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REFINEMENTS AND REVERSES FOR THE RELATIVE OPERATOR ENTROPY $S(A|B)$ WHEN $B \geq A$

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Abstract. In this paper we obtain new refinements and reverse inequalities for the relative operator entropy $S(A|B)$ of two positive invertible operators when $B \geq A$. Applications for the operator entropy $\eta(C)$ in the case of positive contractions C are also given.

1. Introduction

Kamei and Fujii [6], [7] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}, \tag{1.1}$$

which is a relative version of the *operator entropy* considered by Nakamura-Umegaki [15].

For the *entropy function* $\eta(t) = -t\ln t$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A = S(A|1_H) \geq 0$$

for positive contraction A. This shows that the relative operator entropy (1.1) is a relative version of the operator entropy

In [18], A. Uhlmann has shown that the relative operator entropy $S(A|B)$ can be represented as the strong limit

$$S(A|B) = s\text{-}\lim_{t\to 0}\frac{A\sharp_t B - A}{t}, \tag{1.2}$$

where

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},\ \nu \in [0,1]$$

is the *weighted geometric mean* of positive invertible operators A and B. For $\nu = \frac{1}{2}$ we denote $A\sharp B$.

This definition of the weighted geometric mean can be extended for any real number ν with $\nu \neq 0$.

Following [11, p. 149-p. 155], we recall some important properties of relative operator entropy for A and B positive invertible operators:

(i) We have the equalities:

$$S(A|B) = -A^{1/2}\left(\ln A^{1/2}B^{-1}A^{1/2}\right)A^{1/2} = B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}; \tag{1.3}$$

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(ii) We have the inequalities

$$S(A|B) \le A(\ln\|B\| - \ln A) \text{ and } S(A|B) \le B - A; \tag{1.4}$$

(iii) For any C, D positive invertible operators we have that

$$S(A+B|C+D) \ge S(A|C) + S(B|D);$$

(iv) If $B \le C$ then

$$S(A|B) \le S(A|C);$$

(v) If $B_n \downarrow B$ then

$$S(A|B_n) \downarrow S(A|B);$$

(vi) For $\alpha > 0$ we have

$$S(\alpha A|\alpha B) = \alpha S(A|B);$$

(vii) For every operator T we have

$$T^* S(A|B) T \le S(T^*AT|T^*BT).$$

The relative operator entropy is *jointly concave*, namely, for any positive invertible operators A, B, C, D we have

$$S(tA + (1-t)B|tC + (1-t)D) \ge tS(A|C) + (1-t)S(B|D)$$

for any $t \in [0,1]$.

For other results on the relative operator entropy see [3], [8], [12], [13], [14] and [16].

For $t > 0$ and the positive invertible operators A, B we define the *Tsallis relative operator entropy* (see also [10]) by

$$T_t(A|B) := \frac{A\sharp_t B - A}{t}.$$

We observe that, for the function

$$f(x) = \frac{1}{t}\left(1 - x^{-t}\right) = \frac{x^t - 1}{t}x^{-t},\ x > 0,$$

we have

$$\begin{aligned} A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2} &= -AT_t\left(A^{-1}|B^{-1}\right)A = T_t(A|B)\left(A^{-1}\sharp_t B^{-1}\right)A \\ &= T_t(A|B)(A\sharp_t B)^{-1} A \end{aligned}$$

for any positive invertible operators A, B and $t > 0$.

The following result providing upper and lower bounds for relative operator entropy in terms of $T_t(\cdot|\cdot)$ has been obtained in [6] for $0 < t \le 1$. However, it hods for any $t > 0$.

Theorem 1. *Let A, B be two positive invertible operators, then for any $t > 0$ we have*

$$T_t(A|B)(A\sharp_t B)^{-1} A \le S(A|B) \le T_t(A|B). \tag{1.5}$$

In particular, we have

$$A - AB^{-1}A \le S(A|B) \le B - A \text{ [6]} \tag{1.6}$$

and

$$\frac{1}{2}A\left(1_H - \left(B^{-1}A\right)^2\right) \le S(A|B) \le \frac{1}{2}\left(BA^{-1}B - A\right). \tag{1.7}$$

The case $t = \frac{1}{2}$ is of interest as well, since in this case we get from (1.5) that

$$2\left(1_H - A\left(A\sharp B\right)^{-1}\right)A \le S\left(A|B\right) \le 2\left(A\sharp B - A\right) \le B - A. \tag{1.8}$$

This inequality provides a refinement and a reverse for (1.4).

The following upper and lower bounds for the operator entropy also hold for any positive invertible operator C and any $t > 0$:

$$\frac{1}{t}C\left(1_H - C^t\right) \le \eta\left(C\right) \le \frac{1}{t}C^{1-t}\left(1 - C^t\right). \tag{1.9}$$

In particular, we have

$$C\left(1_H - C\right) \le \eta\left(C\right) \le 1_H - C, \tag{1.10}$$

$$\frac{1}{2}C\left(1_H - C^2\right) \le \eta\left(C\right) \le \frac{1}{2}\left(C^{-1} - C\right) \tag{1.11}$$

and

$$2C\left(1_H - C^{1/2}\right) \le \eta\left(C\right) \le 2C^{1/2}\left(1_H - C^{1/2}\right). \tag{1.12}$$

Motivated by the above results, in this paper we obtain new refinements and reverse inequalities for the relative operator entropy $S\left(A|B\right)$ of two positive invertible operators when $B \ge A$. Applications for the operator entropy $\eta\left(C\right)$ in the case of positive contractions C are also given.

2. Some Refinements

We start with the following sequence of scalar inequalities:

Lemma 1. *For any $y \ge 1$ we have the inequalities*

$$\begin{aligned} 0 &\le \frac{y-1}{y} \le \frac{2\left(y-1\right)}{y+1} \le \ln y \le \frac{y-1}{\sqrt{y}} \\ &\le \frac{y-1}{y+1} + \frac{y^2-1}{4y} \le \frac{y^2-1}{2y} \le y-1. \end{aligned} \tag{2.1}$$

Proof. We prove only the third, fourth and fifth inequalities, the other ones are obvious due to the fact that $y \ge 1$.

We use the first *Hermite-Hadamard inequality* for *convex functions*, namely [5]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f\left(t\right)dt, \tag{2.2}$$

where $f : [a, b] \to \mathbb{R}$ is a convex function.

If we take in (2.2) $a = 1$ and $b = y$, then we get the third inequality in (2.1).

It is known that, if $G\left(a, b\right) := \sqrt{ab}$ is the *geometric mean* of a, $b > 0$ and

$$L\left(a, b\right) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a \end{cases}$$

is the *logarithmic mean* of a, b, then

$$G\left(a, b\right) \le L\left(a, b\right). \tag{2.3}$$

Now, if we take in (2.3) $a = 1$ and $b = y$, then we get the fourth inequality in (2.1).

Further, consider the difference

$$\begin{aligned}
\frac{1}{y+1}+\frac{y+1}{4y}-\frac{1}{\sqrt{y}} &= \frac{1}{y+1}-\frac{1}{2\sqrt{y}}+\frac{y+1}{4y}-\frac{1}{2\sqrt{y}} \\
&= \frac{2\sqrt{y}-(y+1)}{2(y+1)\sqrt{y}}+\frac{y+1-2\sqrt{y}}{4y} \\
&= (\sqrt{y}-1)^2\left(\frac{y+1-2\sqrt{y}}{4y(y+1)}\right) \\
&= \frac{(\sqrt{y}-1)^4}{4y(y+1)},
\end{aligned}$$

for $y>0$, which proves the fifth inequality in (2.1). □

The following result provides an improvement of (1.5) in the case that $B \geq A$.

Theorem 2. *Let A, B be two positive invertible operators and $B \geq A$, then for any $t>0$ we have*

$$\begin{aligned}
0 &\leq T_t(A|B)(A\sharp_t B)^{-1}A \\
&\leq 2T_t(A|B)(A\sharp_t B+A)^{-1}A \\
&\leq S(A|B) \leq T_t(A|B)\left(A\sharp_{t/2}B\right)^{-1}A \\
&\leq T_t(A|B)(A\sharp_t B+A)^{-1}A+\frac{1}{2}T_{2t}(A|B)(A\sharp_t B)^{-1}A \\
&\leq T_{2t}(A|B)(A\sharp_t B)^{-1}A \leq T_t(A|B).
\end{aligned} \tag{2.4}$$

Proof. Let $x \geq 1$ and $t>0$, then by taking $y=x^t$ in (2.1) we get

$$\begin{aligned}
0 &\leq \frac{x^t-1}{tx^t} \leq \frac{2(x^t-1)}{t(x^t+1)} \leq \ln x \leq \frac{x^t-1}{tx^{t/2}} \\
&\leq \frac{x^t-1}{t(x^t+1)}+\frac{x^{2t}-1}{4tx^t} \leq \frac{x^{2t}-1}{2tx^t} \leq \frac{x^t-1}{t}.
\end{aligned} \tag{2.5}$$

Using the functional calculus for the operator $X \geq 1_H$, then by (2.5) we get

$$\begin{aligned}
0 &\leq \frac{X^t-1}{t}X^{-t} \leq 2\frac{(X^t-1)}{t}\left(X^t+1\right)^{-1} \leq \ln X \\
&\leq \frac{X^t-1}{t}X^{-t/2} \leq \frac{(X^t-1)}{t}\left(X^t+1\right)^{-1}+\frac{1}{2}\frac{X^{2t}-1}{2t}X^{-t} \\
&\leq \frac{X^{2t}-1}{2t}X^{-t} \leq \frac{X^t-1}{t}.
\end{aligned} \tag{2.6}$$

If $B \geq A$, then by multiplying both sides by $A^{-1/2}$ we get $A^{-1/2}BA^{-1/2} \geq 1_H$ and if we write the inequality for $X = A^{-1/2}BA^{-1/2}$, we get

$$
\begin{aligned}
(2.7) \qquad 0 &\leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq 2\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^t - 1\right)}{t}\left(\left(A^{-1/2}BA^{-1/2}\right)^t + 1\right)^{-1} \\
&\leq \ln\left(A^{-1/2}BA^{-1/2}\right) \\
&\leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t/2} \\
&\leq \frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^t - 1\right)}{t}\left(\left(A^{-1/2}BA^{-1/2}\right)^t + 1\right)^{-1} \\
&+ \frac{1}{2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t} - 1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t} - 1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} \\
&\leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}.
\end{aligned}
$$

Now, by multiplying both sides of (2.7) with $A^{1/2}$, we get

$$
\begin{aligned}
(2.8) \qquad 0 &\leq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} A^{1/2} \\
&\leq 2A^{1/2}\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^t - 1\right)}{t}\left(\left(A^{-1/2}BA^{-1/2}\right)^t + 1\right)^{-1} A^{1/2} \\
&\leq A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \\
&\leq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t/2} A^{1/2} \\
&\leq A^{1/2}\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^t - 1\right)}{t}\left(\left(A^{-1/2}BA^{-1/2}\right)^t + 1\right)^{-1} A^{1/2} \\
&+ \frac{1}{2}A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t} - 1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} A^{1/2} \\
&\leq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t} - 1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} A^{1/2} \\
&\leq A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}A^{1/2}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^t - 1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} A^{1/2} \\
&= T_t\left(A|B\right)\left(A^{-1}\sharp_t B^{-1}\right)A = T_t\left(A|B\right)\left(A\sharp_t B\right)^{-1}A,
\end{aligned}
$$

$$A^{1/2}\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^{t}-1\right)}{t}\left(\left(A^{-1/2}BA^{-1/2}\right)^{t}+1\right)^{-1}A^{1/2}$$
$$=A^{1/2}\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^{t}-1\right)}{t}A^{1/2}A^{-1/2}$$
$$\left(A^{-1/2}\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}+A\right)A^{-1/2}\right)^{-1}A^{1/2}$$
$$=A^{1/2}\frac{\left(\left(A^{-1/2}BA^{-1/2}\right)^{t}-1\right)}{t}A^{1/2}A^{-1/2}$$
$$A^{1/2}\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}+A\right)^{-1}A^{1/2}A^{1/2}$$
$$=T_{t}\left(A|B\right)\left(A\sharp_{t}B+A\right)^{-1}A,$$

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\left(A^{-1/2}BA^{-1/2}\right)^{-t/2}A^{1/2}$$
$$=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}A^{1/2}A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t/2}A^{-1/2}A^{1/2}A^{1/2}$$
$$=T_{t}\left(A|B\right)\left(A^{-1}\sharp_{t/2}B^{-1}\right)A=T_{t}\left(A|B\right)\left(A\sharp_{t/2}B\right)^{-1}A$$

and

$$A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t}-1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t}A^{1/2}$$
$$=A^{1/2}\frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t}-1}{2t}A^{1/2}A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}A^{1/2}A^{1/2}$$
$$=T_{2t}\left(A|B\right)\left(A\sharp_{t}B\right)^{-1}A.$$

By using the inequalities (2.8) we get the desired result (2.4). □

If we take in (2.4) $t=\frac{1}{2}$, then we get the inequalities

$$\begin{aligned}0&\le 2\left(1_{H}-A\left(A\sharp B\right)^{-1}\right)A\\&\le 4\left(A\sharp B-A\right)\left(A\sharp B+A\right)^{-1}A\\&\le S\left(A|B\right)\le 2\left(A\sharp B-A\right)\left(A\sharp_{1/4}B\right)^{-1}A\\&\le 2\left(A\sharp B-A\right)\left(A\sharp B+A\right)^{-1}A+\frac{1}{2}\left(B-A\right)\left(A\sharp B\right)^{-1}A\\&\le\left(B-A\right)\left(A\sharp B\right)^{-1}A\le 2\left(A\sharp B-A\right),\end{aligned}\tag{2.9}$$

for any positive invertible operators with $B\ge A$. This provides a refinement of (1.8).

If we take in (2.4) $t=1$, then we get

$$
\begin{aligned}
0 &\le (B-A)\,B^{-1}A \le 2\,(B-A)\,(B+A)^{-1}\,A \\
&\le S\,(A|B) \le (B-A)\,(A\sharp B)^{-1}\,A \\
&\le (B-A)\,(B+A)^{-1}\,A + \frac{1}{4}\left(B-AB^{-1}A\right) \\
&\le \frac{1}{2}\left(B-AB^{-1}A\right) \le B-A,
\end{aligned} \tag{2.10}
$$

for any positive invertible operators with $B \ge A$. This provides a refinement of (1.6).

If we take in (2.4) $t=2$, then we get

$$
\begin{aligned}
0 &\le \frac{1}{2}\left(BA^{-1}B-A\right)\left(B^{-1}A\right)^2 \\
&\le \left(BA^{-1}B-A\right)\left(BA^{-1}B+A\right)^{-1}A \\
&\le S\,(A|B) \le \frac{1}{2}\left(BA^{-1}B-A\right)B^{-1}A \\
&\le \frac{1}{2}\left(BA^{-1}B-A\right)\left(BA^{-1}B+A\right)^{-1}A + \frac{1}{8}\left(\left(BA^{-1}\right)^3-A\right)\left(B^{-1}A\right)^2 \\
&\le \frac{1}{4}\left(\left(BA^{-1}\right)^3-A\right)\left(B^{-1}A\right)^2 \le \frac{1}{2}\left(BA^{-1}B-A\right),
\end{aligned} \tag{2.11}
$$

for any positive invertible operators with $B \ge A$. This provides a refinement of (1.7).

Corollary 1. *Let C be a positive invertible operator and $C \le 1_H$, then for any $t>0$ we have*

$$
\begin{aligned}
0 &\le \frac{1}{t}C\left(1_H-C^t\right) \\
&\le \frac{2}{t}C\left(1_H-C^t\right)\left(1_H+C^t\right)^{-1} \\
&\le \eta\,(C) \le \frac{1}{t}\left(1_H-C^t\right)C^{3\left(1-\frac{t}{2}\right)} \\
&\le \frac{1}{t}C\left(1_H-C^t\right)\left(1_H+C^t\right)^{-1} + \frac{1}{4t}\left(1_H-C^{2t}\right)C^{1-t} \\
&\le \frac{1}{2t}\left(1_H-C^{2t}\right)C^{1-t} \le \frac{1}{t}C^{1-t}\left(1_H-C^t\right).
\end{aligned} \tag{2.12}
$$

If we take in (2.12) $t=\frac{1}{2}$, then we get

$$
\begin{aligned}
0 &\le 2C\left(1_H-C^{1/2}\right) \\
&\le 4C\left(1_H-C^{1/2}\right)\left(1_H+C^{1/2}\right)^{-1} \\
&\le \eta\,(C) \le 2\left(1_H-C^{1/2}\right)C^{9/4} \\
&\le 2C\left(1_H-C^{1/2}\right)\left(1_H+C^{1/2}\right)^{-1} + \frac{1}{2}\left(1_H-C\right)C^{1/2} \\
&\le \left(1_H-C\right)C^{1/2} \le 2C^{1/2}\left(1_H-C^{1/2}\right),
\end{aligned} \tag{2.13}
$$

for any C be a positive invertible operator with $C \le 1_H$, which is better than (1.12).

If we take in (2.12) $t=1$, then we get

$$\begin{aligned} 0 &\leq C\left(1_H - C\right) \leq 2C\left(1_H - C\right)\left(1_H + C\right)^{-1} \\ &\leq \eta\left(C\right) \leq \left(1_H - C\right)C^{3/2} \\ &\leq C\left(1_H - C\right)\left(1_H + C\right)^{-1} + \frac{1}{4}\left(1_H - C^2\right) \\ &\leq \frac{1}{2}\left(1_H - C^2\right) \leq 1_H - C, \end{aligned} \tag{2.14}$$

for any C be a positive invertible operator with $C \leq 1_H$, which is better than (1.10).

Finally, if we take in (2.12) $t=2$, then we get

$$\begin{aligned} 0 &\leq \frac{1}{2}C\left(1_H - C^2\right) \\ &\leq C\left(1_H - C^2\right)\left(1_H + C^2\right)^{-1} \\ &\leq \eta\left(C\right) \leq \frac{1}{2}\left(1_H - C^2\right) \\ &\leq \frac{1}{2}C\left(1_H - C^2\right)\left(1_H + C^2\right)^{-1} + \frac{1}{8}\left(1_H - C^4\right)C^{-1} \\ &\leq \frac{1}{4}\left(1_H - C^4\right)C^{-1} \leq \frac{1}{t}C^{-1}\left(1_H - C^2\right), \end{aligned} \tag{2.15}$$

for any C be a positive invertible operator with $C \leq 1_H$, which is better than (1.11).

3. Some Reverses

We have:

Lemma 2. *For any $y \geq 1$ we have the inequalities*

$$0 \leq \frac{y^2 - 1}{2y} - \ln y \leq \frac{1}{8}\frac{\left(y-1\right)^3\left(y+1\right)}{y^2} \tag{3.1}$$

and

$$0 \leq \ln y - \frac{2\left(y-1\right)}{y+1} \leq \frac{1}{8}\frac{\left(y-1\right)^3\left(y+1\right)}{y^2}. \tag{3.2}$$

Proof. We use the following reverse of the second Hermite-Hadamard inequality obtained in [2]:

$$0 \leq \frac{f\left(a\right) + f\left(b\right)}{2} - \frac{1}{b-a}\int_a^b f\left(t\right)dt \leq \frac{1}{8}\left(f'_-\left(b\right) - f'_+\left(a\right)\right)\left(b-a\right). \tag{3.3}$$

If we take in this inequality $f\left(t\right) = \frac{1}{t}$, then we get

$$0 \leq \frac{a+b}{2ab} - \frac{\ln b - \ln a}{b-a} \leq \frac{1}{8}\frac{\left(b-a\right)^2\left(b+a\right)}{a^2b^2} \tag{3.4}$$

for any $a, b > 0$.

If in this inequality we take $a=1$ and $b=y\geq 1$, then we get the desired result (3.1).

Further, we use the following reverse of the first Hermite-Hadamard inequality obtained in [1]:

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left(f'_-(b) - f'_+(a)\right)(b-a). \tag{3.5}$$

If we take in this inequality $f(t) = \frac{1}{t}$, then we get

$$0 \leq \frac{\ln b - \ln a}{b-a} - \frac{2}{a+b} \leq \frac{1}{8}\frac{(b-a)^2(b+a)}{a^2b^2} \tag{3.6}$$

for any a, $b > 0$.

If in this inequality we take $a = 1$ and $b = y \geq 1$, then we get the desired result (3.2). □

We also have:

Theorem 3. *Let A, B be two positive invertible operators and $B \geq A$, then for any $t > 0$ we have*

$$\begin{aligned} 0 &\leq T_{2t}(A|B)(A\sharp_t B)^{-1}A - S(A|B) \\ &\leq \frac{1}{8}T_t(A|B)\left(A^{-1} - (A\sharp_t B)^{-1}\right)A\left(A^{-1} - (A\sharp_t B)^{-1}\right)(A\sharp_t B + A) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} 0 &\leq S(A|B) - 2T_t(A|B)(A\sharp_t B + A)^{-1}A \\ &\leq \frac{1}{8}T_t(A|B)\left(A^{-1} - (A\sharp_t B)^{-1}\right)A\left(A^{-1} - (A\sharp_t B)^{-1}\right)(A\sharp_t B + A). \end{aligned} \tag{3.8}$$

Proof. From inequality (3.1) for $y = x^t$ with $x \geq 1$ and $t > 0$, we have

$$0 \leq \frac{x^{2t}-1}{2x^t} - \ln x^t \leq \frac{1}{8}\frac{(x^t-1)^3(x^t+1)}{x^{2t}},$$

that is equivalent to

$$0 \leq \frac{x^{2t}-1}{2t}x^{-t} - \ln x \leq \frac{1}{8}\left(\frac{x^t-1}{t}\right)\left(1-x^{-t}\right)^2\left(x^t+1\right),$$

for any $x \geq 1$ and $t > 0$.

By using the functional calculus, we have

$$\begin{aligned} 0 &\leq \frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t}-1}{2t}\left(A^{-1/2}BA^{-1/2}\right)^{-t} - \ln\left(A^{-1/2}BA^{-1/2}\right) \\ &\leq \frac{1}{8}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\right)\left(1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}\right)^2 \\ &\quad \left(\left(A^{-1/2}BA^{-1/2}\right)^{t}+1\right), \end{aligned} \tag{3.9}$$

for any A, B positive invertible operators with $B \geq A$ and for any $t > 0$.

If we multiply both sides with $A^{1/2}$, then we get

$$0 \le A^{1/2} \frac{\left(A^{-1/2}BA^{-1/2}\right)^{2t} - 1}{2t} \left(A^{-1/2}BA^{-1/2}\right)^{-t} A^{1/2} - A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)\right)A^{1/2} \le \frac{1}{8}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\right)\left(1-\left(A^{-1/2}BA^{-1/2}\right)^{-t}\right)^{2} \left(\left(A^{-1/2}BA^{-1/2}\right)^{t}+1\right)A^{1/2}. \tag{3.10}$$

Observe that

$$\begin{aligned}
&\frac{1}{8}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\right)A^{1/2}A^{-1/2} \\
&\left(A^{1/2}\left(A^{-1}-A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}\right)A^{1/2}\right)^{2} \\
&\left(A^{-1/2}\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}+A\right)A^{-1/2}\right)A^{1/2} \\
&=\frac{1}{8}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\right)A^{1/2} \\
&A^{-1/2}A^{1/2}\left(A^{-1}-A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}\right)A^{1/2} \\
&A^{1/2}\left(A^{-1}-A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}\right)A^{1/2} \\
&A^{-1/2}\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}+A\right)A^{-1/2}A^{1/2} \\
&=\frac{1}{8}A^{1/2}\left(\frac{\left(A^{-1/2}BA^{-1/2}\right)^{t}-1}{t}\right)A^{1/2} \\
&\left(A^{-1}-A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}\right)A \\
&\left(A^{-1}-A^{-1/2}\left(A^{1/2}B^{-1}A^{1/2}\right)^{t}A^{-1/2}\right) \\
&\left(A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{t}A^{1/2}+A\right) \\
&=\frac{1}{8}T_t\left(A|B\right)\left(A^{-1}-\left(A\sharp_t B\right)^{-1}\right)A\left(A^{-1}-\left(A\sharp_t B\right)^{-1}\right)\left(A\sharp_t B+A\right)
\end{aligned}$$

and by (3.10) we get the desired result (3.7).

The inequality (3.8) follows in a similar way and we omit the details. □

If we take in (3.7) and (3.8) $t = \frac{1}{2}$, then we get

$$0 \le (B-A)\left(A\sharp B\right)^{-1}A - S\left(A|B\right) \le \frac{1}{4}\left(A\sharp B - A\right)\left(A^{-1}-\left(A\sharp B\right)^{-1}\right)A\left(A^{-1}-\left(A\sharp B\right)^{-1}\right)\left(A\sharp B+A\right) \tag{3.11}$$

and

$$0 \le S(A|B) - 4(A\sharp B - A)(A\sharp B + A)^{-1} A \le \frac{1}{4}(A\sharp B - A)\left(A^{-1} - (A\sharp B)^{-1}\right) A \left(A^{-1} - (A\sharp B)^{-1}\right)(A\sharp B + A) \tag{3.12}$$

for any positive invertible operators with $B \ge A$.

If we take in (3.7) and (3.8) $t = 1$, then we get

$$0 \le \frac{1}{2}\left(B - AB^{-1}A\right) - S(A|B) \le \frac{1}{8}(B - A)\left(A^{-1} - B^{-1}\right) A \left(A^{-1} - B^{-1}\right)(B + A) \tag{3.13}$$

and

$$0 \le S(A|B) - 2(B - A) + (A\sharp_t B + A)^{-1} A \le \frac{1}{8}(B - A)\left(A^{-1} - B^{-1}\right) A \left(A^{-1} - B^{-1}\right)(B + A) \tag{3.14}$$

for any positive invertible operators with $B \ge A$.

Similar inequalities may be stated if we take $t = 2$ in Theorem 3, however the details are omitted.

Corollary 2. *Let C be a positive invertible operator and $C \le 1_H$, then for any $t > 0$ we have*

$$0 \le \frac{1}{2t}\left(1_H - C^{2t}\right) C^{1-t} - \eta(C) \le \frac{1}{8t} C^{1-2t}\left(1 - C^t\right)^3\left(1 + C^t\right) \tag{3.15}$$

and

$$0 \le \eta(C) - \frac{2}{t} C\left(1_H - C^t\right)\left(1_H + C^t\right)^{-1} \le \frac{1}{8t} C^{1-2t}\left(1 - C^t\right)^3\left(1 + C^t\right). \tag{3.16}$$

If we take in this corollary $t = \frac{1}{2}$, then we get

$$0 \le (1_H - C) C^{1/2} - \eta(C) \le \frac{1}{4}\left(1 - C^{1/2}\right)^3\left(1 + C^{1/2}\right) \tag{3.17}$$

and

$$0 \le \eta(C) - 4C\left(1_H - C^{1/2}\right)\left(1_H + C^{1/2}\right)^{-1} \le \frac{1}{4}\left(1 - C^{1/2}\right)^3\left(1 + C^{1/2}\right) \tag{3.18}$$

while, if we take $t = 1$, then we get

$$0 \le \frac{1}{2}\left(1_H - C^2\right) - \eta(C) \le \frac{1}{8} C^{-1}(1 - C)^3(1 + C) \tag{3.19}$$

and

$$0 \le \eta(C) - 2C(1_H - C)(1_H + C)^{-1} \le \frac{1}{8} C^{-1}(1 - C)^3(1 + C). \tag{3.20}$$

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