write the inequality (2.1) for $\alpha = 1$ and $\beta = \tau$ we get

$$\frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\max\{\tau,1\}} \leq 1-\nu+\nu\tau-\tau^\nu \leq \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\min\{\tau,1\}} \tag{2.9}$$

for any $\tau > 0$ and for any $\nu \in [0,1]$.

If $\tau \in [k,K] \subset (0,\infty)$, then $\max\{\tau,1\} \leq \max\{K,1\}$ and $\min\{k,1\} \leq \min\{\tau,1\}$ and by (2.9) we get

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\frac{\min_{\tau\in[k,K]}(\tau-1)^2}{\max\{K,1\}} &\leq \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\max\{K,1\}} \\ &\leq 1-\nu+\nu\tau-\tau^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{(\tau-1)^2}{\min\{k,1\}} \\ &\leq \frac{1}{2}\nu(1-\nu)\frac{\max_{\tau\in[k,K]}(\tau-1)^2}{\min\{k,1\}}\end{aligned} \tag{2.10}$$

for any $\tau \in [k,K]$ and for any $\nu \in [0,1]$.

Observe that

$$\min_{\tau\in[k,K]}(\tau-1)^2 = \begin{cases}(K-1)^2 \text{ if } K<1, \\ 0 \text{ if } k\leq 1\leq K, \\ (k-1)^2 \text{ if } 1<k\end{cases}$$

and

$$\max_{\tau\in[k,K]}(\tau-1)^2 = \begin{cases}(k-1)^2 \text{ if } K<1, \\ \max\left\{(k-1)^2,(K-1)^2\right\} \text{ if } k\leq 1\leq K, \\ (K-1)^2 \text{ if } 1<k.\end{cases}$$

Then

$$\frac{\min_{\tau\in[k,K]}(\tau-1)^2}{\max\{K,1\}} = c(k,K)$$

and

$$\frac{\max_{\tau\in[k,K]}(\tau-1)^2}{\min\{k,1\}} = C(k,K)$$

as defined by (2.5) and (2.6).

Using the inequality (2.10) we have

$$\frac{1}{2}\nu(1-\nu)c(k,M) \leq \frac{1}{2}\nu(1-\nu)\frac{(z-1)^2}{\max\{M,1\}} \tag{2.11}$$
$$\leq 1-\nu+\nu z-z^{\nu}$$
$$\leq \frac{1}{2}\nu(1-\nu)\frac{(z-1)^2}{\min\{k,1\}}$$
$$\leq \frac{1}{2}\nu(1-\nu)C(k,M)$$

for any real $z \in [k,K] \subset (0,\infty)$ and for any $\nu \in [0,1]$.

Let $u \in A$ with spectrum $\sigma(u) \subset [k,K] \subset (0,\infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\operatorname{Re} z > 0\}$ involved in the inequality (2.11) we conclude that we have in the order of A that

$$\frac{1}{2}\nu(1-\nu)c(k,K) \leq \frac{1}{2}\frac{\nu(1-\nu)}{\max\{K,1\}}(u-1)^2 \tag{2.12}$$
$$\leq 1-\nu+\nu u-u^{\nu}$$
$$\leq \frac{1}{2}\frac{\nu(1-\nu)}{\min\{k,1\}}(u-1)^2$$
$$\leq \frac{1}{2}\nu(1-\nu)C(k,K)$$

for any $\nu \in [0,1]$.

If x, $y \in \operatorname{Inv}(A)$ satisfy the condition (2.2) then, by (2.3), the element $u = \left|yx^{-1}\right|^2 \in \operatorname{Inv}(A)$ and $\sigma(u) \subset \left[m^2, M^2\right] \subset (0,\infty)$.

By (2.12) we then have

$$\frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right) \leq \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}\left(\left|yx^{-1}\right|^2-1\right)^2 \tag{2.13}$$
$$\leq 1-\nu+\nu\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^{\nu}$$
$$\leq \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left(\left|yx^{-1}\right|^2-1\right)^2$$
$$\leq \frac{1}{2}\nu(1-\nu)C\left(m^2,M^2\right)$$

for any $\nu \in [0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\frac{1}{2}\nu(1-\nu)c\left(m^2,M^2\right)|x|^2 \tag{2.14}$$
$$\leq \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}x^*\left(\left|yx^{-1}\right|^2-1\right)^2x$$
$$\leq (1-\nu)|x|^2+\nu x^*\left|yx^{-1}\right|^2x-x^*\left(\left|yx^{-1}\right|^2\right)^{\nu}x$$
$$\leq \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}x^*\left(\left|yx^{-1}\right|^2-1\right)^2x$$
$$\leq \frac{1}{2}\nu(1-\nu)C\left(m^2,M^2\right)|x|^2$$

for any $\nu \in [0,1]$.

Since

$$x^*\left|yx^{-1}\right|^2 x = x^*\left(\left(x^*\right)^{-1} y^* y x^{-1}\right) x = y^* y = |y|^2,$$

$$x^*\left(\left|yx^{-1}\right|^2\right)^{\nu} x = x \circledS_{\nu} y$$

and

$$x^*\left(\left|yx^{-1}\right|^2 - 1\right)^2 x = \left|\left(\left|yx^{-1}\right|^2 - 1\right) x\right|^2$$

for $x,\ y \in \operatorname{Inv}(A)$, then by (2.14) we get the desired result (2.7). □

Corollary 3. *Let A be a unital C^*-algebra. Assume that $x,\ y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (2.2) holds, then we have*

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)\, c\left(m^2, M^2\right)\|x\|^2 &\leq \frac{1}{2}\frac{\nu(1-\nu)}{\max\{M^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \left\||x|^2 \nabla_{\nu} |y|^2 - x \circledS_{\nu} y\right\| \\ &\leq \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \frac{1}{2}\nu(1-\nu)\, C\left(m^2, M^2\right)\|x\|^2 \end{aligned} \tag{2.15}$$

for any $\nu \in [0,1]$.

In particular,

$$\begin{aligned} \frac{1}{8} c\left(m^2, M^2\right)\|x\|^2 &\leq \frac{1}{8}\frac{1}{\max\{M^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \left\||x|^2 \nabla |y|^2 - x \circledS y\right\| \\ &\leq \frac{1}{8}\frac{1}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \frac{1}{8} C\left(m^2, M^2\right)\|x\|^2. \end{aligned} \tag{2.16}$$

Remark 2. *Using the triangle inequality we have*

$$0 \leq \left\||x|^2 \nabla_{\nu} |y|^2\right\| - \|x \circledS_{\nu} y\| \leq \left\||x|^2 \nabla_{\nu} |y|^2 - x \circledS_{\nu} y\right\|$$

and by (2.15) we get the following reverse of the second inequality in (1.9)

$$\begin{aligned} &\left\|(1-\nu)|x|^2 + \nu |y|^2\right\| \\ &\leq \left\|\left|yx^{-1}\right|^{\nu} x\right\|^2 + \frac{1}{2}\frac{\nu(1-\nu)}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \left\|\left|yx^{-1}\right|^{\nu} x\right\|^2 + \frac{1}{2}\nu(1-\nu)\, C\left(m^2, M^2\right)\|x\|^2 \end{aligned} \tag{2.17}$$

provided that x, y and ν are as in Corollary 3.

In particular,

$$\begin{aligned} \frac{1}{2}\left\||x|^2 + |y|^2\right\| &\leq \left\|\left|yx^{-1}\right|^{1/2} x\right\|^2 + \frac{1}{8}\frac{1}{\min\{m^2,1\}}\left\|\left(\left|yx^{-1}\right|^2 - 1\right) x\right\|^2 \\ &\leq \left\|\left|yx^{-1}\right|^{1/2} x\right\|^2 + \frac{1}{8} C\left(m^2, M^2\right)\|x\|^2. \end{aligned} \tag{2.18}$$

Corollary 4. *If $0 < a,\ b \in A$ and $0 < k < K$ are such that*

$$ka \leq b \leq Ka, \tag{2.19}$$

then

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)c(k,K)a &\leq \frac{1}{2}\frac{\nu(1-\nu)}{\max\{K,1\}}\left|\left(\left|b^{1/2}a^{-1/2}\right|^2-1\right)a^{1/2}\right|^2 \\ &\leq a\nabla_\nu b - a\sharp_\nu b \\ &\leq \frac{1}{2}\frac{\nu(1-\nu)}{\min\{k,1\}}\left|\left(\left|b^{1/2}a^{-1/2}\right|^2-1\right)a^{1/2}\right|^2 \\ &\leq \frac{1}{2}\nu(1-\nu)C(k,K)a\end{aligned} \tag{2.20}$$

for any $\nu \in [0,1]$, where $c(k,K)$ and $C(k,K)$ are given by (2.5) and (2.6).

In particular, we have

$$\begin{aligned}\frac{1}{8}c(k,K)a &\leq \frac{1}{8}\frac{1}{\max\{K,1\}}\left|\left(\left|b^{1/2}a^{-1/2}\right|^2-1\right)a^{1/2}\right|^2 \\ &\leq a\nabla b - a\sharp b \\ &\leq \frac{1}{8}\frac{1}{\min\{k,1\}}\left|\left(\left|b^{1/2}a^{-1/2}\right|^2-1\right)a^{1/2}\right|^2 \\ &\leq \frac{1}{8}C(k,K)a.\end{aligned} \tag{2.21}$$

The proof follows by Theorem 3 applied for $x = a^{1/2}$, $y = b^{1/2}$, $M = \sqrt{K}$ and $m = \sqrt{k}$.

3. Some Related Results

We observe that since

$$\max\{\alpha,\beta\}\min\{\alpha,\beta\} = \alpha\beta \text{ for } \alpha,\beta > 0,$$

then the inequality (2.1) can be written in an equivalent form as

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\min\{\alpha,\beta\}\frac{(\beta-\alpha)^2}{\alpha\beta} &\leq (1-\nu)\alpha + \nu\beta - \alpha^{1-\nu}\beta^\nu \\ &\leq \frac{1}{2}\nu(1-\nu)\max\{\alpha,\beta\}\frac{(\beta-\alpha)^2}{\alpha\beta}\end{aligned} \tag{3.1}$$

for any $\alpha,\ \beta > 0$ and $\nu \in [0,1]$.

We define the following coefficients associated with the interval $[k,K] \subset (0,\infty)$:

$$d(k,K) := \begin{cases} \frac{k(K-1)^2}{K} \text{ if } K < 1, \\ 0 \text{ if } k \leq 1 \leq K, \\ \frac{(k-1)^2}{k} \text{ if } 1 < k \end{cases} \tag{3.2}$$

and

$$D(k,K) := \begin{cases} \frac{(k-1)^2}{k} \text{ if } K < 1, \\ \max\left\{\frac{K(k-1)^2}{k}, (K-1)^2\right\} \text{ if } k \leq 1 \leq K, \\ (K-1)^2 \text{ if } 1 < k. \end{cases} \tag{3.3}$$

Theorem 4. *Assume that* $x, y \in \operatorname{Inv}(A)$ *and the constants* $M > m > 0$ *are such that (2.2) is true. Then we have the inequalities*

$$
\begin{aligned}
(3.4)\quad \frac{1}{2}\nu(1-\nu)\, d\left(m^2, M^2\right)|x|^2 &\le \frac{1}{2}\nu(1-\nu)\min\left\{m^2, 1\right\}\left||y|^{-1}\left(|y|^2 - |x|^2\right)\right|^2 \\
&\le |x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y \\
&\le \frac{1}{2}\nu(1-\nu)\max\left\{M^2, 1\right\}\left||y|^{-1}\left(|y|^2 - |x|^2\right)\right|^2 \\
&\le \frac{1}{2}\nu(1-\nu)\, D\left(m^2, M^2\right)|x|^2 .
\end{aligned}
$$

for any $\nu \in [0,1]$, *where the coefficients* $d(\cdot,\cdot)$ *and* $D(\cdot,\cdot)$ *are defined by (3.2) and (3.3).*

In particular, we have

$$
\begin{aligned}
(3.5)\quad \frac{1}{8} d\left(m^2, M^2\right)|x|^2 &\le \frac{1}{8}\min\left\{m^2, 1\right\}\left||y|^{-1}\left(|y|^2 - |x|^2\right)\right|^2 \\
&\le |x|^2 \nabla |y|^2 - x \circledS y \\
&\le \frac{1}{8}\max\left\{M^2, 1\right\}\left||y|^{-1}\left(|y|^2 - |x|^2\right)\right|^2 \\
&\le \frac{1}{8} D\left(m^2, M^2\right)|x|^2 .
\end{aligned}
$$

Proof. If we write the inequality (3.1) for $\alpha = 1$ and $\beta = \tau$ we get

$$
\begin{aligned}
(3.6)\quad \frac{1}{2}\nu(1-\nu)\min\{\tau, 1\}\frac{(\tau-1)^2}{\tau} &\le 1 - \nu + \nu\tau - \tau^\nu \\
&\le \frac{1}{2}\nu(1-\nu)\max\{\tau, 1\}\frac{(\tau-1)^2}{\tau}
\end{aligned}
$$

for any $\tau > 0$ and for any $\nu \in [0,1]$.

If $\tau \in [k, K] \subset (0, \infty)$, then $\max\{\tau, 1\} \le \max\{K, 1\}$ and $\min\{k, 1\} \le \min\{\tau, 1\}$ and by (3.6) we get

$$
\begin{aligned}
(3.7)\quad & \frac{1}{2}\nu(1-\nu)\min\{k, 1\}\min_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau} \\
&\le \frac{1}{2}\nu(1-\nu)\min\{k, 1\}\frac{(\tau-1)^2}{\tau} \\
&\le 1 - \nu + \nu\tau - \tau^\nu \\
&\le \frac{1}{2}\nu(1-\nu)\max\{K, 1\}\frac{(\tau-1)^2}{\tau} \\
&\le \frac{1}{2}\nu(1-\nu)\max\{K, 1\}\max_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}.
\end{aligned}
$$

Consider the function $\delta : (0,\infty) \to (0,\infty)$, $\delta(\tau) = \frac{(\tau-1)^2}{\tau}$. Then

$$
\delta'(\tau) = \frac{2(\tau-1)\tau - (\tau-1)^2}{\tau^2} = \frac{(\tau-1)(\tau+1)}{\tau^2}.
$$

This shows that the function δ is strictly decreasing on $(0,1)$, strictly increasing on $(1,\infty)$, $\delta(1) = 0$ and

$$
\lim_{\tau\to 0+}\delta(\tau) = \lim_{\tau\to\infty}\delta(\tau) = \infty.
$$

By taking into account all possible locations of the interval $[k,K]$ and the number 1 we have

$$\min_{\tau\in[k,K]}\delta(\tau)=\begin{cases}\frac{(K-1)^2}{K} \text{ if } K<1,\\ 0 \text{ if } k\leq 1\leq K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases}$$

and

$$\max_{\tau\in[k,K]}\delta(\tau)=\begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},\frac{(K-1)^2}{K}\right\} \text{ if } k\leq 1\leq K,\\ \frac{(K-1)^2}{K} \text{ if } 1<k.\end{cases}$$

Since

$$\min\{k,1\}\min_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}=\begin{cases}\frac{k(K-1)^2}{K} \text{ if } K<1,\\ 0 \text{ if } k\leq 1\leq K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases}$$

and

$$\max\{K,1\}\max_{\tau\in[k,K]}\frac{(\tau-1)^2}{\tau}=\begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{K(k-1)^2}{k},(K-1)^2\right\} \text{ if } k\leq 1\leq K,\\ (K-1)^2 \text{ if } 1<k,\end{cases}$$

then by (3.7) we have

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d(k,K)&\leq\frac{1}{2}\nu(1-\nu)\min\{k,1\}\left(z+z^{-1}-2\right)\\&\leq 1-\nu+\nu z-z^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)\max\{K,1\}\left(z+z^{-1}-2\right)\\&\leq\frac{1}{2}\nu(1-\nu)D(k,K),\end{aligned}\tag{3.8}$$

for any $z\in[k,K]$ and for any $\nu\in[0,1]$.

Let $u\in A$ with spectrum $\sigma(u)\subset[k,K]\subset(0,\infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\operatorname{Re} z>0\}$ involved in the inequality (3.8) we conclude that we have in the order of A that

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)d(k,K)&\leq\frac{1}{2}\nu(1-\nu)\min\{k,1\}\left(u+u^{-1}-2\right)\\&\leq 1-\nu+\nu u-u^{\nu}\\&\leq\frac{1}{2}\nu(1-\nu)\max\{K,1\}\left(u+u^{-1}-2\right)\\&\leq\frac{1}{2}\nu(1-\nu)D(k,K),\end{aligned}\tag{3.9}$$

for any $\nu\in[0,1]$.

If $x,\ y\in\operatorname{Inv}(A)$ satisfy the condition (2.2) then, by (2.3), the element $u=\left|yx^{-1}\right|^2\in\operatorname{Inv}(A)$ and $\sigma(u)\subset\left[m^2,M^2\right]\subset(0,\infty)$.

By (3.9) we then have

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\,d\left(m^2,M^2\right) \\
&\le \frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(\left|yx^{-1}\right|^2+\left(\left|yx^{-1}\right|^2\right)^{-1}-2\right) \\
&\le 1-\nu+\nu\left|yx^{-1}\right|^2-\left(\left|yx^{-1}\right|^2\right)^{\nu} \\
&\le \frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(\left|yx^{-1}\right|^2+\left(\left|yx^{-1}\right|^2\right)^{-1}-2\right) \\
&\le \frac{1}{2}\nu(1-\nu)\,D\left(m^2,M^2\right),
\end{aligned} \tag{3.10}$$

for any $\nu\in[0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\,d\left(m^2,M^2\right)|x|^2 \\
&\le \frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(x^*\left|yx^{-1}\right|^2x+x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x-2|x|^2\right) \\
&\le (1-\nu)|x|^2+\nu x^*\left|yx^{-1}\right|^2x-x^*\left(\left|yx^{-1}\right|^2\right)^{\nu}x \\
&\le \frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(x^*\left|yx^{-1}\right|^2x+x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x-2|x|^2\right) \\
&\le \frac{1}{2}\nu(1-\nu)\,D\left(m^2,M^2\right)|x|^2,
\end{aligned} \tag{3.11}$$

for any $\nu\in[0,1]$.

Since

$$x^*\left|yx^{-1}\right|^2x=|y|^2,\ x^*\left(\left|yx^{-1}\right|^2\right)^{\nu}x=x\text{Ⓢ}_{\nu}y$$

and

$$\begin{aligned}
x^*\left(\left|yx^{-1}\right|^2\right)^{-1}x &= x^*\left((x^*)^{-1}y^*yx^{-1}\right)^{-1}x=x^*\left(xy^{-1}(y^*)^{-1}x^*\right)x \\
&= x^*xy^{-1}(y^*)^{-1}x^*x=|x|^2|y|^{-2}|x|^2,
\end{aligned}$$

then by (3.11) we get

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\,d\left(m^2,M^2\right)|x|^2 \\
&\le \frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left(|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2\right) \\
&\le |x|^2\nabla_{\nu}|y|^2-x\text{Ⓢ}_{\nu}y \\
&\le \frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left(|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2\right) \\
&\le \frac{1}{2}\nu(1-\nu)\,D\left(m^2,M^2\right)|x|^2.
\end{aligned} \tag{3.12}$$

Observe that

$$
\begin{aligned}
|y|^2+|x|^2|y|^{-2}|x|^2-2|x|^2 &= \left(|y|^2-|x|^2\right)\left(1-|y|^{-2}|x|^2\right)\\
&= \left(|y|^2-|x|^2\right)|y|^{-2}\left(|y|^2-|x|^2\right)\\
&= \left||y|^{-1}\left(|y|^2-|x|^2\right)\right|^2
\end{aligned}
$$

and by (3.12) we get the desired result (3.4). □

Corollary 5. *Let A be a unital C^*-algebra. Assume that $x,\ y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (2.2) holds, then we have*

$$
\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\,d\left(m^2,M^2\right)\|x\|^2\\
&\leq \frac{1}{2}\nu(1-\nu)\min\left\{m^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \left\||x|^2\nabla_\nu|y|^2-x\circledS_\nu y\right\|\\
&\leq \frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \frac{1}{2}\nu(1-\nu)\,D\left(m^2,M^2\right)\|x\|^2
\end{aligned}
\tag{3.13}
$$

for any $\nu\in[0,1]$.

In particular, we have

$$
\begin{aligned}
\frac{1}{8}d\left(m^2,M^2\right)\|x\|^2 &\leq \frac{1}{8}\min\left\{m^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \left\||x|^2\nabla|y|^2-x\circledS y\right\|\\
&\leq \frac{1}{8}\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \frac{1}{8}D\left(m^2,M^2\right)\|x\|^2.
\end{aligned}
\tag{3.14}
$$

Remark 3. *We also have the following reverse of the second inequality in (1.9)*

$$
\begin{aligned}
&\left\|(1-\nu)|x|^2+\nu|y|^2\right\|\\
&\leq \left\|\left|yx^{-1}\right|^\nu x\right\|^2+\frac{1}{2}\nu(1-\nu)\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \left\|\left|yx^{-1}\right|^\nu x\right\|^2+\frac{1}{2}\nu(1-\nu)\,D\left(m^2,M^2\right)\|x\|^2
\end{aligned}
\tag{3.15}
$$

provided that x, y and ν are as in Corollary 3.

In particular,

$$
\begin{aligned}
\frac{1}{2}\left\||x|^2+|y|^2\right\| &\leq \left\|\left|yx^{-1}\right|^{1/2}x\right\|^2+\frac{1}{8}\max\left\{M^2,1\right\}\left\||y|^{-1}\left(|y|^2-|x|^2\right)\right\|^2\\
&\leq \left\|\left|yx^{-1}\right|^{1/2}x\right\|^2+\frac{1}{8}D\left(m^2,M^2\right)\|x\|^2.
\end{aligned}
\tag{3.16}
$$

Corollary 6. *With the assumptions of Corollary 4 we have*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\,d(k,K)\,a &\leq \frac{1}{2}\nu(1-\nu)\min\{k,1\}\left|b^{-1/2}(b-a)\right|^2 \\ &\leq a\nabla_\nu b - a\sharp_\nu b \\ &\leq \frac{1}{2}\nu(1-\nu)\max\{K,1\}\left|b^{-1/2}(b-a)\right|^2 \\ &\leq \frac{1}{2}\nu(1-\nu)\,D(k,K)\,a \end{aligned}\tag{3.17}$$

for any $\nu\in[0,1]$, *where* $d(k,K)$ *and* $D(k,K)$ *are given by (3.2) and (3.3).*

In particular,

$$\begin{aligned}\frac{1}{8}d(k,K)\,a &\leq \frac{1}{8}\min\{k,1\}\left|b^{-1/2}(b-a)\right|^2 \\ &\leq a\nabla b - a\sharp b \leq \frac{1}{8}\max\{K,1\}\left|b^{-1/2}(b-a)\right|^2 \\ &\leq \frac{1}{8}D(k,K)\,a. \end{aligned}\tag{3.18}$$

For an interval $[k,K]$, define the coefficients

$$f(k,K):=\begin{cases}(K-1)^2 \text{ if } K<1,\\ 0 \text{ if } k\leq 1\leq K,\\ \frac{(k-1)^2}{k} \text{ if } 1<k\end{cases}\tag{3.19}$$

and

$$F(k,K):=\begin{cases}\frac{(k-1)^2}{k} \text{ if } K<1,\\ \max\left\{\frac{(k-1)^2}{k},(K-1)^2\right\} \text{ if } k\leq 1\leq K,\\ (K-1)^2 \text{ if } 1<k.\end{cases}\tag{3.20}$$

Theorem 5. *Assume that* $x,\ y\in\operatorname{Inv}(A)$ *and the constants* $M>m>0$ *are such that (2.2) is true. Then we have the inequalities*

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)\,f\left(m^2,M^2\right)|x|^2 &\leq |x|^2\nabla_\nu|y|^2 - x\circledS_\nu y \\ &\leq \frac{1}{2}\nu(1-\nu)\,F\left(m^2,M^2\right)|x|^2\end{aligned}\tag{3.21}$$

for any $\nu\in[0,1]$, *where* $f(\cdot,\cdot)$ *and* $F(\cdot,\cdot)$ *are defined in (3.19) and (3.20).*

In particular, we have

$$\frac{1}{8}f\left(m^2,M^2\right)|x|^2 \leq |x|^2\nabla|y|^2 - x\circledS y \leq \frac{1}{8}F\left(m^2,M^2\right)|x|^2.\tag{3.22}$$

Proof. From (2.9) we get

$$\frac{1}{2}\nu(1-\nu)\,\psi(\tau)\leq 1-\nu+\nu\tau-\tau^\nu\leq\frac{1}{2}\nu(1-\nu)\,\Psi(\tau)\tag{3.23}$$

for any $\tau>0$ and for any $\nu\in[0,1]$, where $\psi(\tau):=\frac{(\tau-1)^2}{\max\{\tau,1\}}$ and $\Psi(\tau):=\frac{(\tau-1)^2}{\min\{\tau,1\}}$.

Observe that

$$\psi(\tau)=\begin{cases}(\tau-1)^2 \text{ if } \tau\in(0,1),\\ \\ \frac{(\tau-1)^2}{\tau} \text{ if } \tau\in[1,\infty)\end{cases}$$

and

$$\Psi(\tau) = \begin{cases} \frac{(\tau-1)^2}{\tau} \text{ if } \tau \in (0,1), \\ \\ (\tau-1)^2 \text{ if } \tau \in [1,\infty). \end{cases}$$

We observe that the functions ψ and Ψ are strictly decreasing on $(0,1)$ and strictly increasing on $[1,\infty)$ with $\psi(1) = \Psi(1) = 0$.

If we consider all possible locations of the interval $[k,K]$ and the number 1 then we get

$$\min_{\tau\in[k,K]} \psi(\tau) = \begin{cases} \psi(K) \text{ if } K < 1, \\ 0 \text{ if } k \le 1 \le K, \\ \psi(k) \text{ if } 1 < k \end{cases} = f(k,K)$$

and

$$\max_{\tau\in[k,K]} \Psi(\tau) = \begin{cases} \Psi(k) \text{ if } K < 1, \\ \max\{\Psi(k), \Psi(K)\} \text{ if } k \le 1 \le K, \\ \Psi(K) \text{ if } 1 < k \end{cases} = F(k,K),$$

then by (3.23) we get

$$\frac{1}{2}\nu(1-\nu) f(k,K) \le 1-\nu+\nu\tau-\tau^{\nu} \le \frac{1}{2}\nu(1-\nu) F(k,K) \tag{3.24}$$

for any $\tau \in [k,K]$ and for any $\nu \in [0,1]$.

By making use of a similar argument as in the proof of Theorem 4 we deduce the desired result (3.21). □

Remark 4. *For $0 < k \le 1 \le K$ we have from (2.6), (3.3) and (3.20) that*

$$C(k,K) = \frac{1}{k}\max\left\{(k-1)^2, (K-1)^2\right\},$$

$$D(k,K) = \max\left\{\frac{K(k-1)^2}{k}, (K-1)^2\right\}$$

and

$$F(k,K) = \max\left\{\frac{(k-1)^2}{k}, (K-1)^2\right\}.$$

We observe that

$$F(k,K) \le C(k,K),\ D(k,K)$$

for $0 < k \le 1 \le K$, which means that the upper bound for the difference $|x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y$ provided by (3.21) is better than the corresponding upper bounds from (2.7) and (3.4).

Corollary 7. *With the assumptions of Corollary 5 we have*

$$\begin{aligned} \frac{1}{2}\nu(1-\nu) f\left(m^2,M^2\right)\|x\|^2 &\le \left\| |x|^2 \nabla_\nu |y|^2 - x \circledS_\nu y \right\| \\ &\le \frac{1}{2}\nu(1-\nu) F\left(m^2,M^2\right)\|x\|^2 \end{aligned} \tag{3.25}$$

for any $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8} f\left(m^2,M^2\right)\|x\|^2 \le \left\| |x|^2 \nabla |y|^2 - x \circledS y \right\| \le \frac{1}{8} F\left(m^2,M^2\right)\|x\|^2. \tag{3.26}$$

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