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MULTIPLICATIVE INEQUALITIES FOR WEIGHTED GEOMETRIC MEAN IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Consider the *quadratic weighted geometric mean*

$$x \circledS_\nu y := \left|\left|yx^{-1}\right|^\nu x\right|^2$$

for invertible elements x, y in a *Hermitian unital Banach $*$-algebra* and real number ν. In this paper, by utilizing some results of Tominaga, Furuichi, Liao-Wu-Zhao, Zuo-Shi-Fujii and the author, we obtain various upper and lower bounds for the positive element $(1-\nu)|x|^2 + \nu|y|^2$ in terms of $x \circledS_\nu y$, where $\nu \in [0,1]$, under various assumptions for the elements x, y involved. Applications for the classical *weighted geometric mean*

$$a \sharp_\nu b := a^{1/2}\left(a^{-1/2} b a^{-1/2}\right)^\upsilon a^{1/2}$$

of positive elements a, b that satisfy the condition $0 < ka \le b \le Ka$ for certain numbers $0 < k < K$, are also given.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \ge 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that a is *positive* and write $a > 0$ if $a \ge 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \ge b$ means that $a - b \ge 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [19] (see also [2, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \ge 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [16], Tanahashi and Uchiyama [21] proved the following fundamental properties (see also [12]):

(i) If $a, b \in A$, then $a \ge 0$, $b \ge 0$ imply $a + b \ge 0$ and $\alpha \ge 0$ implies $\alpha a \ge 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \ge 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \ge b > 0$ or $a > b \ge 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \le a$ if and only if $cbc \le cac$;

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(vi) If $0<a<1$, then $1<a^{-1}$;

(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\leq a$, then $0<a^{-1}\leq b^{-1}$.

Okayasu [16] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b\in A$ and $p\in[0,1]$ then $a>b$ $(a\geq b)$ implies that $a^p>b^p$ $(a^p\geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re}z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}\,dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^\alpha:=\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a)^{-1}\,dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha\in A$. Moreover, since z^α is analytic in $\{\operatorname{Re}z>0\}$, then by (SMT) we have

$$\sigma(a^\alpha)=(\sigma(a))^\alpha=\{z^\alpha:z\in\sigma(a)\}\subset(0,\infty).$$

Following [12], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^\alpha\in A$ with $a^\alpha>0$ and $\left(a^2\right)^{1/2}=a$, [21, Lemma 6];

(ix) If $0<a\in A$ and α, $\beta\in\mathbb{R}$, then $a^\alpha a^\beta=a^{\alpha+\beta}$;

(x) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $(a^\alpha)^{-1}=\left(a^{-1}\right)^\alpha=a^{-\alpha}$;

(xi) If $0<a$, $b\in A$, α, $\beta\in\mathbb{R}$ and $ab=ba$, then $a^\alpha b^\beta=b^\beta a^\alpha$.

We define the following means for $\nu\in[0,1]$, see also [12] for different notations:

$$a\nabla_\nu b:=(1-\nu)a+\nu b,\ a,\ b\in A \tag{A}$$

the *weighted arithmetic mean* of (a,b),

$$a!_\nu b:=\left((1-\nu)a^{-1}+\nu b^{-1}\right)^{-1},\ a,\ b>0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a,b) and

$$a\sharp_\nu b:=a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^\upsilon a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a,b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu=\frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [12], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;
(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)\,c = (c^*ac)!\,(c^*bc) \text{ and } c^*(a\sharp b)\,c = (c^*ac)\,\sharp\,(c^*bc)\,;$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = (a\sharp_\nu b)^{-1}\,.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [10] the following inequality between the weighted means introduced above:

$$a\nabla_\nu b \geq a\sharp_\nu b \geq a!_\nu b \tag{HGA}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

In [21], Tanahashi and Uchiyama obtained the following identity of interest:

Lemma 1. *If $0 < c,\ d$ and λ is a real number, then*

$$(dcd)^\lambda = dc^{1/2}\left(c^{1/2}d^2c^{1/2}\right)^{\lambda-1}c^{1/2}d. \tag{1.1}$$

Using this equality we proved the following fact [8]:

Proposition 1. *For any $0 < a,\ b \in A$ we have*

$$b\sharp_{1-\nu}a = a\sharp_\nu b$$

for any real number ν.

In [8] we introduced the *quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$, as the positive element denoted by $x \circledS_\nu y$ and defined by

$$x \circledS_\nu y := x^*\left((x^*)^{-1}y^*yx^{-1}\right)^\nu x = x^*\left|yx^{-1}\right|^{2\nu}x = \left|\left|yx^{-1}\right|^\nu x\right|^2. \tag{S}$$

When $\nu = 1/2$, we denote $x \circledS_{1/2} y$ by $x \circledS y$ and we have

$$x \circledS y = x^*\left((x^*)^{-1}y^*yx^{-1}\right)^{1/2} x = x^*\left|yx^{-1}\right|x = \left|\left|yx^{-1}\right|^{1/2} x\right|^2.$$

We can also introduce the $1/2$-*quadratic weighted mean* of (x,y) with $x,\ y \in \operatorname{Inv}(A)$ and the real weight $\nu \in \mathbb{R}$ by

$$x \circledS_\nu^{1/2} y := (x \circledS_\nu y)^{1/2} = \left|\left|yx^{-1}\right|^\nu x\right|. \tag{1/2-S}$$

Correspondingly, when $\nu = 1/2$ we denote $x \circledS^{1/2} y$ and we have

$$x \circledS^{1/2} y = \left|\left|yx^{-1}\right|^{1/2} x\right|.$$

The following equalities hold [8]:

Proposition 2. *For any $x,\ y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$ we have*

$$(x \circledS_\nu y)^{-1} = (x^*)^{-1} \circledS_\nu (y^*)^{-1}$$

and

$$\left(x^{-1}\right) \circledS_\nu \left(y^{-1}\right) = (x^* \circledS_\nu y^*)^{-1}\,.$$

If we take in (S) $x = a^{1/2}$ and $y = b^{1/2}$ with a, $b > 0$ then we get

$$a^{1/2}Ⓢ_{\nu}b^{1/2} = a\sharp_{\nu}b$$

for any $\nu \in \mathbb{R}$ that shows that the quadratic weighted mean can be seen as an extension of the weighted geometric mean for positive elements considered in the introduction.

Let x, $y \in \operatorname{Inv}(A)$. If we take in the definition of "$\sharp_{\nu}$" the elements $a = |x|^2 > 0$ and $b = |y|^2 > 0$ we also have for real ν

$$|x|^2 \sharp_{\nu} |y|^2 = |x| \left(|x|^{-1} |y|^2 |x|^{-1}\right)^{\upsilon} |x| = |x| \left||y| |x|^{-1}\right|^{2\upsilon} |x| = \left|\left||y| |x|^{-1}\right|^{\upsilon} |x|\right|^2.$$

It is then natural to ask how the positive elements $xⓈ_{\nu}y$ and $|x|^2 \sharp_{\nu} |y|^2$ do compare, when x, $y \in \operatorname{Inv}(A)$ and $\nu \in \mathbb{R}$?

In [8] we proved the following lemma that provides a slight generalization of Lemma 1.

Lemma 2. *If $0 < c$, $d \in \operatorname{Inv}(A)$ and λ is a real number, then*

$$(dcd^*)^{\lambda} = dc^{1/2} \left(c^{1/2} |d|^2 c^{1/2}\right)^{\lambda-1} c^{1/2}d^*.$$

Remark 1. *The identity (2.18) was proved by. T. Furuta in* [11] *for positive operator c and invertible operator d in the Banach algebra of all bonded linear operators on a Hilbert space by using the polar decomposition of the invertible operator $dc^{1/2}$.*

The following fundamental fact that connects the quadratic weighted geometric mean $Ⓢ_{\nu}$ to the weighted geometric mean $\sharp_{\nu}$ holds [8]:

Theorem 1. *If x, $y \in \operatorname{Inv}(A)$ and λ is a real number, then*

$$xⓈ_{\nu}y = |x|^2 \sharp_{\nu} |y|^2$$

Now, assume that $f(z)$ is analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0, \infty)$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

Lemma 3. *Let $f(z)$ and $g(z)$ be analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0, \infty)$ assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

We have the following inequalities between means [8]:

Theorem 2. *For any x, $y \in \operatorname{Inv}(A)$ and $\nu \in [0, 1]$ we have*

$$|x|^2 \nabla_{\nu} |y|^2 \geq xⓈ_{\nu}y \geq |x|^2 !_{\nu} |y|^2 . \tag{1.2}$$

In particular,

$$|x|^2 \nabla |y|^2 \geq xⓈy \geq |x|^2 ! |y|^2 . \tag{1.3}$$

We can define the weighted means for $\nu \in [0,1]$ and the elements x, $y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ by

$$x\nabla_{\nu}^{1/2}y := \left(|x|^2 \nabla_{\nu} |y|^2\right)^{1/2} = \left((1-\nu)|x|^2 + \nu |y|^2\right)^{1/2}$$

and

$$x!_{\nu}^{1/2}y := \left(|x|^2 !_{\nu} |y|^2\right)^{1/2} = \left((1-\nu)|x|^{-2} + \nu |y|^{-2}\right)^{-1/2}.$$

For $\nu = 1/2$ we consider

$$x\nabla^{1/2}y := \left(|x|^2 \nabla |y|^2\right)^{1/2} = \frac{\sqrt{2}}{2}\left(|x|^2 + |y|^2\right)^{1/2}$$

and

$$x!^{1/2}y := \left(|x|^2 ! |y|^2\right)^{1/2} = \sqrt{2}\left(|x|^{-2} + |y|^{-2}\right)^{-1/2}.$$

We have [8]:

Corollary 1. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Then for any x, $y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$x\nabla_{\nu}^{1/2}y \geq x \circledS_{\nu}^{1/2} y \geq x!_{\nu}^{1/2}y. \tag{1.4}$$

In particular, we have

$$x\nabla^{1/2}y \geq x \circledS^{1/2} y \geq x!^{1/2}y. \tag{1.5}$$

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^*a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [15, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \|b\| \geq \|a\|.$$

Corollary 2. *Let A be a unital C^*-algebra. Then for any x, $y \in \operatorname{Inv}(A)$ and $\nu \in [0,1]$ we have*

$$(1-\nu)\|x\|^2 + \nu\|y\|^2 \geq \left\|(1-\nu)|x|^2 + \nu|y|^2\right\| \geq \left\|\left|yx^{-1}\right|^{\nu} x\right\|^2. \tag{1.6}$$

In particular,

$$\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) \geq \frac{1}{2}\left\||x|^2 + |y|^2\right\| \geq \left\|\left|yx^{-1}\right|^{1/2} x\right\|^2. \tag{1.7}$$

Motivated by the above facts, in this paper we obtain various upper and lower bounds for the positive element $(1-\nu)|x|^2 + \nu|y|^2$ in terms of the quadratic mean $x\circledS_{\nu}y$, namely, inequalities of the form

$$\delta x\circledS_{\nu}y \leq (1-\nu)|x|^2 + \nu|y|^2 \leq \Delta x\circledS_{\nu}y,$$

where $\nu \in [0,1]$ and the numbers $1 \leq \delta < \Delta < \infty$, under various assumptions for the elements involved. Applications for the classical geometric mean $a\sharp_{\nu}b := a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon} a^{1/2}$ of positive elements a, b that satisfy the condition $0 < ka \leq b \leq Ka$ for certain numbers $0 < k < K$, are also given.

2. Some Preliminary Facts

Jensen's inequality for convex function is one of the most known and extensively used inequality in various filed of Modern Mathematics. It is a source of many classical inequalities including the generalized triangle inequality, the arithmetic mean-geometric mean-harmonic mean inequality, the positivity of *relative entropy* in Information Theory, Schannon's inequality, Ky Fan's inequality, Levinson's inequality and other results. For classical and contemporary developments related to the Jensen inequality, see [3], [14], [18] and [9] where further references are provided.

To be more specific, we recall that, if X is a linear space and $C \subseteq X$ a convex subset in X, then for any convex function $f : C \to \mathbb{R}$ and any $z_i \in C$, $r_i \geq 0$ for $i \in \{1, ..., k\}$, $k \geq 2$ with $\sum_{i=1}^{k} r_i = R_k > 0$ one has the *weighted Jensen's inequality:*

$$\frac{1}{R_k}\sum_{i=1}^{k} r_i f(z_i) \geq f\left(\frac{1}{R_k}\sum_{i=1}^{k} r_i z_i\right). \tag{J}$$

If $f : C \to \mathbb{R}$ is strictly convex and $r_i > 0$ for $i \in \{1, ..., k\}$ then the equality case holds in (J) if and only if $z_1 = ... = z_n$.

By $\mathcal{P}_n$ we denote the set of all nonnegative n-tuples $(p_1, ..., p_n)$ with the property that $\sum_{i=1}^{n} p_i = 1$. Consider the *normalised Jensen functional*

$$\mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq 0,$$

where $f : C \to \mathbb{R}$ is a convex function on the convex set C and $\mathbf{x} = (x_1, ..., x_n) \in C^n$ and $\mathbf{p} \in \mathcal{P}_n$.

The following result holds [5]:

Lemma 4. *If* $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n$, $q_i > 0$ *for each* $i \in \{1, ..., n\}$ *then*

$$(0 \leq) \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_n(f, \mathbf{x}, \mathbf{q}) \leq \mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) \leq \max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_n(f, \mathbf{x}, \mathbf{q}). \tag{2.1}$$

In the case $n = 2$, if we put $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ then by (2.1) we get

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q) f(x) + qf(y) - f((1-q) x + qy)] \\ &\leq (1-p) f(x) + pf(y) - f((1-p) x + py) \\ &\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q) f(x) + qf(y) - f((1-q) x + qy)] \end{aligned} \tag{2.2}$$

for any x, $y \in C$.

If we take $q = \frac{1}{2}$ in (2.2), then we get

$$\begin{aligned} &2\min\{t, 1-t\}\left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ &\leq (1-t) f(x) + tf(y) - f((1-t) x + ty) \\ &\leq 2\max\{t, 1-t\}\left[\frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.3}$$

for any x, $y \in C$ and $t \in [0, 1]$.

If we take in (2.2) $f(x) = -\ln x$, then we get

$$\left(\frac{A_q(\alpha,\beta)}{G_q(\alpha,\beta)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \le \frac{A_p(\alpha,\beta)}{G_p(\alpha,\beta)} \le \left(\frac{A_q(\alpha,\beta)}{G_q(\alpha,\beta)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{2.4}$$

for any α, $\beta > 0$ and for any $p \in [0,1]$, $q \in (0,1)$.

This inequality is a particular case for $n = 2$ of the inequality (4.2) from [5].

For $q = \frac{1}{2}$ we have by (2.4) that

$$\left(\frac{A(\alpha,\beta)}{G(\alpha,\beta)}\right)^{2\min\{p,1-p\}} \le \frac{A_p(\alpha,\beta)}{G_p(\alpha,\beta)} \le \left(\frac{A(\alpha,\beta)}{G(\alpha,\beta)}\right)^{2\max\{p,1-p\}} \tag{2.5}$$

for any α, $\beta > 0$ and for any $p \in [0,1]$.

Recall that *Kantorovich's constant* $\mathcal{K}$ is defined by

$$\mathcal{K}(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{K}$$

It is well known that $\mathcal{K}$ is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$, $\mathcal{K}(h) \ge 1$ for any $h > 0$ and $\mathcal{K}(h) = \mathcal{K}\left(\frac{1}{h}\right)$ for any $h > 0$.

The inequality (2.5) can be thus written as

$$\mathcal{K}^{\min\{p,1-p\}}\left(\frac{\alpha}{\beta}\right) \le \frac{A_p(\alpha,\beta)}{G_p(\alpha,\beta)} \le \mathcal{K}^{\max\{p,1-p\}}\left(\frac{\alpha}{\beta}\right). \tag{ZL}$$

The first inequality in (ZL) was obtained by Zou et al. in [23] while the second by Liao et al. [13].

For $q \in (0,1)$ we consider the function $f_q : (0,\infty) \to (0,\infty)$ defined by

$$f_q(h) := \frac{A_q(1,h)}{G_q(1,h)} = \frac{1-q+qh}{h^q} = (1-q)h^{-q} + qh^{1-q}.$$

The function f_q is differentiable and

$$f_q'(h) = (1-q)qh^{-q-1}(h-1),$$

which shows that the function f_q is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$. We have $f_q(1) = 1$, $\lim_{h\to 0+} f_q(h) = +\infty$, $\lim_{h\to\infty} f_q(h) = +\infty$ and $f_q\left(\frac{1}{h}\right) = f_{1-q}(h)$ for any $h > 0$ and $q \in (0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[\ell, L] \subset (0,\infty)$ and the number 1 we get

$$\max_{h\in[\ell,L]} f_q(h) = \begin{cases} f_q(\ell) \text{ if } L < 1, \\ \\ \max\{f_q(\ell), f_q(L)\} \text{ if } \ell \le 1 \le L, \\ \\ f_q(L) \text{ if } 1 < \ell, \end{cases} \tag{2.6}$$

$$= \begin{cases} \frac{A_q(1,\ell)}{G_q(1,\ell)} \text{ if } L < 1, \\ \\ \max\left\{\frac{A_q(1,\ell)}{G_q(1,\ell)}, \frac{A_q(1,L)}{G_q(1,L)}\right\} \text{ if } \ell \le 1 \le L, \\ \\ \frac{A_q(1,L)}{G_q(1,L)} \text{ if } 1 < \ell \end{cases}$$

and

$$\min_{h\in[\ell,L]} f_q(h) = \begin{cases} f_q(L) \text{ if } L<1, \\ \\ 1 \text{ if } \ell \le 1 \le L \\ \\ f_q(\ell) \text{ if } 1<\ell, \end{cases} = \begin{cases} \frac{A_q(1,L)}{G_q(1,L)} \text{ if } L<1, \\ \\ 1 \text{ if } \ell \le 1 \le L, \\ \\ \frac{A_q(1,\ell)}{G_q(1,\ell)} \text{ if } 1<\ell. \end{cases} \tag{2.7}$$

We then have the following fact:

Lemma 5. *For any $p\in[0,1]$, $q\in(0,1)$ and $h\in[\ell,L]\subset(0,\infty)$ we have the bounds*

$$\gamma_{p,q}(\ell,L) \le \frac{A_p(1,h)}{G_p(1,h)} \le \Gamma_{p,q}(\ell,L)$$

where

$$\Gamma_{p,q}(\ell,L) := \begin{cases} \left(\frac{A_q(1,\ell)}{G_q(1,\ell)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \textit{ if } L<1, \\ \\ \max\left\{\left(\frac{A_q(1,\ell)}{G_q(1,\ell)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}, \left(\frac{A_q(1,L)}{G_q(1,L)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}\right\} \\ \textit{if } \ell\le 1\le L, \\ \\ \left(\frac{A_q(1,L)}{G_q(1,L)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \textit{ if } 1<\ell \end{cases} \tag{2.8}$$

and

$$\gamma_{p,q}(\ell,L) := \begin{cases} \left(\frac{A_q(1,L)}{G_q(1,L)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \textit{ if } L<1, \\ \\ 1 \textit{ if } \ell\le 1\le L, \\ \\ \left(\frac{A_q(1,\ell)}{G_q(1,\ell)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \textit{ if } 1<\ell. \end{cases} \tag{2.9}$$

We observe that for $q=1/2$, we get, see also (ZL), that

$$\gamma_{p,1/2}(\ell,L) \le \frac{A_p(1,h)}{G_p(1,h)} \le \Gamma_{p,1/2}(\ell,L), \tag{2.10}$$

for $p\in[0,1]$ and $h\in[\ell,L]$, where

$$\Gamma_{p,1/2}(\ell,L) = \begin{cases} \mathcal{K}^{\max\{p,1-p\}}(\ell) \text{ if } L<1, \\ \\ \max\left\{\mathcal{K}^{\max\{p,1-p\}}(\ell), \mathcal{K}^{\max\{p,1-p\}}(L)\right\} \\ \text{if } \ell\le 1\le L, \\ \\ \mathcal{K}^{\max\{p,1-p\}}(L) \text{ if } 1<\ell \end{cases} \tag{2.11}$$

and

$$\gamma_{p,1/2}(\ell,L) := \begin{cases} \mathcal{K}^{\min\{p,1-p\}}(L) \text{ if } L<1, \\ \\ 1 \text{ if } \ell\le 1\le L, \\ \\ \mathcal{K}^{\min\{p,1-p\}}(\ell) \text{ if } 1<\ell, \end{cases} \tag{2.12}$$

where $\mathcal{K}$ is Kantorovich's constant.

By taking $q = p \in (0,1)$ in Lemma 5 we get

$$\gamma_p(\ell, L) \le \frac{A_p(1,h)}{G_p(1,h)} \le \Gamma_p(\ell, L) \tag{2.13}$$

where

$$\Gamma_p(\ell, L) := \begin{cases} \frac{A_p(1,\ell)}{G_p(1,\ell)} \text{ if } L < 1, \\ \\ \max\left\{\frac{A_p(1,\ell)}{G_p(1,\ell)}, \frac{A_p(1,L)}{G_p(1,L)}\right\} \\ \text{if } \ell \le 1 \le L, \\ \\ \frac{A_p(1,L)}{G_p(1,L)} \text{ if } 1 < \ell \end{cases} \tag{2.14}$$

and

$$\gamma_p(\ell, L) := \begin{cases} \frac{A_p(1,L)}{G_p(1,L)} \text{ if } L < 1, \\ \\ 1 \text{ if } \ell \le 1 \le L, \\ \\ \frac{A_p(1,\ell)}{G_p(1,\ell)} \text{ if } 1 < \ell. \end{cases} \tag{2.15}$$

We recall that *Specht's ratio* is defined by

$$\mathcal{S}(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{2.16}$$

It is well known that $\lim_{h\to 1} \mathcal{S}(h) = 1$, $\mathcal{S}(h) = \mathcal{S}\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is *strictly decreasing* on $(0,1)$ and *strictly increasing* on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's

$$\mathcal{S}\left(\left(\frac{\alpha}{\beta}\right)^r\right)\alpha^{1-\nu}\beta^{\nu} \le (1-\nu)\alpha + \nu\beta \le \mathcal{S}\left(\frac{\alpha}{\beta}\right)\alpha^{1-\nu}\beta^{\nu}, \tag{2.17}$$

where α, $\beta > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (2.17) is due to Tominaga [22] while the first one is due to Furuichi [12].

We then have the inequalities

$$\mathcal{S}(h^r)\,h^{\nu} \le 1 - \nu + \nu h \le \mathcal{S}(h)\,h^{\nu}, \tag{2.18}$$

for any $h > 0$ and $\nu \in [0,1]$, where $r = \min\{1-\nu, \nu\}$.

In [23] the authors also showed that

$$\mathcal{K}^r(h) \ge \mathcal{S}(h^r) \text{ for } h > 0 \text{ and } r \in \left[0, \frac{1}{2}\right]$$

implying that the lower bound in (ZL) is better than the lower bound from (2.17).

Using the properties of the function $\mathcal{S}$ we can conclude that

$$\sigma_p(\ell, L) \le \frac{A_p(1,h)}{G_p(1,h)} \le \Sigma(\ell, L) \tag{2.19}$$

for $p \in [0,1]$ and $h \in [\ell, L]$, where

$$\Sigma(\ell, L) := \max_{h \in [\ell, L]} \mathcal{S}(h) = \begin{cases} \mathcal{S}(\ell) \text{ if } L < 1, \\ \max\{\mathcal{S}(\ell), \mathcal{S}(L)\} \text{ if } \ell \le 1 \le L, \\ \mathcal{S}(L) \text{ if } 1 < \ell \end{cases}$$

and

$$\sigma_p(\ell, L) := \min_{h \in [\ell, L]} \mathcal{S}(h) = \begin{cases} \mathcal{S}\left(L^{\min\{p,1-p\}}\right) \text{ if } L < 1, \\ 1 \text{ if } \ell \le 1 \le L, \\ \mathcal{S}\left(\ell^{\min\{p,1-p\}}\right) \text{ if } 1 < \ell. \end{cases}$$

In [6] we obtained the following *exponential upper bound*

$$(1 \le) \frac{(1-\nu)\alpha + \nu\beta}{\alpha^{1-\nu}\beta^{\nu}} \le \exp\left[4\nu(1-\nu)\left(\mathcal{K}\left(\frac{\alpha}{\beta}\right) - 1\right)\right], \tag{2.20}$$

giving the inequality

$$\frac{A_p(1,h)}{G_p(1,h)} \le \Psi_p(\ell, L) \tag{2.21}$$

where

$$\Psi_p(\ell, L) := \max_{h \in [\ell, L]} \{\exp[4p(1-p)(\mathcal{K}(h) - 1)]\} \tag{2.22}$$

$$= \begin{cases} \exp[4p(1-p)(\mathcal{K}(\ell) - 1)] \text{ if } L < 1, \\ \\ \exp[4p(1-p)(\max\{\mathcal{K}(\ell), \mathcal{K}(L)\} - 1)] \\ \text{if } \ell \le 1 \le L, \\ \\ \exp[4p(1-p)(\mathcal{K}(L) - 1)] \text{ if } 1 < \ell. \end{cases}$$

For $p \in [0,1]$ and the interval $[\ell, L] \subset (0, \infty)$, we define the following composite coefficients

$$\Theta_p(\ell, L) := \min\left\{\Gamma_{p,1/2}(\ell, L), \Gamma_p(\ell, L), \Sigma(\ell, L), \Psi_p(\ell, L)\right\} \tag{2.23}$$

and

$$\theta_p(\ell, L) := \max\left\{\gamma_{p,1/2}(\ell, L), \gamma_p(\ell, L), \sigma_p(\ell, L)\right\}. \tag{2.24}$$

Then from (2.10), (2.13), (2.19) and (2.21) we have the double inequality

$$\theta_p(\ell, L) h^p \le 1 - p + ph \le \Theta_p(\ell, L) h^p \tag{2.25}$$

for any $p \in [0,1]$ and $h \in [\ell, L] \subset (0, \infty)$.

3. Multiplicative Inequalities for the Quadratic Geometric Mean

Assume that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that

$$M \ge \left|yx^{-1}\right| \ge m. \tag{3.1}$$

The inequality (3.1) is equivalent to

$$M^2 \ge \left|yx^{-1}\right|^2 = (x^*)^{-1} |y|^2 x^{-1} \ge m^2. \tag{3.2}$$

If we multiply at left with x^* and at right with x we get the equivalent relation

$$M^2 |x|^2 \ge |y|^2 \ge m^2 |x|^2. \tag{3.3}$$

We have:

Theorem 3. *Let A be a Hermitian unital Banach $*$-algebra. Assume that $x, y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (3.1) is true. Then we have the inequalities*

$$\theta_\nu\left(m^2, M^2\right) x \circledS_\nu y \leq |x|^2 \nabla_\nu |y|^2 \leq \Theta_\nu\left(m^2, M^2\right) x \circledS_\nu y \tag{3.4}$$

for any $\nu \in [0,1]$.

Proof. Using the inequality (2.25) we have

$$\theta_\nu(k, K) z^\nu \leq 1 - \nu + \nu z \leq \Theta_\nu(k, K) z^\nu \tag{3.5}$$

for any real $z \in [k, K] \subset (0, \infty)$ and for any $\nu \in [0,1]$, where the coefficients $\theta_\nu(k, K)$ and $\Theta_\nu(k, K)$ are defined by (2.24) and (2.23).

Let $u \in A$ with spectrum $\sigma(u) \subset [k, K] \subset (0, \infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\operatorname{Re} z > 0\}$ involved in the inequality (3.5) we conclude that we have in the order of A that

$$\theta_\nu(k, K) u^\nu \leq 1 - \nu + \nu u \leq \Theta_\nu(k, K) u^\nu. \tag{3.6}$$

If $x, y \in \operatorname{Inv}(A)$ satisfy the condition (3.1) then, by (3.2), the element $u = \left|yx^{-1}\right|^2 \in \operatorname{Inv}(A)$ and $\sigma(u) \subset \left[m^2, M^2\right] \subset (0, \infty)$.

By (3.6) we then have

$$\begin{aligned} \theta_\nu\left(m^2, M^2\right)\left(\left|yx^{-1}\right|^2\right)^\nu &\leq 1 - \nu + \nu\left|yx^{-1}\right|^2 \\ &\leq \Theta_\nu\left(m^2, M^2\right)\left(\left|yx^{-1}\right|^2\right)^\nu, \end{aligned} \tag{3.7}$$

for any $\nu \in [0,1]$.

If we multiply this inequality at left with x^* and at right with x we get

$$\begin{aligned} \theta_\nu\left(m^2, M^2\right) x^*\left(\left|yx^{-1}\right|^2\right)^\nu x &\leq (1-\nu)|x|^2 + \nu x^*\left|yx^{-1}\right|^2 x \\ &\leq \Theta_\nu\left(m^2, M^2\right) x^*\left(\left|yx^{-1}\right|^2\right)^\nu x, \end{aligned} \tag{3.8}$$

for any $\nu \in [0,1]$.

Since

$$x^*\left|yx^{-1}\right|^2 x = x^*\left(\left(x^*\right)^{-1} y^* y x^{-1}\right) x = y^* y = |y|^2,$$

and

$$x^*\left(\left|yx^{-1}\right|^2\right)^\nu x = x \circledS_\nu y$$

for $x, y \in \operatorname{Inv}(A)$, then by (3.8) we get the desired result (3.4). □

Remark 2. *For $\nu = 1/2$, let us consider*

$$\begin{aligned} \Theta\left(m^2, M^2\right) &:= \Theta_{1/2}\left(m^2, M^2\right) \\ &= \min\left\{\Gamma_{1/2,1/2}\left(m^2, M^2\right), \Gamma_{1/2}\left(m^2, M^2\right), \Sigma\left(m^2, M^2\right), \Psi_{1/2}\left(m^2, M^2\right)\right\} \end{aligned}$$

where

$$\Gamma_{1/2,1/2}\left(m^2,M^2\right) = \begin{cases} \mathcal{K}^{1/2}\left(m^2\right) \text{ if } M<1, \\ \\ \max\left\{\mathcal{K}^{1/2}\left(m^2\right),\mathcal{K}^{1/2}\left(M^2\right)\right\} \\ \text{if } m\le 1\le M, \\ \\ \mathcal{K}^{1/2}\left(M^2\right) \text{ if } 1<m \end{cases}$$
$$= \Gamma_{1/2}\left(m^2,M^2\right),$$

$$\Sigma\left(m^2,M^2\right) = \begin{cases} \mathcal{S}\left(m^2\right) \text{ if } M<1, \\ \max\left\{\mathcal{S}\left(m^2\right),\mathcal{S}\left(M^2\right)\right\} \text{ if } m\le 1\le M, \\ \mathcal{S}\left(M^2\right) \text{ if } 1<m \end{cases}$$

and

$$\Psi_{1/2}\left(m^2,M^2\right) = \begin{cases} \exp\left[\mathcal{K}\left(m^2\right)-1\right] \text{ if } M<1, \\ \\ \exp\left[\max\left\{\mathcal{K}\left(m^2\right),\mathcal{K}\left(M^2\right)\right\}-1\right] \\ \text{if } m\le 1\le M, \\ \\ \exp\left[\mathcal{K}\left(M^2\right)-1\right] \text{ if } 1<m. \end{cases}$$

Also, let us put

$$\theta\left(m^2,M^2\right) := \theta_{1/2}\left(m^2,M^2\right)$$
$$= \min\left\{\gamma_{1/2,1/2}\left(m^2,M^2\right),\gamma_{1/2}\left(m^2,M^2\right),\sigma_{1/2}\left(m^2,M^2\right)\right\}$$

where

$$\gamma_{1/2,1/2}\left(m^2,M^2\right) = \begin{cases} \mathcal{K}^{1/2}\left(M^2\right) \text{ if } M<1, \\ \\ 1 \text{ if } m\le 1\le M, \\ \\ \mathcal{K}^{1/2}\left(m^2\right) \text{ if } 1<m, \end{cases}$$
$$= \gamma_{1/2}\left(m^2,M^2\right),$$

and

$$\sigma_{1/2}\left(m^2,M^2\right) = \begin{cases} \mathcal{S}\left(M\right) \text{ if } M<1, \\ 1 \text{ if } m\le 1\le M, \\ \mathcal{S}\left(m\right) \text{ if } 1<m. \end{cases}$$

Then by (3.4) written by $\nu = 1/2$ we get the simple inequality

$$\theta\left(m^2,M^2\right) x \circledS y \le |x|^2 \nabla |y|^2 \le \Theta\left(m^2,M^2\right) x \circledS y \tag{3.9}$$

provided that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (3.1) is true.

With the notations $x\nabla_\nu^{1/2}y$, $x\nabla^{1/2}y$, $x\circledS_\nu^{1/2}y$ and $x\circledS^{1/2}y$ from the introduction, we can state:

Corollary 3. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Assume that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (3.1) is true. Then we have the inequalities*

$$\theta_\nu^{1/2}\left(m^2,M^2\right) x \circledS_\nu^{1/2} y \le x\nabla_\nu^{1/2} y \le \Theta_\nu\left(m^2,M^2\right) x\circledS_\nu^{1/2} y \tag{3.10}$$

for any $\nu \in [0,1]$.

In particular,

$$\theta^{1/2}\left(m^2, M^2\right) x \circledS^{1/2} y \leq x \nabla^{1/2} y \leq \Theta\left(m^2, M^2\right) x \circledS^{1/2} y. \tag{3.11}$$

The proof follows by Okayasu's theorem from the introduction and the inequality (3.4) in which we take the square root.

Corollary 4. *Let A be a unital C^*-algebra. Assume that x, $y \in \operatorname{Inv}(A)$ and the constants $M > m > 0$ are such that (3.1) is true. Then we have the inequalities*

$$\begin{aligned} \theta_\nu\left(m^2, M^2\right) \left\| \left|yx^{-1}\right|^\nu x \right\|^2 &\leq \left\| (1-\nu)|x|^2 + \nu |y|^2 \right\| \\ &\leq \Theta_\nu\left(m^2, M^2\right) \left\| \left|yx^{-1}\right|^\nu x \right\|^2, \end{aligned} \tag{3.12}$$

for any $\nu \in [0,1]$.

In particular, we have

$$\begin{aligned} \theta\left(m^2, M^2\right) \left\| \left|yx^{-1}\right|^{1/2} x \right\|^2 &\leq \frac{1}{2} \left\| |x|^2 + |y|^2 \right\| \\ &\leq \Theta\left(m^2, M^2\right) \left\| \left|yx^{-1}\right|^{1/2} x \right\|^2. \end{aligned} \tag{3.13}$$

We also have the following result for positive elements:

Corollary 5. *Let A be a Hermitian unital Banach $*$-algebra. If $0 < a$, $b \in A$ and $0 < k < K$ are such that*

$$ka \leq b \leq Ka, \tag{3.14}$$

then

$$\theta_\nu(k,K)\, a \sharp_\nu b \leq a \nabla_\nu b \leq \Theta_\nu(k,K)\, a \sharp_\nu b \tag{3.15}$$

for any $\nu \in [0,1]$, where $\theta_\nu(k,K)$ and $\Theta_\nu(k,K)$ are given by (2.24) and (2.23).

In particular, we have

$$\theta(k,K)\, a \sharp b \leq a \nabla b \leq \Theta(k,K)\, a \sharp b \tag{3.16}$$

where $\theta(k,K) = \theta_{1/2}(k,K)$ and $\Theta(k,K) = \Theta_{1/2}(k,K)$.

The proof follows by Theorem 3 applied for $x = a^{1/2}$, $y = b^{1/2}$, $M = \sqrt{K}$ and $m = \sqrt{k}$.

4. Related Exponential Bounds

Further on, we also have the exponential inequalities:

Lemma 6. *For any α, $\beta > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1 - \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}\right)^2\right] \\ &\leq \frac{(1-\nu)\alpha + \nu\beta}{\alpha^{1-\nu}\beta^\nu} \\ &\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{\alpha,\beta\}}{\min\{\alpha,\beta\}} - 1\right)^2\right]. \end{aligned} \tag{4.1}$$

These inequalities were obtained in current form in [7] and for $\alpha<\beta$, via a different technique, in [1].

We have:

Theorem 4. *Let A be a Hermitian unital Banach $*$-algebra. Assume that x, $y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (3.1) is true. Then we have the inequalities*

$$\begin{aligned}(4.2)\qquad &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\right)^2\right]x\circledS_\nu y\\ &\le |x|^2\nabla_\nu |y|^2\\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1\right)^2\right]x\circledS_\nu y\end{aligned}$$

any $\nu\in[0,1]$.

In particular, we have

$$\begin{aligned}(4.3)\qquad &\exp\left[\frac{1}{8}\left(1-\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\right)^2\right]x\circledS y\\ &\le |x|^2\nabla |y|^2\\ &\le \exp\left[\frac{1}{8}\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1\right)^2\right]x\circledS y.\end{aligned}$$

Proof. From the inequality (4.1) we have

$$\begin{aligned}(4.4)\qquad &\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,z\}}{\max\{1,z\}}\right)^2\right]z^\nu\\ &\le 1-\nu+\nu z\\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,z\}}{\min\{1,z\}}-1\right)^2\right]z^\nu\end{aligned}$$

for any $z>0$ and any $\nu\in[0,1]$.

If $z\in\left[m^2,M^2\right]\subset(0,\infty)$ then

$$0\le\frac{\max\{1,z\}}{\min\{1,z\}}-1\le\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1$$

and

$$0\le 1-\frac{\min\left\{1,M^2\right\}}{\max\left\{1,m^2\right\}}\le 1-\frac{\min\{1,z\}}{\max\{1,z\}},$$

which implies that

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,z\}}{\min\{1,z\}}-1\right)^2\right]\\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\left\{1,M^2\right\}}{\min\left\{1,m^2\right\}}-1\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M^2\}}{\max\{1,m^2\}}\right)^2\right]\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,z\}}{\max\{1,z\}}\right)^2\right].\end{aligned}$$

By (4.4) we then have

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M^2\}}{\max\{1,m^2\}}\right)^2\right]z^\nu\\&\leq 1-\nu+\nu z\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,M^2\}}{\min\{1,m^2\}}-1\right)^2\right]z^\nu\end{aligned}\tag{4.5}$$

for any $z\in[m^2,M^2]$ and any $\nu\in[0,1]$.

Let $u\in A$ with spectrum $\sigma(u)\subset[m^2,M^2]\subset(0,\infty)$. Then by applying Lemma 3 for the corresponding analytic functions in the right half open plane $\{\mathrm{Re}\,z>0\}$ involved in the inequality (4.5) we conclude that we have in the order of A that

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,M^2\}}{\max\{1,m^2\}}\right)^2\right]u^\nu\\&\leq 1-\nu+\nu u\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,M^2\}}{\min\{1,m^2\}}-1\right)^2\right]u^\nu\end{aligned}\tag{4.6}$$

for any $\nu\in[0,1]$.

Now, on making use of a similar argument to the one in the proof of Theorem 3 we deduce the desired result (4.2). We omit the details. □

Corollary 6. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution. Assume that x, $y\in\mathrm{Inv}(A)$ and the constants $M>m>0$ are such that (3.1) is true. Then we have the inequalities*

$$\begin{aligned}&\exp\left[\frac{1}{4}\nu(1-\nu)\left(1-\frac{\min\{1,M^2\}}{\max\{1,m^2\}}\right)^2\right]x\circledS_\nu^{1/2}y\\&\leq x\nabla_\nu^{1/2}y\\&\leq\exp\left[\frac{1}{4}\nu(1-\nu)\left(\frac{\max\{1,M^2\}}{\min\{1,m^2\}}-1\right)^2\right]x\circledS_\nu^{1/2}y\end{aligned}\tag{4.7}$$

for any $\nu\in[0,1]$.

In particular, we have

$$\exp\left[\frac{1}{16}\left(1-\frac{\min\left\{1,M^{2}\right\}}{\max\left\{1,m^{2}\right\}}\right)^{2}\right]x\circledS^{1/2}y \tag{4.8}$$
$$\leq x\nabla^{1/2}y$$
$$\leq \exp\left[\frac{1}{16}\left(\frac{\max\left\{1,M^{2}\right\}}{\min\left\{1,m^{2}\right\}}-1\right)^{2}\right]x\circledS^{1/2}y.$$

We have the norm inequalities:

Corollary 7. *Let A be a unital C^{*}-algebra. Assume that $x,\ y\in\operatorname{Inv}(A)$ and the constants $M>m>0$ are such that (3.1) is true. Then we have the inequalities*

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\left\{1,M^{2}\right\}}{\max\left\{1,m^{2}\right\}}\right)^{2}\right]\left\|\left|yx^{-1}\right|^{\nu}x\right\|^{2} \tag{4.9}$$
$$\leq\left\|(1-\nu)\left|x\right|^{2}+\nu\left|y\right|^{2}\right\|$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\left\{1,M^{2}\right\}}{\min\left\{1,m^{2}\right\}}-1\right)^{2}\right]\left\|\left|yx^{-1}\right|^{\nu}x\right\|^{2},$$

for any $\nu\in[0,1]$.

In particular, we have

$$\exp\left[\frac{1}{8}\left(1-\frac{\min\left\{1,M^{2}\right\}}{\max\left\{1,m^{2}\right\}}\right)^{2}\right]\left\|\left|yx^{-1}\right|^{1/2}x\right\|^{2}\leq\frac{1}{2}\left\|\left|x\right|^{2}+\left|y\right|^{2}\right\| \tag{4.10}$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{\max\left\{1,M^{2}\right\}}{\min\left\{1,m^{2}\right\}}-1\right)^{2}\right]\left\|\left|yx^{-1}\right|^{1/2}x\right\|^{2}.$$

We also have the following result for positive elements:

Corollary 8. *Let A be a Hermitian unital Banach $*$-algebra. If $0<a,\ b\in A$ and $0<k<K$ are such that the condition (3.14) is valid, then*

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^{2}\right]a\sharp_{\nu}b\leq a\nabla_{\nu}b \tag{4.11}$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^{2}\right]a\sharp_{\nu}b$$

for any $\nu\in[0,1]$.

In particular, we have

$$\exp\left[\frac{1}{8}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^{2}\right]a\sharp b\leq a\nabla b \tag{4.12}$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^{2}\right]a\sharp b.$$

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