Received 04/11/16

Most General Fractional Self Adjoint Operator Representation formulae and Operator Poincaré and Sobolev type and other basic Inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

We give here many very general fractional self adjoint operator Poincaré and Sobolev type and other basic inner product inequalities to various directions. Initially we give several very general fractional representation formulae in the self adjoint operator sense. Inequalities are based in the self adjoint operator order over a Hilbert space.

2010 AMS Subject Classification: 26A33, 26D10, 26D20, 47A60, 47A67.
Key Words and Phrases: Self adjoint operator, Hilbert space, Poincaré inequality, Sobolev inequality, generalized fractional derivative.

1 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [11, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f) \Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [10], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U) x, y \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, y \rangle), \tag{1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \ \text{ and } \ g_{x,y}(M) = \langle x, y \rangle, \tag{2}$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U) x, x \rangle = \int_{m-0}^{M} f(\lambda) \, d(\langle E_\lambda x, x \rangle), \quad \forall \, x \in H. \tag{3}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda) \, dE_\lambda. \tag{4}$$

Above, $m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U)$, $M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U)$. The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda (U), \ \forall \lambda \in \mathbb{R},$$

is a projection which reduces U, with

$$\varphi_\lambda (s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases} \tag{5}$$

The spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [12], pp. 256-266, and for more details see there pp. 157-266. See also [9].

Some more basics are given (we follow [10], pp. 1-5):

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x \rangle \in \mathbb{R}$, $\forall \ x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x \in H: \|x\|=1} |\langle Ax, x \rangle| . \tag{6}$$

Let A, B be selfadjoint operators on H. Then $A \leq B$ iff $\langle Ax, x \rangle \leq \langle Bx, x \rangle$, $\forall$ $x \in H$.

In particular, A is called positive if $A \geq 0$.

Denote by

$$\mathcal{P} := \left\{ \varphi (s) := \sum_{k=0}^{n} \alpha_k s^k | n \geq 0, \, \alpha_k \in \mathbb{C}, \, 0 \leq k \leq n \right\} . \tag{7}$$

If $A \in \mathcal{B}(H)$ is selfadjoint, and $\varphi (s) \in \mathcal{P}$ has real coefficients, then $\varphi (A)$ is selfadjoint, and

$$\|\varphi (A)\| = \max \{ |\varphi (\lambda)| , \lambda \in Sp (A) \} . \tag{8}$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup \{ |\varphi (\lambda)| , \lambda \in Sp (A) \} . \tag{9}$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi (A)$ is selfadjoint, then $\|\varphi (A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi (A)$ and $|\varphi| (A) = |\varphi (A)|$ are selfadjoint operators (by [10], p. 4, Theorem 7).

Hence it holds

$$\| |\varphi (A)| \| = \| |\varphi| \|_A = \sup \{ | |\varphi (\lambda)| | , \lambda \in Sp (A) \} \tag{10}$$

$$= \sup \{ |\varphi (\lambda)| , \lambda \in Sp (A) \} = \|\varphi\|_A = \|\varphi (A)\| ,$$

that is

$$\| |\varphi (A)| \| = \|\varphi (A)\| . \tag{11}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}. \quad (12)$$

2 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain many very general fractional operator representation formulae, and many very general fractional operator Poincaré and Sobolev type inequalities, and many other basic fractional operator inner product inequalities, in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself). All of our functions next in this article are real valued.

We mention the following general Taylor formula

Theorem 1 *([2], p. 400) Let $f, f', ..., f^{(n)}; g, g'$ be continuous from $[m, M]$ into $\mathbb{R}$, $n \in \mathbb{N}$. Assume $\left(g^{-1}\right)^{(k)}$, $k = 0, 1, ..., n$, are continuous. Then*

$$f(\lambda) = f(m) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(m))}{k!} (g(\lambda) - g(m))^k + R_n(m, \lambda), \quad (13)$$

where

$$R_n(m, \lambda) = \frac{1}{(n-1)!} \int_m^\lambda (g(\lambda) - g(s))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(s)) \, g'(s) \, ds \quad (14)$$

$$= \frac{1}{(n-1)!} \int_{g(m)}^{g(\lambda)} (g(\lambda) - t)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (t) \, dt, \quad \forall \, \lambda \in [m, M].$$

We present the operator representation formula

Theorem 2 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$ (the interior of I) and $n \in \mathbb{N}$. We consider $f \in C^n([m, M])$, $g \in C^1([m, M])$, such that there exist $\left(g^{-1}\right)^{(k)}$, $k = 0, 1, ..., n$, that are continuous, where $f, g : I \to \mathbb{R}$.*

Then

$$f(A) = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(m))}{k!} \left(g(A) - g(m) 1_H\right)^k + R_n(f, g, m, M), \quad (15)$$

where

$$R_n(f, g, m, M) =$$

$$\frac{1}{(n-1)!} \int_{m-0}^{M} \left(\int_m^\lambda (g(\lambda) - g(s))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(s)) g'(s) \, ds \right) dE_\lambda \quad (16)$$

$$= \frac{1}{(n-1)!} \int_{m-0}^{M} \left(\int_{g(m)}^{g(\lambda)} (g(\lambda) - t)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (t) \, dt \right) dE_\lambda.$$

Proof. We integrate (13), (14) against E_λ to get

$$\int_{m-0}^{M} f(\lambda) \, dE_\lambda = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(m))}{k!} \int_{m-0}^{M} (g(\lambda) - g(m))^k \, dE_\lambda$$

$$+ \int_{m-0}^{M} R_n(m, \lambda) \, dE_\lambda. \quad (17)$$

By the spectral representation theorem we obtain

$$f(A) = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(m))}{k!} \left(g(A) - g(m) 1_H\right)^k +$$

$$\int_{m-0}^{M} R_n(m, \lambda) \, dE_\lambda, \quad (18)$$

proving the claim. ■

Note 3 *(to Theorem 2) By [2], p. 401, if $f^{(k)}(m) = 0$, for $k = 0, 1, ..., n-1$, then $\left(f \circ g^{-1}\right)^{(k)} (g(m)) = 0$, all $k = 0, 1, ..., n-1$. In that case it holds*

$$f(A) = R_n(f, g, m, M). \quad (19)$$

We need

Definition 4 *([3]) Let* $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ *the ceiling of the number. Here* $g \in AC([m, M])$ *(absolutely continuous functions) and it is strictly increasing. We assume that* $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([m, M])$. *We define the left generalized* g*-fractional derivative of* f *of order* α *as follows:*

$$\left(D_{m+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_m^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{20}$$

$x \geq m$, *where* Γ *is the gamma function.*

If $\alpha \notin \mathbb{N}$, *by [3], we have that* $D_{m+;g}^{\alpha} f \in C([m, M])$.

We set

$$D_{m+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \tag{21}$$

$$\left(D_{m+;g}^{0} f\right)(x) := f(x), \quad \forall\, x \in [m, M].$$

When $g = id$, *then*

$$D_{m+;g}^{\alpha} f = D_{m+;id}^{\alpha} f = D_{*m}^{\alpha} f, \tag{22}$$

the usual left Caputo fractional derivative [1], p. 270, and [8], p. 50.

We need the following g-left fractional generalized Taylor's formula:

Theorem 5 *([3]) Let* g *be strictly increasing function and* $g \in AC([m, M])$. *We assume that* $\left(f \circ g^{-1}\right) \in AC^n([g(m), g(M)])$, *where* $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$ *(it means* $\left(f \circ g^{-1}\right)^{(n-1)} \in AC([g(m), g(M)])$, *and implies that* $f \in C([m, M])$*). Also we assume that* $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([m, M])$. *Then*

$$f(\lambda) = f(m) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(m))}{k!} (g(\lambda) - g(m))^k + \tag{23}$$

$$\frac{1}{\Gamma(\alpha)} \int_m^\lambda (g(\lambda) - g(t))^{\alpha-1} g'(t) \left(D_{m+;g}^{\alpha} f\right)(t)\, dt,$$

$\forall\, \lambda \in [m, M]$.

Calling $R_\alpha^{(1)}(m, \lambda)$ *the remainder of (23), we get that*

$$R_\alpha^{(1)}(m, \lambda) = \frac{1}{\Gamma(\alpha)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{\alpha-1} \left(\left(D_{m+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \tag{24}$$

$\forall\, \lambda \in [m, M]$.

$R_\alpha^{(1)}(m, \lambda)$ is a continuous function in $\lambda \in [m, M]$.

We present the following operator left fractional representation formula

Theorem 6 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$ and $n \in \mathbb{N}$, with $n := \lceil \alpha \rceil$, $\alpha > 0$. Let $f, g : I \to \mathbb{R}$. Assume that g is strictly increasing and $g \in AC([m, M])$, and $(f \circ g^{-1}) \in AC^n([g(m), g(M)])$, and $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([m, M])$. Then*

$$f(A) = \sum_{k=0}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(m))}{k!} (g(A) - g(m) 1_H)^k + R_\alpha^{(1)}(f, g, \alpha, m, M), \tag{25}$$

where

$$R_\alpha^{(1)}(f, g, \alpha, m, M) :=$$

$$\frac{1}{\Gamma(\alpha)} \int_{m-0}^{M} \left(\int_m^\lambda (g(\lambda) - g(t))^{\alpha-1} g'(t) \left(D_{m+;g}^\alpha f\right)(t)\, dt \right) dE_\lambda$$

$$= \frac{1}{\Gamma(\alpha)} \int_{m-0}^{M} \left(\int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{\alpha-1} \left(\left(D_{m+;g}^\alpha f\right) \circ g^{-1}\right)(z)\, dz \right) dE_\lambda. \tag{26}$$

Proof. We integrate (23) against E_λ to get

$$\int_{m-0}^{M} f(\lambda)\, dE_\lambda = \sum_{k=0}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(m))}{k!} \int_{m-0}^{M} (g(\lambda) - g(m))^k\, dE_\lambda$$

$$+ \int_{m-0}^{M} R_\alpha^{(1)}(m, \lambda)\, dE_\lambda. \tag{27}$$

By the spectral representation theorem we obtain

$$f(A) = \sum_{k=0}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(m))}{k!} (g(A) - g(m) 1_H)^k +$$

$$\int_{m-0}^{M} R_\alpha^{(1)}(m, \lambda)\, dE_\lambda, \tag{28}$$

proving the claim. ■

Note 7 *(to Theorem 6) If $(f \circ g^{-1})^{(k)}(g(m)) = 0$, for $k = 0, 1, ..., n-1$, then*

$$f(A) = R_\alpha^{(1)}(f, g, \alpha, m, M). \tag{29}$$

We need

Definition 8 *([3]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$. Here $g \in AC([m, M])$ and it is strictly increasing. We assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([m, M])$. We define the right generalized g-fractional derivative of f of order α as follows:*

$$\left(D_{M-;g}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^M (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{30}$$

all $x \in [m, M]$.

If $\alpha \notin \mathbb{N}$, by [3], we get that $\left(D_{M-;g}^{\alpha} f\right) \in C([m, M])$.

We set

$$\left(D_{M-;g}^{n} f\right)(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \tag{31}$$

$$\left(D_{M-;g}^{0} f\right)(x) := f(x), \ \ \forall\, x \in [m, M].$$

When $g = id$, then

$$\left(D_{M-;g}^{\alpha} f\right)(x) = \left(D_{M-;id}^{\alpha} f\right)(x) = \left(D_{M-}^{\alpha} f\right)(x), \tag{32}$$

the usual right Caputo fractional derivative, [2], p. 336-337.

We will use the g-right generalized fractional Taylor's formula:

Theorem 9 *([3]) Let g be strictly increasing function and $g \in AC([m, M])$. We assume that $\left(f \circ g^{-1}\right) \in AC^n([g(m), g(M)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_{\infty}([m, M])$. Then*

$$f(\lambda) = f(M) + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} (g(\lambda) - g(M))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_{\lambda}^{M} (g(t) - g(\lambda))^{\alpha-1} g'(t) \left(D_{M-;g}^{\alpha} f\right)(t)\, dt, \tag{33}$$

all $m \leq \lambda \leq M$.

Calling $R_{\alpha}^{(2)}(M, \lambda)$ the remainder of (33), we get that

$$R_{\alpha}^{(2)}(M, \lambda) = \frac{1}{\Gamma(\alpha)} \int_{g(\lambda)}^{g(M)} (z - g(\lambda))^{\alpha-1} \left(\left(D_{M-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \tag{34}$$

$\forall\, \lambda \in [m, M]$.

$R_{\alpha}^{(2)}(M, \lambda)$ is a continuous function in $\lambda \in [m, M]$.

We present the following operator right fractional representation formula

Theorem 10 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$ and $n \in \mathbb{N}$, with*

$n := \lceil \alpha \rceil$, $\alpha > 0$. Let $f, g : I \to \mathbb{R}$. Assume that g is strictly increasing and $g \in AC([m, M])$, and $\left(f \circ g^{-1}\right) \in AC^n([g(m), g(M)])$, and $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([m, M])$. Then

$$f(A) = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} (g(A) - g(M) 1_H)^k + R_\alpha^{(2)}(f, g, \alpha, m, M), \tag{35}$$

where

$$R_\alpha^{(2)}(f, g, \alpha, m, M) :=$$

$$\frac{1}{\Gamma(\alpha)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (g(t) - g(\lambda))^{\alpha-1} g'(t) \left(D_{M-;g}^{\alpha} f\right)(t)\, dt \right) dE_\lambda$$

$$= \frac{1}{\Gamma(\alpha)} \int_{m-0}^{M} \left(\int_{g(\lambda)}^{g(M)} (z - g(\lambda))^{\alpha-1} \left(\left(D_{M-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz \right) dE_\lambda. \tag{36}$$

Proof. We integrate (33) against E_λ to get

$$\int_{m-0}^{M} f(\lambda)\, dE_\lambda = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} \int_{m-0}^{M} (g(\lambda) - g(M))^k\, dE_\lambda$$

$$+ \int_{m-0}^{M} R_\alpha^{(2)}(M, \lambda)\, dE_\lambda. \tag{37}$$

By the spectral representation theorem we obtain

$$f(A) = \sum_{k=0}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} (g(A) - g(M) 1_H)^k +$$

$$\int_{m-0}^{M} R_\alpha^{(2)}(M, \lambda)\, dE_\lambda, \tag{38}$$

proving the claim. ■

Note 11 *(to Theorem 10) If $\left(f \circ g^{-1}\right)^{(k)}(g(M)) = 0$, for $k = 0, 1, ..., n-1$, then*

$$f(A) = R_\alpha^{(2)}(f, g, \alpha, m, M). \tag{39}$$

We make

Background 12 *([4]) Let $g : [m, M] \to \mathbb{R}$ be a strictly increasing function. Let $f \in C^n([m, M])$, $n \in \mathbb{N}$. Assume that $g \in C^1([m, M])$, and $g^{-1} \in C^n([g(m), g(M)])$. Call $l := f \circ g^{-1} : [g(m), g(M)] \to \mathbb{R}$. It is clear that $l, l', ..., l^{(n)}$ are continuous from $[g(m), g(M)]$ into $f([m, M]) \subseteq \mathbb{R}$.*

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number. Clearly when $0 < \nu < 1$, $[\nu] = 0$.

Next we follow [1], pp. 7-9.

Let $h \in C([g(m), g(M)])$, we define the left Riemann-Liouville fractional integral

$$\left(J_{\nu}^{g(m)} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{g(m)}^{z} (z-t)^{\nu-1} h(t)\, dt, \tag{40}$$

for $g(m) \leq z \leq g(M)$.

We set $J_0^{g(m)} h = h$.

Let $\overline{\alpha} := \nu - [\nu]$ ($0 < \overline{\alpha} < 1$). We define the subspace $C_{g(m)}^{\nu}([g(m), g(M)])$ of $C^{[\nu]}([g(m), g(M)])$ as

$$C_{g(m)}^{\nu}([g(m), g(M)]) :=$$

$$\left\{h \in C^{[\nu]}([g(m), g(M)]) : J_{1-\overline{\alpha}}^{g(m)} h^{([\nu])} \in C^{1}([g(m), g(M)])\right\}. \tag{41}$$

So let $h \in C_{g(m)}^{\nu}([g(m), g(M)])$; we define the left g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(m), g(M)]$ as

$$D_{g(m)}^{\nu} h := \left(J_{1-\overline{\alpha}}^{g(m)} h^{([\nu])}\right)'. \tag{42}$$

Clearly, for $h \in C_{g(m)}^{\nu}([g(m), g(M)])$, there exists

$$\left(D_{g(m)}^{\nu} h\right)(z) = \frac{1}{\Gamma(1-\overline{\alpha})} \frac{d}{dz} \int_{g(m)}^{z} (z-t)^{-\overline{\alpha}} h^{([\nu])}(t)\, dt, \tag{43}$$

for all $g(m) \leq z \leq g(M)$.

In particular, when $f \circ g^{-1} \in C_{g(m)}^{\nu}([g(m), g(M)])$ we have that

$$\left(D_{g(m)}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{1}{\Gamma(1-\overline{\alpha})} \frac{d}{dz} \int_{g(m)}^{z} (z-t)^{-\overline{\alpha}} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{44}$$

for all $g(m) \leq z \leq g(M)$.

We have that

$$D_{g(m)}^{n}\left(f \circ g^{-1}\right) = \left(f \circ g^{-1}\right)^{(n)}, \tag{45}$$

and

$$D_{g(m)}^{0}\left(f \circ g^{-1}\right) = f \circ g^{-1}. \tag{46}$$

We mention the following left generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 13 *([4]) Let $f \circ g^{-1} \in C_{g(m)}^{\nu}([g(m), g(M)])$.*

(i) if $\nu \geq 1$*, then*

$$f(\lambda) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(m))}{k!} (g(\lambda) - g(m))^k + \tag{47}$$

$$\frac{1}{\Gamma(\nu)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - t)^{\nu-1} \left(D_{g(m)}^{\nu} \left(f \circ g^{-1}\right)\right)(t)\, dt,$$

all $\lambda \in [m, M]$,

(ii) if $0 < \nu < 1$*, then*

$$f(\lambda) = \frac{1}{\Gamma(\nu)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - t)^{\nu-1} \left(D_{g(m)}^{\nu} \left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{48}$$

all $\lambda \in [m, M]$.

By the change of variable method, see [13], we may rewrite the remainder of (47), (48), as

$$R_{\nu}^{(3)}(m, \lambda) := \frac{1}{\Gamma(\nu)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - t)^{\nu-1} \left(D_{g(m)}^{\nu} \left(f \circ g^{-1}\right)\right)(t)\, dt = \tag{49}$$

$$\frac{1}{\Gamma(\nu)} \int_{m}^{\lambda} (g(\lambda) - g(s))^{\nu-1} \left(D_{g(m)}^{\nu} \left(f \circ g^{-1}\right)\right)(g(s))\, g'(s)\, ds,$$

all $\lambda \in [m, M]$.

We present the following operator left fractional representation formula

Theorem 14 *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval of* $\mathbb{R}$ *with* $[m, M] \subset \overset{\circ}{I}$ *and* $n \in \mathbb{N}$*, with* $n := [\nu]$, $\nu > 0$*. Let* $f, g : I \to \mathbb{R}$*. Assume that* $g : [m, M] \to \mathbb{R}$ *is strictly increasing function,* $f \in C^n([m, M])$, $g \in C^1([m, M])$*, and* $g^{-1} \in C^n([g(m), g(M)])$*. Suppose also that* $f \circ g^{-1} \in C_{g(m)}^{\nu}([g(m), g(M)])$*. Then*

(i) if $\nu \geq 1$*, then*

$$f(A) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(m))}{k!} (g(A) - g(m) 1_H)^k + R_{\nu}^{(3)}(f, g, \nu, m, M), \tag{50}$$

(ii) if $0 < \nu < 1$*, then*

$$f(A) = R_{\nu}^{(3)}(f, g, \nu, m, M). \tag{51}$$

Here it is

$$R_{\nu}^{(3)}(f, g, \nu, m, M) :=$$

$$\frac{1}{\Gamma(\nu)}\int_{m-0}^{M}\left(\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{\nu-1}\left(D_{g(m)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt\right)dE_{\lambda}=$$

$$\frac{1}{\Gamma(\nu)}\int_{m-0}^{M}\left(\int_{m}^{\lambda}(g(\lambda)-g(s))^{\nu-1}\left(D_{g(m)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(s))\,g'(s)\,ds\right)dE_{\lambda}. \tag{52}$$

Proof. We integrate (47), (48) against E_λ and use the spectral representation theorem, as in Theorem 6. ■

Note 15 *If $\nu \geq 1$ and $f^{(k)}(m)=0$, then $\left(f\circ g^{-1}\right)^{(k)}(g(m))=0$, all $k=0,1,...,[\nu]-1$, (see [2], p. 401), and*

$$f(A)=R_{\nu}^{(3)}(f,g,\nu,m,M). \tag{53}$$

We need

Background 16 *Let g,f,l,ν,n,h as in Background 12. Here we follow [2], pp. 345-348.*

We define the right Riemann-Liouville fractional integral as

$$\left(J_{g(M)-}^{\nu}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z}^{g(M)}(t-z)^{\nu-1}h(t)\,dt, \tag{54}$$

for $g(m)\leq z\leq g(M)$.

We set $J_{g(M)-}^{0}h=h$.

Let $\overline{\alpha}:=\nu-[\nu]$ $(0<\overline{\alpha}<1)$. We define the subspace $C_{g(M)-}^{\nu}([g(m),g(M)])$ of $C^{[\nu]}([g(m),g(M)])$ as

$$C_{g(M)-}^{\nu}([g(m),g(M)]):=$$

$$\left\{h\in C^{[\nu]}([g(m),g(M)]):J_{g(M)-}^{1-\overline{\alpha}}h^{([\nu])}\in C^{1}([g(m),g(M)])\right\}. \tag{55}$$

So let $h\in C_{g(M)-}^{\nu}([g(m),g(M)])$; we define the right g-generalized fractional derivative of h of order ν, of Canavati type, over $[g(m),g(M)]$ as

$$D_{g(M)-}^{\nu}h:=(-1)^{n-1}\left(J_{g(M)-}^{1-\overline{\alpha}}h^{([\nu])}\right)'. \tag{56}$$

Clearly, for $h\in C_{g(M)-}^{\nu}([g(m),g(M)])$, there exists

$$\left(D_{g(M)-}^{\nu}h\right)(z)=\frac{(-1)^{n-1}}{\Gamma(1-\overline{\alpha})}\frac{d}{dz}\int_{z}^{g(M)}(t-z)^{-\overline{\alpha}}h^{([\nu])}(t)\,dt, \tag{57}$$

for all $g(m)\leq z\leq g(M)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(M)-}([g(m), g(M)])$ *we have that*

$$\left(D^{\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\overline{\alpha})} \frac{d}{dz} \int_{z}^{g(M)} (t-z)^{-\overline{\alpha}} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{58}$$

for all $g(m) \leq z \leq g(M)$.

We get that

$$\left(D^{n}_{g(M)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^{n} \left(f \circ g^{-1}\right)^{(n)}(z), \tag{59}$$

and

$$\left(D^{0}_{g(M)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z), \tag{60}$$

all $z \in [g(m), g(M)]$.

We need the following right generalized g-fractional, of Canavati type, Taylor's formula:

Theorem 17 *([4]) Let* $f \circ g^{-1} \in C^{\nu}_{g(M)-}([g(m), g(M)])$.

(i) if $\nu \geq 1$*, then*

$$f(\lambda) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} (g(\lambda) - g(M))^{k} +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(\lambda)}^{g(M)} (t - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{61}$$

all $m \leq \lambda \leq M$*,*

(ii) if $0 < \nu < 1$*, we get*

$$f(\lambda) = \frac{1}{\Gamma(\nu)} \int_{g(\lambda)}^{g(M)} (t - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{62}$$

all $m \leq \lambda \leq M$.

By change of variable, see [13], we may rewrite the remainder of (61), (62), as

$$R^{(4)}_{\nu}(M, \lambda) := \frac{1}{\Gamma(\nu)} \int_{g(\lambda)}^{g(M)} (t - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(t)\, dt =$$

$$\frac{1}{\Gamma(\nu)} \int_{\lambda}^{M} (g(s) - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(g(s))\, g'(s)\, ds, \tag{63}$$

all $m \leq \lambda \leq M$.

We present the following operator right fractional representation formula

Theorem 18 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$ and $n \in \mathbb{N}$, with $n := [\nu]$, $\nu > 0$. Let $f, g : I \to \mathbb{R}$. Assume that $g : [m, M] \to \mathbb{R}$ is strictly increasing function, $f \in C^n([m, M])$, $g \in C^1([m, M])$, and $g^{-1} \in C^n([g(m), g(M)])$. Suppose also that $f \circ g^{-1} \in C^{\nu}_{g(M)-}([g(m), g(M)])$. Then*

(i) if $\nu \geq 1$, then

$$f(A) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(M))}{k!} (g(A) - g(M) 1_H)^k + R_\nu^{(4)}(f, g, \nu, m, M), \tag{64}$$

(ii) if $0 < \nu < 1$, then

$$f(A) = R_\nu^{(4)}(f, g, \nu, m, M). \tag{65}$$

Here it is

$$R_\nu^{(4)}(f, g, \nu, m, M) :=$$

$$\frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_{g(\lambda)}^{g(M)} (t - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-} \left(f \circ g^{-1}\right)\right)(t)\, dt \right) dE_\lambda =$$

$$\frac{1}{\Gamma(\nu)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (g(s) - g(\lambda))^{\nu-1} \left(D^{\nu}_{g(M)-} \left(f \circ g^{-1}\right)\right)(g(s))\, g'(s)\, ds \right) dE_\lambda. \tag{66}$$

Proof. We integrate (61), (62) against E_λ and use the spectral representation theorem, as in Theorem 10. ∎

Note 19 *If $\nu \geq 1$ and $f^{(k)}(M) = 0$, then $\left(f \circ g^{-1}\right)^{(k)}(g(M)) = 0$, all $k = 0, 1, ..., [\nu] - 1$, (see [2], p. 401), and*

$$f(A) = R_\nu^{(4)}(f, g, \nu, m, M). \tag{67}$$

We need

Background 20 *Let $f : [m, M] \to \mathbb{R} : f^{(\overline{m})} \in L_\infty([m, M])$, the left Caputo fractional derivative ([8], p. 50) of order $\alpha \notin \mathbb{N}$, $\alpha > 0$, $\overline{m} = \lceil \alpha \rceil$ ($\lceil \cdot \rceil$ ceiling) is defined as follows:*

$$\left(D^{\alpha}_{*m} f\right)(x) = \frac{1}{\Gamma(\overline{m} - \alpha)} \int_m^x (x - t)^{\overline{m} - \alpha - 1} f^{(\overline{m})}(t)\, dt, \tag{68}$$

$\forall\ x \in [m, M]$.

Let $n \in \mathbb{N}$, we denote

$$D^{n\alpha}_{*m} = D^{\alpha}_{*m} D^{\alpha}_{*m} ... D^{\alpha}_{*m} \quad \text{(n-times)}. \tag{69}$$

Let us assume now that

$$D_{*m}^{k\alpha}f\in C\left([m,M]\right),\quad k=0,1,...,n+1;\ n\in\mathbb{N},\ 0<\alpha\leq 1. \tag{70}$$

By [5], [14], we mention the following generalized fractional Caputo type Taylor's formula:

$$f\left(\lambda\right)=\sum_{i=0}^{n}\frac{\left(\lambda-m\right)^{i\alpha}}{\Gamma\left(i\alpha+1\right)}\left(D_{*m}^{i\alpha}f\right)(m)+$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{m}^{\lambda}\left(\lambda-t\right)^{(n+1)\alpha-1}\left(D_{*m}^{(n+1)\alpha}f\right)(t)\,dt, \tag{71}$$

$\forall\ \lambda\in[m,M]$.

We give the following operator left fractional representation formula

Theorem 21 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp\left(A\right)\subseteq[m,M]$ for some real numbers $m<M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m,M]\subset\overset{\circ}{I}$. Here $f:I\to\mathbb{R}$. Furthermore assume that $f'\in L_\infty\left([m,M]\right)$, and $D_{*m}^{k\alpha}f\in C\left([m,M]\right)$, $k=0,1,...,n+1$; $n\in\mathbb{N}$, $0<\alpha\leq1$. Then*

$$f\left(A\right)=\sum_{i=0}^{n}\frac{\left(D_{*m}^{i\alpha}f\right)(m)}{\Gamma\left(i\alpha+1\right)}\left(A-m1_H\right)^{i\alpha}+$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{m-0}^{M}\left(\int_{m}^{\lambda}\left(\lambda-t\right)^{(n+1)\alpha-1}\left(D_{*m}^{(n+1)\alpha}f\right)(t)\,dt\right)dE_\lambda. \tag{72}$$

Proof. We use (71) and the spectral representation theorem, as in Theorem 6. ■

Note 22 *(to Theorem 21) If $\left(D_{*m}^{i\alpha}f\right)(m)=0$, $i=0,1,...,n$, then*

$$f\left(A\right)=\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{m-0}^{M}\left(\int_{m}^{\lambda}\left(\lambda-t\right)^{(n+1)\alpha-1}\left(D_{*m}^{(n+1)\alpha}f\right)(t)\,dt\right)dE_\lambda. \tag{73}$$

We need

Background 23 *The right Caputo fractional derivative of order $\alpha>0$, $\overline{m}=\lceil\alpha\rceil$, $f\in AC^{\overline{m}}\left([m,M]\right)$ is defined as follows (see [2], p. 336):*

$$\left(D_{M-}^{\alpha}f\right)(x)=\frac{\left(-1\right)^{\overline{m}}}{\Gamma\left(\overline{m}-\alpha\right)}\int_{x}^{M}\left(z-x\right)^{\overline{m}-\alpha-1}f^{(\overline{m})}(z)\,dz, \tag{74}$$

$\forall\ x \in [m, M]$, *with*

$$D_{M-}^{\overline{m}} f(x) := (-1)^{\overline{m}} f^{(\overline{m})}(x). \tag{75}$$

Denote by

$$D_{M-}^{n\alpha} = D_{M-}^{\alpha} D_{M-}^{\alpha} ... D_{M-}^{\alpha} \quad \text{(n-times)},\ n \in \mathbb{N}. \tag{76}$$

We need the following right generalized fractional Taylor's formula

Theorem 24 *([3]) Suppose that $f \in AC([m, M])$ and $D_{M-}^{k\alpha} f \in C([m, M])$, for $k = 0, 1, ..., n+1$, where $0 < \alpha \leq 1$. Then*

$$f(\lambda) = \sum_{i=0}^{n} \frac{(M-\lambda)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{M-}^{i\alpha} f\right)(M) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{\lambda}^{M} (z-\lambda)^{(n+1)\alpha-1} \left(D_{M-}^{(n+1)\alpha} f\right)(z)\, dz, \tag{77}$$

$\forall\ \lambda \in [m, M]$.

We give the following operator right fractional representation formula

Theorem 25 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$. Here $f : I \to \mathbb{R}$. Furthermore assume that $f \in AC([m, M])$, and $D_{M-}^{k\alpha} f \in C([m, M])$, for $k = 0, 1, ..., n+1$, where $0 < \alpha \leq 1$. Then*

$$f(A) = \sum_{i=0}^{n} \frac{(M1_H - A)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{M-}^{i\alpha} f\right)(M) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (z-\lambda)^{(n+1)\alpha-1} \left(D_{M-}^{(n+1)\alpha} f\right)(z)\, dz \right) dE_\lambda. \tag{78}$$

Proof. Use of (77) and spectral representation theorem, as in Theorem 21. ∎

Note 26 *(to Theorem 25) If $\left(D_{M-}^{i\alpha} f\right)(M) = 0$, $i = 0, 1, ..., n$, then*

$$f(A) = \frac{1}{\Gamma((n+1)\alpha)} \int_{m-0}^{M} \left(\int_{\lambda}^{M} (z-\lambda)^{(n+1)\alpha-1} \left(D_{M-}^{(n+1)\alpha} f\right)(z)\, dz \right) dE_\lambda. \tag{79}$$

Background 27 *([3]) Denote by ($\alpha > 0$)*

$$D_{m+;g}^{n\alpha} := D_{m+;g}^{\alpha} D_{m+;g}^{\alpha} ... D_{m+;g}^{\alpha} \quad \text{(n-times)},\ n \in \mathbb{N}. \tag{80}$$

By convention $D_{m+;g}^{0} = I$ (identity operator).

We need the following left general fractional Taylor's formula

Theorem 28 *([3]) Let g be strictly increasing and $g \in AC([m, M])$. Suppose that $F_k := D^{k\alpha}_{m+;g} f$, for $k = 0, 1, ..., n+1$, fulfill: $F_k \circ g^{-1} \in AC([g(m), g(M)])$ and $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([m, M])$, where $0 < \alpha \leq 1$. Then*

$$f(\lambda) = \sum_{i=0}^{n} \frac{(g(\lambda) - g(m))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{m+;g} f\right)(m) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_m^\lambda (g(\lambda) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{m+;g} f\right)(t)\, dt, \tag{81}$$

$\forall\ \lambda \in [m, M]$.

We give the following operator general left fractional representation formula

Theorem 29 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$. Here $f, g : I \to \mathbb{R}$. Furthermore we assume that g is strictly increasing and $g \in AC([m, M])$. Suppose that $F_k := D^{k\alpha}_{m+;g} f$, for $k = 0, 1, ..., n+1$, fulfill: $F_k \circ g^{-1} \in AC([g(m), g(M)])$, and $\left(F_k \circ g^{-1}\right)' \circ g \in L_\infty([m, M])$, where $0 < \alpha \leq 1$. Then*

$$f(A) = \sum_{i=0}^{n} \frac{\left(D^{i\alpha}_{m+;g} f\right)(m)}{\Gamma(i\alpha + 1)} (g(A) - g(m) 1_H)^{i\alpha} +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{m-0}^{M} \left(\int_m^\lambda (g(\lambda) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{m+;g} f\right)(t)\, dt \right) dE_\lambda. \tag{82}$$

Proof. Use of (81) and spectral representation theorem. ■

Note 30 *(to Theorem 29) If $\left(D^{i\alpha}_{m+;g} f\right)(m) = 0$, $i = 0, 1, ..., n$, then*

$$f(A) = \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_{m-0}^{M} \left(\int_m^\lambda (g(\lambda) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{m+;g} f\right)(t)\, dt \right) dE_\lambda. \tag{83}$$

We need

Background 31 *([3]) Denote by ($\alpha > 0$)*

$$D^{n\alpha}_{M-;g} := D^\alpha_{M-;g} D^\alpha_{M-;g} ... D^\alpha_{M-;g} \quad \text{(n-times)},\ n \in \mathbb{N}. \tag{84}$$

By convention $D^0_{M-;g} = I$ (identity operator).

We need the following right general fractional Taylor's formula

Theorem 32 *([3]) Let g be strictly increasing and $g \in AC([m, M])$. Suppose that $F_k := D^{k\alpha}_{M-;g} f$, for $k = 0, 1, ..., n+1$, fulfill: $F_k \circ g^{-1} \in AC([g(m), g(M)])$ and $(F_k \circ g^{-1})' \circ g \in L_\infty([m, M])$, where $0 < \alpha \leq 1$. Then*

$$f(\lambda) = \sum_{i=0}^{n} \frac{(g(M) - g(\lambda))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{M-;g} f\right)(M) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_\lambda^M (g(t) - g(\lambda))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{M-;g} f\right)(t)\, dt, \tag{85}$$

$\forall\ \lambda \in [m, M]$.

We give the following operator general right fractional representation formula

Theorem 33 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval of $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$. Here $f, g : I \to \mathbb{R}$. Furthermore we assume that g is strictly increasing and $g \in AC([m, M])$. Suppose that $F_k := D^{k\alpha}_{M-;g} f$, for $k = 0, 1, ..., n+1$, fulfill: $F_k \circ g^{-1} \in AC([g(m), g(M)])$, and $(F_k \circ g^{-1})' \circ g \in L_\infty([m, M])$, where $0 < \alpha \leq 1$. Then*

$$f(A) = \sum_{i=0}^{n} \frac{\left(D^{i\alpha}_{M-;g} f\right)(M)}{\Gamma(i\alpha + 1)} (g(M) 1_H - g(A))^{i\alpha} +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{m-0}^{M} \left(\int_\lambda^M (g(t) - g(\lambda))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{M-;g} f\right)(t)\, dt \right) dE_\lambda. \tag{86}$$

Proof. Use of (85) and spectral representation theorem. ∎

Note 34 *(to Theorem 33) If $\left(D^{i\alpha}_{M-;g} f\right)(M) = 0$, $i = 0, 1, ..., n$, then*

$$f(A) = \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_{m-0}^{M} \left(\int_\lambda^M (g(t) - g(\lambda))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{M-;g} f\right)(t)\, dt \right) dE_\lambda. \tag{87}$$

We need

Background 35 *([4]) Denote by*

$$D^{\overline{m}\nu}_{g(m)} = D^{\nu}_{g(m)} D^{\nu}_{g(m)} ... D^{\nu}_{g(m)} \quad (\overline{m}\text{-times}),\ \overline{m} \in \mathbb{N}. \tag{88}$$

We will use the left fractional Taylor's formula

Theorem 36 *([4]) Let* $0 < \nu < 1$.

Assume that $\left(D_{g(m)}^{i\nu} \left(f \circ g^{-1} \right) \right) \in C_{g(m)}^{\nu} \left([g(m), g(M)] \right)$, $i = 0, 1, ..., \overline{m}$. *Assume also that* $\left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) \in C \left([g(m), g(M)] \right)$. *Then*

$$f(\lambda) = \frac{1}{\Gamma((\overline{m}+1)\nu)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{(\overline{m}+1)\nu - 1} \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(z)\, dz$$

$$= \frac{1}{\Gamma((\overline{m}+1)\nu)} \int_{m}^{\lambda} (g(\lambda) - g(s))^{(\overline{m}+1)\nu - 1} \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(g(s))\, g'(s)\, ds, \tag{89}$$

all $m \leq \lambda \leq M$.

We present the operator left fractional representation formula

Theorem 37 *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $Sp(A) \subseteq [m, M]$ *for some real numbers* $m < M$, $\{E_\lambda\}_\lambda$ *be its spectral family,* I *be a closed subinterval on* $\mathbb{R}$ *with* $[m, M] \subset \overset{\circ}{I}$, *and* $0 < \nu < 1$. *Let* $f, g : I \to \mathbb{R}$. *Assume that* $g : [m, M] \to \mathbb{R}$ *is strictly increasing function,* $f \in C^1([m, M])$, $g \in C^1([m, M])$, *and* $g^{-1} \in C^1([g(m), g(M)])$. *Furthermore we suppose that* $\left(D_{g(m)}^{i\nu} \left(f \circ g^{-1} \right) \right) \in C_{g(m)}^{\nu} \left([g(m), g(M)] \right)$, $i = 0, 1, ..., \overline{m}$, *and* $\left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) \in C \left([g(m), g(M)] \right)$. *Then*

$$f(A) = \frac{1}{\Gamma((\overline{m}+1)\nu)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{(\overline{m}+1)\nu - 1} \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(z)\, dz \right) dE_\lambda \tag{90}$$

$$= \frac{1}{\Gamma((\overline{m}+1)\nu)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{m}^{\lambda} (g(\lambda) - g(s))^{(\overline{m}+1)\nu - 1} \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(g(s))\, g'(s)\, ds \right) dE_\lambda.$$

Proof. Use of (89). ∎

We need

Background 38 *([4]) Denote by*

$$D_{g(M)-}^{\overline{m}\nu} = D_{g(M)-}^{\nu} D_{g(M)-}^{\nu} ... D_{g(M)-}^{\nu} \quad (\overline{m}\text{-times}),\ \overline{m} \in \mathbb{N}. \tag{91}$$

We will use the right fractional Taylor's formula

Theorem 39 *([4]) Let $0 < \nu < 1$.*

Assume that $\left(D_{g(M)-}^{i\nu} \left(f \circ g^{-1} \right) \right) \in C_{g(M)-}^{\nu} \left([g(m), g(M)] \right)$, *for all* $i = 0, 1, ..., \overline{m}$. *Assume also that* $\left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) \in C \left([g(m), g(M)] \right)$. *Then*

$$f(\lambda) = \frac{1}{\Gamma((\overline{m}+1)\nu)} \int_{g(\lambda)}^{g(M)} (z - g(\lambda))^{(\overline{m}+1)\nu - 1} \left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(z)\, dz$$

$$= \frac{1}{\Gamma((\overline{m}+1)\nu)} \int_{\lambda}^{M} (g(s) - g(\lambda))^{(\overline{m}+1)\nu - 1} \left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(g(s))\, g'(s)\, ds, \tag{92}$$

all $m \leq \lambda \leq M$.

We present the operator right fractional representation formula

Theorem 40 *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$ for some real numbers $m < M$, $\{E_\lambda\}_\lambda$ be its spectral family, I be a closed subinterval on $\mathbb{R}$ with $[m, M] \subset \overset{\circ}{I}$, and $0 < \nu < 1$. Let $f, g : I \to \mathbb{R}$. Assume that $g : [m, M] \to \mathbb{R}$ is strictly increasing function, $f \in C^1([m, M])$, $g \in C^1([m, M])$, and $g^{-1} \in C^1([g(m), g(M)])$. Furthermore we suppose that $\left(D_{g(M)-}^{i\nu} \left(f \circ g^{-1} \right) \right) \in C_{g(M)-}^{\nu} \left([g(m), g(M)] \right)$, for all $i = 0, 1, ..., \overline{m}$, and $\left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) \in C \left([g(m), g(M)] \right)$. Then*

$$f(A) = \frac{1}{\Gamma((\overline{m}+1)\nu)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{g(\lambda)}^{g(M)} (z - g(\lambda))^{(\overline{m}+1)\nu - 1} \left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(z)\, dz \right) dE_\lambda$$

$$= \frac{1}{\Gamma((\overline{m}+1)\nu)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{\lambda}^{M} (g(s) - g(\lambda))^{(\overline{m}+1)\nu - 1} \left(D_{g(M)-}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right)(g(s))\, g'(s)\, ds \right) dE_\lambda. \tag{93}$$

Proof. Use of (92). ■

Note 41 *From now on in this article let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.*

We make

Remark 42 *(to Theorems 1, 2) Assume $f^{(k)}(m)=0$, for $k=0,1,...,n-1$, then $\left(f\circ g^{-1}\right)^{(k)}(g(m))=0$, all $k=0,1,...,n-1$, and*

$$f(\lambda)=\frac{1}{(n-1)!}\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt, \tag{94}$$

$\forall\ \lambda\in[m,M]$. Hence, if $g(\lambda)\geq g(m)$, we have

$$|f(\lambda)|\leq\frac{1}{(n-1)!}\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{n-1}\left|\left(f\circ g^{-1}\right)^{(n)}(t)\right|dt$$

$$\leq\frac{1}{(n-1)!}\left(\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{p(n-1)}\,dt\right)^{\frac{1}{p}}\left(\int_{g(m)}^{g(\lambda)}\left|\left(f\circ g^{-1}\right)^{(n)}(t)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$=\frac{1}{(n-1)!}\frac{(g(\lambda)-g(m))^{(n-1)+\frac{1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}}\left(\int_{g(m)}^{g(\lambda)}\left|\left(f\circ g^{-1}\right)^{(n)}(t)\right|^{q}dt\right)^{\frac{1}{q}}=$$

$$\frac{1}{(n-1)!}\frac{(g(\lambda)-g(m))^{n-\frac{1}{q}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,[g(m),g(\lambda)]}. \tag{95}$$

We have proved that (if $g(\lambda)\geq g(m)$)

$$\left|\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt\right|\leq$$

$$\frac{(g(\lambda)-g(m))^{n-\frac{1}{q}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,[g(m),g(\lambda)]}. \tag{96}$$

Next, if $g(\lambda)\leq g(m)$, then

$$|f(\lambda)|=\frac{1}{(n-1)!}\left|\int_{g(\lambda)}^{g(m)}(g(\lambda)-t)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt\right| \tag{97}$$

$$\leq\frac{1}{(n-1)!}\int_{g(\lambda)}^{g(m)}(t-g(\lambda))^{n-1}\left|\left(f\circ g^{-1}\right)^{(n)}(t)\right|dt$$

$$\leq\frac{1}{(n-1)!}\left(\int_{g(\lambda)}^{g(m)}(t-g(\lambda))^{p(n-1)}\,dt\right)^{\frac{1}{p}}\left(\int_{g(\lambda)}^{g(m)}\left|\left(f\circ g^{-1}\right)^{(n)}(t)\right|^{q}dt\right)^{\frac{1}{q}} \tag{98}$$

$$=\frac{1}{(n-1)!}\frac{(g(m)-g(\lambda))^{n-\frac{1}{q}}}{(p(n-1)+1)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,[g(\lambda),g(m)]}.$$

We have proved that (if $g(\lambda)\leq g(m)$)

$$\left|\int_{g(m)}^{g(\lambda)}(g(\lambda)-t)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt\right|\leq$$

$$\frac{\left(g\left(m\right)-g\left(\lambda\right)\right)^{n-\frac{1}{q}}}{\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,\left[g(\lambda),g(m)\right]}. \tag{99}$$

Conclusion: *it holds*

$$\left|\int_{g(m)}^{g(\lambda)}\left(g\left(\lambda\right)-t\right)^{n-1}\left(f\circ g^{-1}\right)^{(n)}\left(t\right)dt\right|\leq$$

$$\frac{\left|g\left(m\right)-g\left(\lambda\right)\right|^{n-\frac{1}{q}}}{\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,g([m,M])},. \tag{100}$$

$\forall\ \lambda\in\left[m,M\right].$

By Note 3, we have

$$f\left(A\right)=\frac{1}{\left(n-1\right)!}\int_{m-0}^{M}\left(\int_{g(m)}^{g(\lambda)}\left(g\left(\lambda\right)-t\right)^{n-1}\left(f\circ g^{-1}\right)^{(n)}\left(t\right)dt\right)dE_{\lambda}, \tag{101}$$

which means

$$\left\langle f\left(A\right)x,x\right\rangle=\frac{1}{\left(n-1\right)!}\int_{m-0}^{M}\left(\int_{g(m)}^{g(\lambda)}\left(g\left(\lambda\right)-t\right)^{n-1}\left(f\circ g^{-1}\right)^{(n)}\left(t\right)dt\right)d\left\langle E_{\lambda}x,x\right\rangle, \tag{102}$$

$\forall\ x\in H.$

It is well known that ([10]) $g_x\left(\lambda\right):=\left\langle E_{\lambda}x,x\right\rangle$ *is nondecreasing and right continuous in* λ *on* $\left[m,M\right].$

Therefore it holds

$$\begin{aligned}\left|\left\langle f\left(A\right)x,x\right\rangle\right| &\overset{(102)}{\leq}\frac{1}{\left(n-1\right)!}\int_{m-0}^{M}\left|\int_{g(m)}^{g(\lambda)}\left(g\left(\lambda\right)-t\right)^{n-1}\left(f\circ g^{-1}\right)^{(n)}\left(t\right)dt\right|d\left\langle E_{\lambda}x,x\right\rangle\\ &\overset{(100)}{\leq}\frac{1}{\left(n-1\right)!}\int_{m-0}^{M}\frac{\left|g\left(m\right)-g\left(\lambda\right)\right|^{n-\frac{1}{q}}}{\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,g([m,M])}d\left\langle E_{\lambda}x,x\right\rangle \qquad (103)\\ &=\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,g([m,M])}}{\left(n-1\right)!\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}\left(\int_{m-0}^{M}\left|g\left(m\right)-g\left(\lambda\right)\right|^{n-\frac{1}{q}}d\left\langle E_{\lambda}x,x\right\rangle\right)\\ &=\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\right\|_{q,g([m,M])}}{\left(n-1\right)!\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}\left\langle\left|g\left(m\right)1_{H}-g\left(A\right)\right|^{n-\frac{1}{q}}x,x\right\rangle, \qquad (104)\end{aligned}$$

$\forall\ x\in H.$

We have proved

Theorem 43 *Here all as in Theorem 2, with $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{q, g([m,M])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}} \left\langle |g(m)1_H - g(A)|^{n-\frac{1}{q}} x, x \right\rangle, \quad (105)$$

$\forall\ x \in H$.

Inequality (105) means that

$$\|f(A)\| \leq \frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{q, g([m,M])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}} \left\| |g(m)1_H - g(A)|^{n-\frac{1}{q}} \right\|, \quad (106)$$

and in particular,

$$f(A) \leq \frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{q, g([m,M])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}} |g(m)1_H - g(A)|^{n-\frac{1}{q}}. \quad (107)$$

Remark 44 *(to Theorems 5, 6) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\alpha \notin \mathbb{N}$.*

If $\left(f \circ g^{-1}\right)^{(k)}(g(m)) = 0$, for $k = 0, 1, ..., n-1$, then

$$f(\lambda) = \frac{1}{\Gamma(\alpha)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{\alpha-1} \left(\left(D_{m+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\,dz, \quad (108)$$

$\forall\ \lambda \in [m, M]$.

Hence we have

$$|f(\lambda)| \leq \frac{1}{\Gamma(\alpha)} \int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{\alpha-1} \left| \left(\left(D_{m+;g}^{\alpha} f\right) \circ g^{-1}\right)(z) \right| dz \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{p(\alpha-1)}\,dz \right)^{\frac{1}{p}} \left(\int_{g(m)}^{g(\lambda)} \left| \left(\left(D_{m+;g}^{\alpha} f\right) \circ g^{-1}\right)(z) \right|^q dz \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{(g(\lambda) - g(m))^{\frac{(p(\alpha-1)+1)}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left(D_{m+;g}^{\alpha} f\right) \circ g^{-1} \right\|_{q, [g(m), g(M)]}. \quad (109)$$

We have proved that

$$\left| \int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{\alpha-1} \left(\left(D_{m+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\,dz \right| \leq \quad (110)$$

$$\frac{(g(\lambda) - g(m))^{\alpha-\frac{1}{q}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left(D_{m+;g}^{\alpha} f\right) \circ g^{-1} \right\|_{q, [g(m), g(M)]},$$

$\forall\ \lambda \in [m, M]$, *with $\alpha > \frac{1}{q}$.*

By Note 7 we have

$$f(A)=\frac{1}{\Gamma(\alpha)}\int_{m-0}^{M}\left(\int_{g(m)}^{g(\lambda)}(g(\lambda)-z)^{\alpha-1}\left(\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz\right)dE_{\lambda}, \tag{111}$$

which means

$$\langle f(A)x,x\rangle=$$

$$\frac{1}{\Gamma(\alpha)}\int_{m-0}^{M}\left(\int_{g(m)}^{g(\lambda)}(g(\lambda)-z)^{\alpha-1}\left(\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz\right)d\langle E_{\lambda}x,x\rangle, \tag{112}$$

$\forall\ x\in H$.

Therefore it holds

$$|\langle f(A)x,x\rangle|\overset{(112)}{\leq}$$

$$\frac{1}{\Gamma(\alpha)}\int_{m-0}^{M}\left|\int_{g(m)}^{g(\lambda)}(g(\lambda)-z)^{\alpha-1}\left(\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz\right|d\langle E_{\lambda}x,x\rangle\overset{(110)}{\leq}$$

$$\frac{1}{\Gamma(\alpha)}\int_{m-0}^{M}\frac{(g(\lambda)-g(m))^{\alpha-\frac{1}{q}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}d\langle E_{\lambda}x,x\rangle=$$

$$\frac{\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\int_{m-0}^{M}(g(\lambda)-g(m))^{\alpha-\frac{1}{q}}d\langle E_{\lambda}x,x\rangle=$$

$$\frac{\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\langle(g(A)-g(m)1_{H})^{\alpha-\frac{1}{q}}x,x\right\rangle, \tag{113}$$

$\forall\ x\in H$.

We have proved

Theorem 45 *Here all as in Theorem 6, with* $\left(f\circ g^{-1}\right)^{(k)}(g(m))=0$, *for* $k=0,1,...,n-1$, $\alpha>\frac{1}{q}$, $\alpha\notin\mathbb{N}$. *Then*

$$|\langle f(A)x,x\rangle|\leq\frac{\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\langle(g(A)-g(m)1_{H})^{\alpha-\frac{1}{q}}x,x\right\rangle, \tag{114}$$

$\forall\ x\in H$.

Inequality (114) means that

$$\|f(A)\|\leq\frac{\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|(g(A)-g(m)1_{H})^{\alpha-\frac{1}{q}}\right\|, \tag{115}$$

and in particular

$$f(A)\leq\frac{\left\|\left(D_{m+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}(g(A)-g(m)1_{H})^{\alpha-\frac{1}{q}}. \tag{116}$$

We also present

Theorem 46 *Here all as in Theorem 10, with* $\left(f\circ g^{-1}\right)^{(k)}(g(M))=0$, *for* $k=0,1,...,n-1$, $\alpha>\frac{1}{q}$, $\alpha\notin\mathbb{N}$. *Then*

$$|\langle f(A)x,x\rangle|\leq\frac{\left\|\left(D_{M-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\langle (g(M)1_H-g(A))^{\alpha-\frac{1}{q}}x,x\right\rangle, \tag{117}$$

$\forall\ x\in H$.

Inequality (117) means that

$$\|f(A)\|\leq\frac{\left\|\left(D_{M-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|(g(M)1_H-g(A))^{\alpha-\frac{1}{q}}\right\|, \tag{118}$$

and in particular

$$f(A)\leq\frac{\left\|\left(D_{M-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}(g(M)1_H-g(A))^{\alpha-\frac{1}{q}}. \tag{119}$$

Proof. Similar to Theorem 45. ∎

We give

Theorem 47 *Here all as in Theorem 14.*

If $\nu\geq1$, *we assume that* $\left(f\circ g^{-1}\right)^{(k)}(g(m))=0$, $k=0,1,...,[\nu]-1$, *and always* $\nu>\frac{1}{q}$. *Then*

$$|\langle f(A)x,x\rangle|\leq\frac{\left\|D_{g(m)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left\langle (g(A)-g(m)1_H)^{\nu-\frac{1}{q}}x,x\right\rangle, \tag{120}$$

$\forall\ x\in H$.

Inequality (120) means that

$$\|f(A)\|\leq\frac{\left\|D_{g(m)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}\left\|(g(A)-g(m)1_H)^{\nu-\frac{1}{q}}\right\|, \tag{121}$$

and in particular

$$f(A)\leq\frac{\left\|D_{g(m)}^{\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}}(g(A)-g(m)1_H)^{\nu-\frac{1}{q}}. \tag{122}$$

Proof. Similar to Theorem 45. ∎

We give

Theorem 48 *Here all as in Theorem 18.*

If $\nu \geq 1$*, we assume that* $\left(f \circ g^{-1}\right)^{(k)} (g(M)) = 0,\ k = 0, 1, ..., [\nu] - 1$*, and always* $\nu > \frac{1}{q}$*. Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\left\| D^{\nu}_{g(M)-} \left(f \circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \left\langle \left(g(M)1_H - g(A)\right)^{\nu-\frac{1}{q}} x, x\right\rangle, \tag{123}$$

$\forall\ x \in H$.

Inequality (123) means that

$$\|f(A)\| \leq \frac{\left\| D^{\nu}_{g(M)-} \left(f \circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \left\| \left(g(M)1_H - g(A)\right)^{\nu-\frac{1}{q}} \right\|, \tag{124}$$

and in particular

$$f(A) \leq \frac{\left\| D^{\nu}_{g(M)-} \left(f \circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma(\nu)\left(p(\nu-1)+1\right)^{\frac{1}{p}}} \left(g(M)1_H - g(A)\right)^{\nu-\frac{1}{q}}. \tag{125}$$

Proof. Similar to Theorem 45. ■

We make

Remark 49 *(to Theorem 21) Assume* $\left(D^{i\alpha}_{*m} f\right)(m) = 0,\ i = 0, 1, ..., n$*, then*

$$f(\lambda) = \frac{1}{\Gamma((n+1)\alpha)} \int_m^{\lambda} (\lambda - t)^{(n+1)\alpha - 1} \left(D^{(n+1)\alpha}_{*m} f\right)(t)\, dt, \tag{126}$$

$\forall\ \lambda \in [m, M]$.

We obtain

$$|f(\lambda)| \leq \frac{1}{\Gamma((n+1)\alpha)} \int_m^{\lambda} (\lambda - t)^{(n+1)\alpha - 1} \left|\left(D^{(n+1)\alpha}_{*m} f\right)(t)\right| dt \leq \tag{127}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \left(\int_m^{\lambda} (\lambda - t)^{p((n+1)\alpha-1)} dt\right)^{\frac{1}{p}} \left(\int_m^{\lambda} \left|\left(D^{(n+1)\alpha}_{*m} f\right)(t)\right|^q dt\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma((n+1)\alpha)} \frac{(\lambda - m)^{\frac{(p((n+1)\alpha-1)+1)}{p}}}{\left(p((n+1)\alpha - 1) + 1\right)^{\frac{1}{p}}} \left\| D^{(n+1)\alpha}_{*m} f \right\|_{q,[m,M]}. \tag{128}$$

We have proved that

$$\left| \int_m^{\lambda} (\lambda - t)^{(n+1)\alpha - 1} \left(D^{(n+1)\alpha}_{*m} f\right)(t)\, dt \right| \leq$$

$$\frac{(\lambda - m)^{(n+1)\alpha - \frac{1}{q}}}{(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}, \tag{129}$$

$\forall\ \lambda \in [m, M]$, *under* $\alpha > \frac{1}{q(n+1)}$.

By Note 22 we have

$$\langle f(A)x, x\rangle = \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{m}^{\lambda} (\lambda - t)^{(n+1)\alpha - 1} \left(D_{*m}^{(n+1)\alpha} f \right)(t)\, dt \right) d\langle E_\lambda x, x\rangle, \tag{130}$$

$\forall\ x \in H$.

Therefore

$$\langle f(A)x, x\rangle \overset{(129)}{\leq}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{m-0}^{M} \frac{\left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}}{(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} (\lambda - m)^{(n+1)\alpha - \frac{1}{q}}\, d\langle E_\lambda x, x\rangle =$$

$$\frac{\left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\langle (A - m1_H)^{(n+1)\alpha - \frac{1}{q}} x, x \right\rangle, \tag{131}$$

$\forall\ x \in H$.

We have proved:

Theorem 50 *Here all as in Theorem 21, with* $\left(D_{*m}^{i\alpha} f\right)(m) = 0$, $i = 0, 1, ..., n$. *Then*

$$|\langle f(A)x, x\rangle| \leq \frac{\left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\langle (A - m1_H)^{(n+1)\alpha - \frac{1}{q}} x, x \right\rangle, \tag{132}$$

$\forall\ x \in H$.

Inequality (132) means that

$$\|f(A)\| \leq \frac{\left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\| (A - m1_H)^{(n+1)\alpha - \frac{1}{q}} \right\|, \tag{133}$$

and in particular,

$$f(A) \leq \frac{\left\| D_{*m}^{(n+1)\alpha} f \right\|_{q,[m,M]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} (A - m1_H)^{(n+1)\alpha - \frac{1}{q}}, \tag{134}$$

all inequalities here under $\alpha > \frac{1}{(n+1)q}$.

It follows

Theorem 51 *Here all as in Theorem 25, with* $\left(D_{M-}^{i\alpha}f\right)(M)=0$, $i=0,1,...,n$. *Then*

$$\left|\left\langle f(A)x,x\right\rangle\right| \leq \frac{\left\|D_{M-}^{(n+1)\alpha}f\right\|_{q,[m,M]}}{\Gamma\left((n+1)\alpha\right)\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\langle\left(M1_H-A\right)^{(n+1)\alpha-\frac{1}{q}}x,x\right\rangle, \tag{135}$$

$\forall\ x\in H$.

Inequality (135) means that

$$\left\|f(A)\right\| \leq \frac{\left\|D_{M-}^{(n+1)\alpha}f\right\|_{q,[m,M]}}{\Gamma\left((n+1)\alpha\right)\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(M1_H-A\right)^{(n+1)\alpha-\frac{1}{q}}\right\|, \tag{136}$$

and in particular,

$$f(A) \leq \frac{\left\|D_{M-}^{(n+1)\alpha}f\right\|_{q,[m,M]}}{\Gamma\left((n+1)\alpha\right)\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left(M1_H-A\right)^{(n+1)\alpha-\frac{1}{q}}, \tag{137}$$

all here under $\alpha>\frac{1}{(n+1)q}$.

Proof. As in Theorem 50. ■

We make

Remark 52 *(to Theorem 29) Assume* $\left(D_{m+;g}^{i\alpha}f\right)(m)=0$, $i=0,1,...,n$, *then we have that*

$$f(\lambda)=\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_m^{\lambda}\left(g(\lambda)-g(t)\right)^{(n+1)\alpha-1}g'(t)\left(D_{m+;g}^{(n+1)\alpha}f\right)(t)\,dt \overset{(by\ [13])}{=}$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\int_{g(m)}^{g(\lambda)}\left(g(\lambda)-z\right)^{(n+1)\alpha-1}\left(\left(D_{m+;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right)(z)\,dz, \tag{138}$$

$\forall\ \lambda\in[m,M]$.

Hence it holds (with $\alpha>\frac{1}{q(n+1)}$*)*

$$\left|f(\lambda)\right| \leq \frac{1}{\Gamma\left((n+1)\alpha\right)}\left(\int_{g(m)}^{g(\lambda)}\left(g(\lambda)-z\right)^{p((n+1)\alpha-1)}dz\right)^{\frac{1}{p}}\cdot$$

$$\left(\int_{g(m)}^{g(\lambda)}\left|\left(\left(D_{m+;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right)(z)\right|^{q}dz\right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma\left((n+1)\alpha\right)}\frac{\left(g(\lambda)-g(m)\right)^{(n+1)\alpha-\frac{1}{q}}}{\left(p\left((n+1)\alpha-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(D_{m+;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}, \tag{139}$$

$\forall\ \lambda \in [m, M]$.

By Note 30, we have

$$\langle f(A)x, x\rangle = \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_{m-0}^{M} \left(\int_{g(m)}^{g(\lambda)} (g(\lambda) - z)^{(n+1)\alpha - 1} \left(\left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right)(z)\, dz \right) d\langle E_\lambda x, x\rangle, \tag{140}$$

$\forall\ x \in H$.

Therefore we get

$$|\langle f(A)x, x\rangle| \overset{(139)}{\leq} \frac{1}{\Gamma((n+1)\alpha)} \cdot$$

$$\int_{m-0}^{M} \left[\frac{(g(\lambda) - g(m))^{(n+1)\alpha - \frac{1}{q}}}{(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\| \left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right\|_{q,[g(m),g(M)]} \right] d\langle E_\lambda x, x\rangle \tag{141}$$

$$= \frac{\left\| \left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\langle (g(A) - g(m) 1_H)^{(n+1)\alpha - \frac{1}{q}} x, x \right\rangle,$$

$\forall\ x \in H$.

We have proved:

Theorem 53 *Here all as in Theorem 29, with* $\left(D_{m+;g}^{i\alpha} f \right)(m) = 0$, $i = 0, 1, ..., n$. *Then*

$$|\langle f(A)x, x\rangle| \leq$$

$$\frac{\left\| \left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\langle (g(A) - g(m) 1_H)^{(n+1)\alpha - \frac{1}{q}} x, x \right\rangle, \tag{142}$$

$\forall\ x \in H$.

Inequality (142) means that

$$\|f(A)\| \leq \frac{\left\| \left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} \left\| (g(A) - g(m) 1_H)^{(n+1)\alpha - \frac{1}{q}} \right\|, \tag{143}$$

and in particular,

$$f(A) \leq \frac{\left\| \left(D_{m+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha - 1) + 1)^{\frac{1}{p}}} (g(A) - g(m) 1_H)^{(n+1)\alpha - \frac{1}{q}}, \tag{144}$$

all inequalities here under $\alpha > \frac{1}{(n+1)q}$.

It follows

Theorem 54 *Here all as in Theorem 33, with* $\left(D_{M-;g}^{i\alpha}f\right)(M)=0$, $i=0,1,...,n$. *Then*

$$|\langle f(A)x,x\rangle| \leq$$

$$\frac{\left\|\left(D_{M-;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}\left\langle (g(M)1_H-g(A))^{(n+1)\alpha-\frac{1}{q}}x,x\right\rangle, \tag{145}$$

$\forall\ x\in H$.

Inequality (145) means that

$$\|f(A)\| \leq \frac{\left\|\left(D_{M-;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}\left\|(g(M)1_H-g(A))^{(n+1)\alpha-\frac{1}{q}}\right\|, \tag{146}$$

and in particular,

$$f(A) \leq \frac{\left\|\left(D_{M-;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right\|_{q,[g(m),g(M)]}}{\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}}(g(M)1_H-g(A))^{(n+1)\alpha-\frac{1}{q}}, \tag{147}$$

all inequalities here under $\alpha>\frac{1}{(n+1)q}$.

Proof. As in Theorem 53. ■

We give

Theorem 55 *Here all as in Theorem 37 with* $\nu>\frac{1}{q(\overline{m}+1)}$. *Then*

$$|\langle f(A)x,x\rangle| \leq$$

$$\frac{\left\|D_{g(m)}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma((\overline{m}+1)\nu)(p((\overline{m}+1)\nu-1)+1)^{\frac{1}{p}}}\left\langle (g(A)-g(m)1_H)^{(\overline{m}+1)\nu-\frac{1}{q}}x,x\right\rangle, \tag{148}$$

$\forall\ x\in H$.

Inequality (148) means that

$$\|f(A)\| \leq \frac{\left\|D_{g(m)}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma((\overline{m}+1)\nu)(p((\overline{m}+1)\nu-1)+1)^{\frac{1}{p}}}\left\|(g(A)-g(m)1_H)^{(\overline{m}+1)\nu-\frac{1}{q}}\right\|, \tag{149}$$

and in particular,

$$f(A) \leq \frac{\left\|D_{g(m)}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma((\overline{m}+1)\nu)(p((\overline{m}+1)\nu-1)+1)^{\frac{1}{p}}}(g(A)-g(m)1_H)^{(\overline{m}+1)\nu-\frac{1}{q}}. \tag{150}$$

Proof. As in Theorem 53. ■

It follows

Theorem 56 *Here all as in Theorem 40 with $\nu > \frac{1}{q(\overline{m}+1)}$. Then*

$$|\langle f(A)x,x\rangle| \leq$$

$$\frac{\left\| D_{g(M)-}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma\left(\left(\overline{m}+1\right)\nu\right)\left(p\left(\left(\overline{m}+1\right)\nu-1\right)+1\right)^{\frac{1}{p}}}\left\langle \left(g(M)1_H-g(A)\right)^{(\overline{m}+1)\nu-\frac{1}{q}}x,x\right\rangle, \tag{151}$$

$\forall\ x\in H$.

Inequality (151) means that

$$\|f(A)\| \leq \frac{\left\| D_{g(M)-}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma\left(\left(\overline{m}+1\right)\nu\right)\left(p\left(\left(\overline{m}+1\right)\nu-1\right)+1\right)^{\frac{1}{p}}}\left\|\left(g(M)1_H-g(A)\right)^{(\overline{m}+1)\nu-\frac{1}{q}}\right\|, \tag{152}$$

and in particular,

$$f(A) \leq \frac{\left\| D_{g(M)-}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right\|_{q,[g(m),g(M)]}}{\Gamma\left(\left(\overline{m}+1\right)\nu\right)\left(p\left(\left(\overline{m}+1\right)\nu-1\right)+1\right)^{\frac{1}{p}}}\left(g(M)1_H-g(A)\right)^{(\overline{m}+1)\nu-\frac{1}{q}}. \tag{153}$$

Proof. As in Theorem 53. ■

We need

Definition 57 *Let the real valued function $f\in C([m,M])$, and we consider*

$$g(t)=\int_m^t f(z)\,dz,\ \ \forall\ t\in[m,M], \tag{154}$$

then $g\in C([m,M])$.

We denote by

$$\int_{m1_H}^{A} f := \Phi(g) = g(A). \tag{155}$$

We understand and write that ($r>0$)

$$g^r(A)=\Phi(g^r) =: \left(\int_{m1_H}^{A} f\right)^r. \tag{156}$$

Clearly $\left(\int_{m1_H}^{A} f\right)^r$ is a self adjoint operator on H, for any $r>0$.

Definition 58 *Let $f : [m, M] \to \mathbb{R}$ be continuous. We consider*

$$g(t) = \int_t^M f(z)\,dz, \ \ \forall\ t \in [m, M], \tag{157}$$

then $g \in C([m, M])$.

We denote by

$$\int_A^{M 1_H} f := \Phi(g) = g(A). \tag{158}$$

We denote also

$$g^r(A) = \Phi(g^r) =: \left(\int_A^{M 1_H} f\right)^r, \quad r > 0. \tag{159}$$

Clearly $\left(\int_A^{M 1_H} f\right)^r$ is a self adjoint operator on H, for any $r > 0$.

We mention a left fractional Poincaré type inequality:

Theorem 59 *(by [7], pp. 385-386) Let $g \in C^1([m, M])$ and strictly increasing. Suppose that $F_\rho := D_{m+;g}^{\rho\alpha} f$, for $\rho = 0, 1, ..., n+1$, $n \in \mathbb{N}$, fulfill: $F_\rho \circ g^{-1} \in AC([g(m), g(M)])$ and $\left(F_\rho \circ g^{-1}\right)' \circ g \in L_\infty([m, M])$, where α as in (160) next.*

Assume that $\left(D_{m+;g}^{i\alpha} f\right)(m) = 0$, $i = 0, 1, ..., n$. Let $\gamma > 0$ with $\lceil \gamma \rceil = \overline{m}$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We further assume that

$$1 \geq \alpha > \max\left(\frac{\overline{m} + (k-1)\gamma}{n+1}, \frac{k\gamma q + 1}{(n+1)q}\right), \tag{160}$$

where $k \in \mathbb{N}$.

Then

$$\int_m^\lambda \left|D_{m+;g}^{k\gamma} f(t)\right|^q dt \leq \frac{(g(\lambda) - g(m))^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (\lambda - m)}{(\Gamma((n+1)\alpha - k\gamma))^q (p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{q}{p}}} \cdot$$

$$\left(\int_{g(m)}^{g(\lambda)} \left|\left(\left(D_{m+;g}^{(n+1)\alpha} f\right) \circ g^{-1}\right)(z)\right|^q dz\right) \tag{161}$$

$$= \frac{(g(\lambda) - g(m))^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (\lambda - m)}{(\Gamma((n+1)\alpha - k\gamma))^q (p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{q}{p}}} \cdot$$

$$\left(\int_m^\lambda \left|\left(D_{m+;g}^{(n+1)\alpha} f\right)(t)\right|^q g'(t)\,dt\right),$$

$\forall\ \lambda \in [m, M]$.

Here we have that $\left(D_{m+;g}^{k\gamma} f\right), \left(D_{m+;g}^{(n+1)\alpha} f\right) \in C([m, M])$.

Using (161) and properties (P) and (ii), we derive the operator left fractional Poincaré inequality:

Theorem 60 *All as in Theorem 59. Then*

$$\int_{m1_H}^{A} \left| D_{m+;g}^{k\gamma} f \right|^q \leq$$

$$\frac{\left(g(A) - g(m) 1_H\right)^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (A - m1_H)}{\left(\Gamma\left((n+1)\alpha - k\gamma\right)\right)^q \left(p\left((n+1)\alpha - k\gamma - 1\right) + 1\right)^{\frac{q}{p}}} \left(\int_{m1_H}^{A} \left| D_{m+;g}^{(n+1)\alpha} f \right|^q g' \right). \tag{162}$$

We mention a right fractionalPoincaré type inequality:

Theorem 61 *(by [7], p. 387) Let $g \in C^1([m, M])$ and strictly increasing. Suppose that $F_\rho := D_{M-;g}^{\rho\alpha} f$, for $\rho = 0, 1, ..., n+1$, $n \in \mathbb{N}$, fulfill: $F_\rho \circ g^{-1} \in AC([g(m), g(M)])$ and $\left(F_\rho \circ g^{-1}\right)' \circ g \in L_\infty([m, M])$, where α as in (163) next. Assume that $\left(D_{M-;g}^{i\alpha} f\right)(M) = 0$, $i = 0, 1, ..., n$. Let $\gamma > 0$ with $\lceil \gamma \rceil = \overline{m}$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We further assume that*

$$1 \geq \alpha > \max\left(\frac{\overline{m} + (k-1)\gamma}{n+1}, \frac{k\gamma q + 1}{(n+1) q} \right), \tag{163}$$

where $k \in \mathbb{N}$. Then

$$\int_{\lambda}^{M} \left| D_{M-;g}^{k\gamma} f(t) \right|^q dt \leq \frac{\left(g(M) - g(\lambda)\right)^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (M - \lambda)}{\left(\Gamma\left((n+1)\alpha - k\gamma\right)\right)^q \left(p\left((n+1)\alpha - k\gamma - 1\right) + 1\right)^{\frac{q}{p}}} \cdot \tag{164}$$

$$\left(\int_{g(\lambda)}^{g(M)} \left| \left(\left(D_{M-;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right)(z) \right|^q dz \right)$$

$$= \frac{\left(g(M) - g(\lambda)\right)^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (M - \lambda)}{\left(\Gamma\left((n+1)\alpha - k\gamma\right)\right)^q \left(p\left((n+1)\alpha - k\gamma - 1\right) + 1\right)^{\frac{q}{p}}} \cdot$$

$$\left(\int_{\lambda}^{M} \left| \left(D_{M-;g}^{(n+1)\alpha} f \right)(t) \right|^q g'(t)\, dt \right),$$

$\forall\ \lambda \in [m, M]$.

Here we have that $\left(D_{M-;g}^{k\gamma} f\right), \left(D_{M-;g}^{(n+1)\alpha} f\right) \in C([m, M])$.

Using (164) and properties (P) and (ii), we derive the operator right fractional Poincaré inequality:

Theorem 62 *All as in Theorem 61. Then*

$$\int_A^{M1_H} \left| D_{M-;g}^{k\gamma} f \right|^q \leq$$

$$\frac{(g(M)1_H - g(A))^{q((n+1)\alpha - k\gamma - 1) + \frac{q}{p}} (M1_H - A)}{(\Gamma((n+1)\alpha - k\gamma))^q (p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{q}{p}}} \left(\int_A^{M1_H} \left| D_{M-;g}^{(n+1)\alpha} f \right|^q g' \right). \tag{165}$$

We mention the following Sobolev type left fractional inequality:

Theorem 63 *(by [6], pp. 493-495) Let* $0 < \alpha < 1$, $f : [m, M] \to \mathbb{R}$ *such that* $f' \in L_\infty([m, M])$. *Assume that* $D_{*m}^{\overline{k}\alpha} f \in C([m, M])$, $\overline{k} = 0, 1, ..., n+1$; $n \in \mathbb{N}$. *Suppose that* $(D_{*m}^{i\alpha} f)(m) = 0$, *for* $i = 0, 2, 3, ..., n$. *Let* $\gamma > 0$ *with* $\lceil \gamma \rceil = \overline{m}$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *We further assume that* ($k \in \mathbb{N}$)

$$1 > \alpha > \max\left(\frac{\overline{m} + (k-1)\gamma}{n+1}, \frac{k\gamma q + 1}{(n+1) q} \right), \tag{166}$$

Let $r \geq 1$. *Then*

$$\left(\int_m^\lambda \left| (D_{*m}^{k\gamma} f)(t) \right|^r dt \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma((n+1)\alpha - k\gamma)} \cdot$$

$$\frac{(\lambda - m)^{(n+1)\alpha - k\gamma - \frac{1}{q} + \frac{1}{r}}}{(p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{1}{p}} \left[r\left((n+1)\alpha - k\gamma - \frac{1}{q} \right) + 1 \right]^{\frac{1}{r}}} \cdot$$

$$\left(\int_m^\lambda \left| \left(D_{*m}^{(n+1)\alpha} f \right)(t) \right|^q dt \right)^{\frac{1}{q}}, \tag{167}$$

$\forall \lambda \in [m, M]$.

Here $\left(D_{*m}^{k\gamma} f \right) \in C([m, M])$.

Applying (167), using properties (P) and (ii), we get the following operator left fractional Sobolev type inequality:

Theorem 64 *All as in Theorem 63. Then*

$$\left(\int_{m1_H}^A \left| D_{*m}^{k\gamma} f \right|^r \right)^{\frac{1}{r}} \leq \frac{1}{\Gamma((n+1)\alpha - k\gamma)} \cdot$$

$$\frac{(A - m1_H)^{(n+1)\alpha - k\gamma - \frac{1}{q} + \frac{1}{r}}}{(p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{1}{p}} \left[r\left((n+1)\alpha - k\gamma - \frac{1}{q} \right) + 1 \right]^{\frac{1}{r}}} \cdot$$

$$\left(\int_{m1_H}^A \left| D_{*m}^{(n+1)\alpha} f \right|^q \right)^{\frac{1}{q}}. \tag{168}$$

We mention the following Sobolev type right fractional inequality:

Theorem 65 *(by [6], p. 496) Let* $0 < \alpha < 1$, $f : [m, M] \to \mathbb{R}$ *such that* $f' \in L_\infty([m, M])$. *Assume that* $D_{M-}^{\overline{k}\alpha} f \in C([m, M])$, $\overline{k} = 0, 1, ..., n+1$; $n \in \mathbb{N}$. *Suppose that* $\left(D_{M-}^{i\alpha} f\right)(M) = 0$, *for* $i = 0, 2, 3, ..., n$. *Let* $\gamma > 0$ *with* $\lceil \gamma \rceil = \overline{m}$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *We further assume that* ($k \in \mathbb{N}$)

$$1 > \alpha > \max\left(\frac{\overline{m} + (k-1)\gamma}{n+1}, \frac{k\gamma q + 1}{(n+1)q}\right), \tag{169}$$

Let $r \geq 1$. *Then*

$$\left(\int_\lambda^M \left|\left(D_{M-}^{k\gamma} f\right)(t)\right|^r dt\right)^{\frac{1}{r}} \leq \frac{1}{\Gamma((n+1)\alpha - k\gamma)} \cdot$$
$$\frac{(M-\lambda)^{(n+1)\alpha - k\gamma - \frac{1}{q} + \frac{1}{r}}}{(p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{1}{p}} \left[r\left((n+1)\alpha - k\gamma - \frac{1}{q}\right) + 1\right]^{\frac{1}{r}}} \cdot \tag{170}$$
$$\left(\int_\lambda^M \left|\left(D_{M-}^{(n+1)\alpha} f\right)(t)\right|^q dt\right)^{\frac{1}{q}},$$

$\forall\ \lambda \in [m, M]$.

Here $\left(D_{M-}^{k\gamma} f\right) \in C([m, M])$.

Applying (170), using properties (P) and (ii), we get the following operator right fractional Sobolev type inequality:

Theorem 66 *All as in Theorem 65. Then*

$$\left(\int_A^{M1_H} \left|D_{M-}^{k\gamma} f\right|^r\right)^{\frac{1}{r}} \leq \frac{1}{\Gamma((n+1)\alpha - k\gamma)} \cdot$$
$$\frac{(M1_H - A)^{(n+1)\alpha - k\gamma - \frac{1}{q} + \frac{1}{r}}}{(p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{1}{p}} \left[r\left((n+1)\alpha - k\gamma - \frac{1}{q}\right) + 1\right]^{\frac{1}{r}}} \cdot$$
$$\left(\int_A^{M1_H} \left|D_{M-}^{(n+1)\alpha} f\right|^q\right)^{\frac{1}{q}}. \tag{171}$$

We need the following left fractional Poincaré type inequality:

Theorem 67 *Let* $\alpha > 0$, $\lceil \alpha \rceil = n$, $\alpha \notin \mathbb{N}$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *with* $\alpha > \frac{1}{q}$, *and* g *be strictly increasing with* $g \in C^1([m, M])$. *We assume that* $\left(f \circ g^{-1}\right) \in AC^n([g(m), g(M)])$ *and* $\left(f \circ g^{-1}\right)^{(n)} \circ g \in L_\infty([m, M])$, *and* $\left(f \circ g^{-1}\right)^{(k)}(g(m)) = 0$, $k = 0, 1, ..., n-1$. *Then*

$$\int_m^\lambda |f(t)|^q \, dt \leq \frac{\|g\|_{\infty,[m,M]}^{q\alpha - 1} (\lambda - m)^{q\alpha}}{(\Gamma(\alpha))^q (p(\alpha - 1) + 1)^{q-1} q\alpha} \left(\int_m^\lambda \left| \left(D_{m+;g}^\alpha f\right)(t) \right|^q g'(t) \, dt \right), \tag{172}$$

$\forall \lambda \in [m, M]$.

Proof. By Theorem 5, since $\left(f \circ g^{-1}\right)^{(k)}(g(m)) = 0$, for $k = 0, 1, ..., n-1$, we get that

$$f(\lambda_1) = \frac{1}{\Gamma(\alpha)} \int_{g(m)}^{g(\lambda_1)} (g(\lambda_1) - z)^{\alpha - 1} \left(\left(D_{m+;g}^\alpha f\right) \circ g^{-1} \right)(z) \, dz, \tag{173}$$

$\forall \lambda_1 \in [m, M]$. Let $m \leq \lambda_1 \leq \lambda \leq M$. Hence we have

$$|f(\lambda_1)| \leq \frac{1}{\Gamma(\alpha)} \int_{g(m)}^{g(\lambda_1)} (g(\lambda_1) - z)^{\alpha - 1} \left| \left(\left(D_{m+;g}^\alpha f\right) \circ g^{-1} \right)(z) \right| dz \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_{g(m)}^{g(\lambda_1)} (g(\lambda_1) - z)^{p(\alpha - 1)} \, dz \right)^{\frac{1}{p}} \left(\int_{g(m)}^{g(\lambda_1)} \left| \left(\left(D_{m+;g}^\alpha f\right) \circ g^{-1} \right)(z) \right|^q dz \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{(g(\lambda_1) - g(m))^{\frac{(p(\alpha-1)+1)}{p}}}{(p(\alpha - 1) + 1)^{\frac{1}{p}}} \left(\int_{g(m)}^{g(\lambda)} \left| \left(\left(D_{m+;g}^\alpha f\right) \circ g^{-1} \right)(z) \right|^q dz \right)^{\frac{1}{q}} \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{\|g\|_{\infty,[m,M]}^{\alpha - \frac{1}{q}} (\lambda_1 - m)^{\alpha - \frac{1}{q}}}{(p(\alpha - 1) + 1)^{\frac{1}{p}}} \left(\int_m^\lambda \left| \left(D_{m+;g}^\alpha f\right)(t) \right|^q g'(t) \, dt \right)^{\frac{1}{q}}. \tag{174}$$

We have proved that

$$|f(\lambda_1)|^q \leq \frac{\|g\|_{\infty,[m,M]}^{q\alpha - 1} (\lambda_1 - m)^{q\alpha - 1}}{(\Gamma(\alpha))^q (p(\alpha - 1) + 1)^{q-1}} \left(\int_m^\lambda \left| \left(D_{m+;g}^\alpha f\right)(t) \right|^q g'(t) \, dt \right). \tag{175}$$

Consequently, by integration of (175), we obtain

$$\int_m^\lambda |f(\lambda_1)|^q \, d\lambda_1 \leq \frac{\|g\|_{\infty,[m,M]}^{q\alpha - 1} (\lambda - m)^{q\alpha - 1}}{(\Gamma(\alpha))^q (p(\alpha - 1) + 1)^{q-1} q\alpha} \left(\int_m^\lambda \left| \left(D_{m+;g}^\alpha f\right)(t) \right|^q g'(t) \, dt \right), \tag{176}$$

proving the claim. ■

Using (172), and properties (P) and (ii), we obtain the following operator Poincaré type left fractional inequality:

Theorem 68 *All as in Theorem 67. Then*

$$\left(\int_{m1_H}^{A} |f|^q\right) \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{q\alpha-1}}{(\Gamma(\alpha))^q (p(\alpha-1)+1)^{q-1} q\alpha} (A - m1_H)^{q\alpha} \left(\int_{m1_H}^{A} \left|D_{m+;g}^{\alpha} f\right|^q g'\right). \quad (177)$$

We need the following left fractional Sobolev type inequality:

Theorem 69 *All as in Theorem 67 and $r \geq 1$. Then*

$$\left(\int_m^{\lambda} |f(t)|^r dt\right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{\left(\alpha-\frac{1}{q}\right)} (\lambda - m)^{\left(\alpha-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left(\int_m^{\lambda} \left|\left(D_{m+;g}^{\alpha} f\right)(t)\right|^q g'(t)\, dt\right)^{\frac{1}{q}}, \quad (178)$$

$\forall\ \lambda \in [m, M]$.

Proof. As in the proof of Theorem 67 we find that ($m \leq \lambda_1 \leq \lambda \leq M$)

$$|f(\lambda_1)| \leq \frac{\|g\|_{\infty,[m,M]}^{\alpha-\frac{1}{q}} (\lambda_1 - m)^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_m^{\lambda} \left|\left(D_{m+;g}^{\alpha} f\right)(t)\right|^q g'(t)\, dt\right)^{\frac{1}{q}}. \quad (179)$$

Hence, by $r \geq 1$, we obtain

$$|f(\lambda_1)|^r \leq \frac{\|g\|_{\infty,[m,M]}^{r\left(\alpha-\frac{1}{q}\right)} (\lambda_1 - m)^{r\left(\alpha-\frac{1}{q}\right)}}{(\Gamma(\alpha))^r (p(\alpha-1)+1)^{\frac{r}{p}}} \left(\int_m^{\lambda} \left|\left(D_{m+;g}^{\alpha} f\right)(t)\right|^q g'(t)\, dt\right)^{\frac{r}{q}}. \quad (180)$$

Consequently, it holds

$$\int_m^{\lambda} |f(\lambda_1)|^r d\lambda_1 \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{r\left(\alpha-\frac{1}{q}\right)} (\lambda - m)^{r\left(\alpha-\frac{1}{q}\right)+1}}{(\Gamma(\alpha))^r (p(\alpha-1)+1)^{\frac{r}{p}} \left(r\left(a-\frac{1}{q}\right)+1\right)} \left(\int_m^{\lambda} \left|\left(D_{m+;g}^{\alpha} f\right)(t)\right|^q g'(t)\, dt\right)^{\frac{r}{q}}. \quad (181)$$

So that proving

$$\left(\int_m^{\lambda} |f(\lambda_1)|^r d\lambda_1\right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{\left(\alpha-\frac{1}{q}\right)} (\lambda-m)^{\left(\alpha-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(r\left(a-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}\left(\int_m^{\lambda}\left|\left(D_{m+;g}^{\alpha}f\right)(t)\right|^q g'(t)\,dt\right)^{\frac{1}{q}}. \tag{182}$$

■

Using (178), and properties (P) and (ii), we obtain the following operator Sobolev type left fractional inequality:

Theorem 70 *All as in Theorem 69. Then*

$$\left(\int_{m1_H}^{A}|f|^r\right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\alpha-\frac{1}{q}\right)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}\cdot$$

$$(A-m1_H)^{\left(\alpha-\frac{1}{q}+\frac{1}{r}\right)}\left(\int_{m1_H}^{A}\left|\left(D_{m+;g}^{\alpha}f\right)\right|^q g'\right)^{\frac{1}{q}}. \tag{183}$$

We need the following right fractional Poincaré type inequality:

Theorem 71 *Let $\alpha>0$, $\lceil\alpha\rceil=n$, $\alpha\notin\mathbb{N}$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $\alpha>\frac{1}{q}$, and g be strictly increasing with $g\in C^1([m,M])$. We assume that $\left(f\circ g^{-1}\right)\in AC^n([g(m),g(M)])$ and $\left(f\circ g^{-1}\right)^{(n)}\circ g\in L_\infty([m,M])$, and $\left(f\circ g^{-1}\right)^{(k)}(g(M))=0$, $k=0,1,...,n-1$. Then*

$$\int_{\lambda}^{M}|f(t)|^q\,dt \leq \frac{\|g\|_{\infty,[m,M]}^{q\alpha-1}(M-\lambda)^{q\alpha}}{(\Gamma(\alpha))^q(p(\alpha-1)+1)^{q-1}q\alpha}\left(\int_{\lambda}^{M}\left|\left(D_{M-;g}^{\alpha}f\right)(t)\right|^q g'(t)\,dt\right), \tag{184}$$

$\forall\ \lambda\in[m,M]$.

Proof. Similar to Theorem 67. ■

We derive the following Poincaré type right fractional inequality:

Theorem 72 *All as in Theorem 71. Then*

$$\left(\int_{A}^{M1_H}|f|^q\right)\leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{q\alpha-1}}{(\Gamma(\alpha))^q(p(\alpha-1)+1)^{q-1}q\alpha}(M1_H-A)^{q\alpha}\left(\int_{A}^{M1_H}\left|D_{M-;g}^{\alpha}f\right|^q g'\right). \tag{185}$$

We need the following right fractional Sobolev type inequality:

Theorem 73 *All as in Theorem 71, $r \geq 1$. Then*

$$\left(\int_{\lambda}^{M} |f(t)|^{r} dt \right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{\left(\alpha-\frac{1}{q}\right)} (M-\lambda)^{\left(\alpha-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \left(\int_{\lambda}^{M} \left| \left(D_{M-;g}^{\alpha} f \right)(t) \right|^{q} g'(t)\, dt \right)^{\frac{1}{q}}, \tag{186}$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 69. ■

We derive the following operator Sobolev type right fractional inequality:

Theorem 74 *All as in Theorem 73. Then*

$$\left(\int_{A}^{M1_H} |f|^{r} \right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\alpha-\frac{1}{q}\right)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}.$$

$$(M1_H - A)^{\left(\alpha-\frac{1}{q}+\frac{1}{r}\right)} \left(\int_{A}^{M1_H} \left| \left(D_{M-;g}^{\alpha} f \right) \right|^{q} g' \right)^{\frac{1}{q}}. \tag{187}$$

We give the following left fractional Poincaré type inequality:

Theorem 75 *Here $n \in \mathbb{N}$ with $n = [\nu]$, $\nu > 0$. Assume that $g : [m, M] \to \mathbb{R}$ is strictly increasing function, $f \in C^{n}([m, M])$, $g \in C^{1}([m, M])$, and $g^{-1} \in C^{n}([g(m), g(M)])$. Suppose also that $f \circ g^{-1} \in C_{g(m)}^{\nu}([g(m), g(M)])$. If $\nu \geq 1$, we assume that $f^{(k)}(m) = 0$, all $k = 0, 1, ..., [\nu] - 1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. Then*

$$\int_{m}^{\lambda} |f(t)|^{q} dt \leq \tag{188}$$

$$\frac{\|g\|_{\infty,[m,M]}^{q\nu-1} (\lambda-m)^{q\nu}}{(\Gamma(\nu))^{q} (p(\nu-1)+1)^{q-1} q\nu} \left(\int_{m}^{\lambda} \left| \left(D_{g(m)}^{\nu} \left(f \circ g^{-1} \right) \right)(g(s)) \right|^{q} g'(t)\, dt \right),$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 67. ■

We give the following operator Poincaré type left fractional inequality:

Theorem 76 *All as in Theorem 75. Then*

$$\left(\int_{m1_H}^{A} |f|^{q} \right) \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{q\nu-1}}{(\Gamma(\nu))^q (p(\nu-1)+1)^{q-1} q\nu} (A - m1_H)^{q\nu} \left(\int_{m1_H}^{A} \left| \left(D_{g(m)}^{\nu} (f \circ g^{-1}) \right) \circ g \right|^q g' \right). \tag{189}$$

We need the following left fractional Sobolev type inequality:

Theorem 77 *All as in Theorem 75 and $r \geq 1$. Then*

$$\left(\int_m^{\lambda} |f(t)|^r dt \right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\nu-\frac{1}{q}\right)} (\lambda - m)^{\left(\nu-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu - \frac{1}{q}\right) + 1 \right)^{\frac{1}{r}}} \cdot$$

$$\left(\int_m^{\lambda} \left| \left(D_{g(m)}^{\nu} (f \circ g^{-1}) \right) (g(s)) \right|^q g'(t)\, dt \right)^{\frac{1}{q}}, \tag{190}$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 69. ■

We give the corresponding operator Sobolev type left fractional inequality:

Theorem 78 *All as in Theorem 77. Then*

$$\left(\int_{m1_H}^{A} |f|^r \right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\nu-\frac{1}{q}\right)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu - \frac{1}{q}\right) + 1 \right)^{\frac{1}{r}}} \cdot$$

$$(A - m1_H)^{\left(\nu-\frac{1}{q}+\frac{1}{r}\right)} \left(\int_{m1_H}^{A} \left| \left(D_{g(m)}^{\nu} (f \circ g^{-1}) \right) \circ g \right|^q g' \right)^{\frac{1}{q}}. \tag{191}$$

Proof. Using (190). ■

We give the following right fractional Poincaré type inequality:

Theorem 79 *Here $n \in \mathbb{N}$ with $n = [\nu]$, $\nu > 0$. Assume that $g : [m, M] \to \mathbb{R}$ is strictly increasing function, $f \in C^n([m, M])$, $g \in C^1([m, M])$, and $g^{-1} \in C^n([g(m), g(M)])$. Suppose also that $f \circ g^{-1} \in C_{g(M)-}^{\nu}([g(m), g(M)])$. If $\nu \geq 1$, we assume that $f^{(k)}(M) = 0$, all $k = 0, 1, ..., [\nu] - 1$. Let $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$, $\nu > \frac{1}{q}$. Then*

$$\int_{\lambda}^{M} |f(t)|^q dt \leq \frac{\|g\|_{\infty,[m,M]}^{q\nu-1} (M - \lambda)^{q\nu}}{(\Gamma(\nu))^q (p(\nu-1)+1)^{q-1} q\nu} \cdot \tag{192}$$

$$\left(\int_{\lambda}^{M} \left| \left(D_{g(M)-}^{\nu} (f \circ g^{-1}) \right) (g(s)) \right|^q g'(t)\, dt \right),$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 67. ■

We give the following operator Poincaré type right fractional inequality:

Theorem 80 *All as in Theorem 79. Then*

$$\left(\int_A^{M1_H} |f|^q\right) \leq \frac{\|g\|_{\infty,[m,M]}^{q\nu-1}}{(\Gamma(\nu))^q (p(\nu-1)+1)^{q-1} q\nu} \cdot \tag{193}$$

$$(M1_H - A)^{q\nu} \left(\int_A^{M1_H} \left| \left(D_{g(M)-}^{\nu} \left(f \circ g^{-1}\right)\right) \circ g \right|^q g'\right).$$

We need the following right fractional Sobolev type inequality:

Theorem 81 *All as in Theorem 79, $r \geq 1$. Then*

$$\left(\int_\lambda^M |f(t)|^r \, dt\right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\nu-\frac{1}{q}\right)} (M-\lambda)^{\left(\nu-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \cdot$$

$$\left(\int_\lambda^M \left| \left(D_{g(M)-}^{\nu} \left(f \circ g^{-1}\right)\right) (g(s)) \right|^q g'(t) \, dt\right)^{\frac{1}{q}}, \tag{194}$$

$\forall \ \lambda \in [m, M]$.

Proof. Similar to Theorem 69. ■

We give the corresponding operator Sobolev type right fractional inequality:

Theorem 82 *All as in Theorem 81. Then*

$$\left(\int_A^{M1_H} |f|^r\right)^{\frac{1}{r}} \leq \frac{\|g\|_{\infty,[m,M]}^{\left(\nu-\frac{1}{q}\right)}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \cdot$$

$$(M1_H - A)^{\left(\nu-\frac{1}{q}+\frac{1}{r}\right)} \left(\int_A^{M1_H} \left| \left(D_{g(M)-}^{\nu} \left(f \circ g^{-1}\right)\right) \circ g \right|^q g'\right)^{\frac{1}{q}}. \tag{195}$$

Proof. Using (194). ■

We give the following Poincaré type left fractional inequality:

Theorem 83 *Let $g : [m, M] \to \mathbb{R}$ be strictly increasing, $f \in C^1([m, M])$, $g \in C^1([m, M])$, and $g^{-1} \in C^1([g(m), g(M)])$, $0 < \nu < 1$. Suppose that $\left(D_{g(m)}^{i\nu} \left(f \circ g^{-1}\right)\right) \in C_{g(m)}^{\nu}([g(m), g(M)])$, $i = 0, 1, ..., \overline{m}$, and $\left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1}\right)\right)$ $\in C([g(m), g(M)])$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{(\overline{m}+1)q}$. Then*

$$\int_m^\lambda |f(t)|^q \, dt \leq \frac{\|g\|_{\infty,[m,M]}^{q(\overline{m}+1)\nu-1}}{(\Gamma((\overline{m}+1)\nu))^q (p((\overline{m}+1)\nu-1)+1)^{q-1} q(\overline{m}+1)\nu} \cdot \tag{196}$$

$$(\lambda - m)^{q(\overline{m}+1)\nu} \left(\int_m^\lambda \left| \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) (g(s)) \right|^q g'(t)\, dt \right),$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 67. ■

We give the corresponding operator Poincaré type left fractional inequality:

Theorem 84 *All as in Theorem 83. Then*

$$\left(\int_{m1_H}^{A} |f|^q \right) \leq \frac{\|g\|_{\infty,[m,M]}^{q(\overline{m}+1)\nu-1}}{(\Gamma((\overline{m}+1)\nu))^q (p((\overline{m}+1)\nu-1)+1)^{q-1} q(\overline{m}+1)\nu} \cdot$$

$$(A - m1_H)^{q(\overline{m}+1)\nu} \left(\int_{m1_H}^{A} \left| D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \circ g \right|^q g' \right). \tag{197}$$

We need the following left fractional Sobolev type inequality:

Theorem 85 *All as in Theorem 83, and $r \geq 1$. Then*

$$\left(\int_m^\lambda |f(t)|^r\, dt \right)^{\frac{1}{r}} \leq \tag{198}$$

$$\frac{\|g\|_{\infty,[m,M]}^{(\overline{m}+1)\nu-\frac{1}{q}} (\lambda - m)^{\left((\overline{m}+1)\nu-\frac{1}{q}+\frac{1}{r}\right)}}{\Gamma((\overline{m}+1)\nu)(p((\overline{m}+1)\nu-1)+1)^{\frac{1}{p}} \left(r\left((\overline{m}+1)\nu - \frac{1}{q} \right) + 1 \right)^{\frac{1}{r}}} \cdot$$

$$\left(\int_m^\lambda \left| \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) (g(s)) \right|^q g'(t)\, dt \right)^{\frac{1}{q}},$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 69. ■

We give the corresponding operator Sobolev type left fractional inequality:

Theorem 86 *All as in Theorem 85. Then*

$$\left(\int_{m1_H}^{A} |f|^r \right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{(\overline{m}+1)\nu-\frac{1}{q}}}{\Gamma((\overline{m}+1)\nu)(p((\overline{m}+1)\nu-1)+1)^{\frac{1}{p}} \left(r\left((\overline{m}+1)\nu - \frac{1}{q} \right) + 1 \right)^{\frac{1}{r}}} \cdot$$

$$(A - m1_H)^{\left((\overline{m}+1)\nu-\frac{1}{q}+\frac{1}{r}\right)} \left(\int_{m1_H}^{A} \left| \left(D_{g(m)}^{(\overline{m}+1)\nu} \left(f \circ g^{-1} \right) \right) \circ g \right|^q g' \right)^{\frac{1}{q}}. \tag{199}$$

Proof. Using (198). ■

We give the following Poincaré type right fractional inequality:

Theorem 87 *Let $g : [m, M] \to \mathbb{R}$ be strictly increasing, $f \in C^1([m, M])$, $g \in C^1([m, M])$, and $g^{-1} \in C^1([g(m), g(M)])$, $0 < \nu < 1$. Suppose that $\left(D^{i\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(M)-}([g(m), g(M)])$, $i = 0, 1, ..., \overline{m}$, and $\left(D^{(\overline{m}+1)\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right) \in C([g(m), g(M)])$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\nu > \frac{1}{(\overline{m}+1)q}$. Then*

$$\int_{\lambda}^{M} |f(t)|^q \, dt \leq \frac{\|g\|^{q(\overline{m}+1)\nu-1}_{\infty,[m,M]}}{\left(\Gamma((\overline{m}+1)\nu)\right)^q \left(p((\overline{m}+1)\nu-1)+1\right)^{q-1} q(\overline{m}+1)\nu} \cdot \tag{200}$$

$$(M-\lambda)^{q(\overline{m}+1)\nu} \left(\int_{\lambda}^{M} \left|\left(D^{(\overline{m}+1)\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(g(s))\right|^q g'(t)\, dt\right),$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 67. ■

We give the corresponding operator Poincaré type right fractional inequality:

Theorem 88 *All as in Theorem 87. Then*

$$\left(\int_{A}^{M1_H} |f|^q\right) \leq \frac{\|g\|^{q(\overline{m}+1)\nu-1}_{\infty,[m,M]}}{\left(\Gamma((\overline{m}+1)\nu)\right)^q \left(p((\overline{m}+1)\nu-1)+1\right)^{q-1} q(\overline{m}+1)\nu} \cdot$$

$$(M1_H - A)^{q(\overline{m}+1)\nu} \left(\int_{A}^{M1_H} \left|D^{(\overline{m}+1)\nu}_{g(M)-}\left(f \circ g^{-1}\right) \circ g\right|^q g'\right). \tag{201}$$

We need the following right fractional Sobolev type inequality:

Theorem 89 *All as in Theorem 87, $r \geq 1$. Then*

$$\left(\int_{\lambda}^{M} |f(t)|^r \, dt\right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|^{(\overline{m}+1)\nu-\frac{1}{q}}_{\infty,[m,M]}}{\Gamma((\overline{m}+1)\nu)\left(p((\overline{m}+1)\nu-1)+1\right)^{\frac{1}{p}}\left(r\left((\overline{m}+1)\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \cdot$$

$$(M-\lambda)^{\left((\overline{m}+1)\nu-\frac{1}{q}+\frac{1}{r}\right)} \left(\int_{\lambda}^{M} \left|\left(D^{(\overline{m}+1)\nu}_{g(M)-}\left(f \circ g^{-1}\right)\right)(g(s))\right|^q g'(t)\, dt\right)^{\frac{1}{q}}, \tag{202}$$

$\forall\ \lambda \in [m, M]$.

Proof. Similar to Theorem 69. ■

We finish with the corresponding operator Sobolev type right fractional inequality:

Theorem 90 *All as in Theorem 89. Then*

$$\left(\int_{A}^{M1_H} |f|^r\right)^{\frac{1}{r}} \leq$$

$$\frac{\|g\|_{\infty,[m,M]}^{(\overline{m}+1)\nu-\frac{1}{q}}}{\Gamma\left((\overline{m}+1)\nu\right)\left(p\left((\overline{m}+1)\nu-1\right)+1\right)^{\frac{1}{p}}\left(r\left((\overline{m}+1)\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}.$$

$$(M1_H - A)^{\left((\overline{m}+1)\nu-\frac{1}{q}+\frac{1}{r}\right)}\left(\int_{A}^{M1_H}\left|\left(D_{g(M)-}^{(\overline{m}+1)\nu}\left(f\circ g^{-1}\right)\right)\circ g\right|^q g'\right)^{\frac{1}{q}}. \quad (203)$$

Proof. Using (202). ■

References

[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[2] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, 2011.

[3] G. Anastassiou, *Advanced Fractional Taylor's formulae*, J. of Computational Analysis and Applications, 21 (7) (2016), 1185-1204.

[4] G. Anastassiou, *Generalized Canavati type Fractional Taylor's formulae*, J. of Computational Analysis and Applications, 21 (7) (2016), 1205-1212.

[5] G. Anastassiou and I.K. Argyros, *A convergence Analysis for a certain family of extended iterative methods: Part II. Applications to fractional calculus*, Annales Univ. Sci. Budapest, sect. comp., accepted, 2015.

[6] G. Anastassiou and O. Duman, Editors, *Intelligent Mathematics II: Applied Mathematics and Approximation Theory*, Springer, Heidelberg, 2016.

[7] G. Anastassiou and O. Duman, Editors, *Computational Analysis*, AMAT, Ankara, May 2015, Springer, New York, 2016.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, New York, 2010.

[9] S.S. Dragomir, *Inequalities for functions of selfadjoint operators on Hilbert Spaces*, ajmaa.org/RGMIA/monographs/InFuncOp.pdf, 2011.

[10] S. Dragomir, *Operator inequalities of Ostrowski and Trapezoidal type*, Springer, New York, 2012.

[11] T. Furuta, J. Mićić Hot, J. Pečaric, Y. Seo, *Mond-Pečaric Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[12] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc., New York, 1969.

[13] Rong-Qing Jia, *Chapter 3. Absolutely Continuous Functions*, https://www.ualberta.ca/~rjia/Math418/Notes/Chap.3.pdf.

[14] Z.M. Odibat, N.J. Shawagleh, *Generalized Taylor's formula*, Appl. Math. Comput. 186, (2007), 286-293.