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INEQUALITIES OF MCCARTHY'S TYPE IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. We establish in this paper some inequalities of McCarthy's type that hold for bounded linear operators on Hilbert spaces in the more general setting of Hermitian unital Banach $*$-algebra and positive linear functionals.

1. Introduction

Let A be a nonnegative operator on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$, namely $\langle Ax, x\rangle \geq 0$ for any $x \in H$. We write this as $A \geq 0$.

By the use of the spectral resolution of A and the Hölder inequality, C. A. McCarthy [15] proved that

$$\langle Ax, x\rangle^p \leq \langle A^p x, x\rangle, \ p \in (1, \infty) \tag{1.1}$$

and

$$\langle A^p x, x\rangle \leq \langle Ax, x\rangle^p, \ p \in (0, 1) \tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator on H with

$$mI \leq A \leq MI, \tag{1.3}$$

where I is the *identity operator* and m, M are real numbers with $m < M$. In [9, Theorem 3] Fujii et al. obtained the following interesting ratio inequality that provides a reverse of the *Hölder-McCarthy inequality* (1.1) for an operator A that satisfy the condition (1.3) with $m > 0$

$$\langle A^p x, x\rangle \leq \left\{ \frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}} \right\}^p \langle Ax, x\rangle^p, \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$, where $q = p/(p-1)$, $p > 1$.

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If A satisfies the condition (1.3) with $m \geq 0$, then we also have the additive reverse of (1.1) that has been obtained by the author in 2008, see [5]

$$
\begin{aligned}
0 &\leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p \\
&\leq p \begin{cases} \frac{1}{2}(M-m)\left[\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}\left(M^{p-1}-m^{p-1}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right),
\end{aligned} \tag{1.5}
$$

for any $x \in H$ with $\|x\| = 1$, where $p > 1$.

For various related inequalities, see [8]-[10] and [14].

Motivated by the above results we establish in this paper some similar inequalities in the more general setting of Hermitian unital Banach $*$-algebra and positive linear functionals.

2. Some Facts on Hermitian $*$-Algebras

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [18] (see also [1, Theorem 41.5])

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [17], Tanahashi and Uchiyama [20] proved the following fundamental properties (see also [7]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [17] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1}\, dz.$$

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1}\, dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $\left(a^2\right)^{1/2} = a$, [20, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = \left(a^{-1}\right)^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(z)$ is analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0, \infty)$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A.

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in the right half open plane $\{\operatorname{Re} z > 0\}$ and for the interval $I \subset (0, \infty)$ assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

3. McCarthy's Type Inequalities

We have the following result:

Theorem 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.*

(i) If $p \in (0,1)$ and $a \geq 0$, then

$$\psi^{p}(a) \geq \psi(a^{p}) \geq 0; \tag{3.1}$$

(ii) If $q \geq 1$ and $b \geq 0$, then

$$\psi(b^{q}) \geq \psi^{q}(b) \geq 0. \tag{3.2}$$

Proof. (i) Using the arithmetic mean-geometric mean inequality for positive real numbers, we have

$$(1-p)\alpha + p\beta \geq \alpha^{1-p}\beta^{p}$$

for any $\alpha, \beta \geq 0$.

Fix $p \in (0,1)$ and $\alpha \geq 0$ and apply Lemma 1 for $f(z) = (1-p)\alpha + pz$ and $g(z) = \alpha^{1-p}z^{p}$ to get in the order of A that

$$(1-p)\alpha + pu \geq \alpha^{1-p}u^{p} \tag{3.3}$$

for any $u \in A$ with $u > 0$ and $p \in (0,1)$, $\alpha \geq 0$.

If we take the functional ψ for $u = a$ in (3.3) we get the scalar inequality

$$(1-p)\alpha + p\psi(a) \geq \alpha^{1-p}\psi(a^{p}) \tag{3.4}$$

for any $p \in (0,1)$, $\alpha \geq 0$.

If $\psi(a) > 0$, then by taking $\alpha = \psi(a)$ we get $\psi(a) \geq \psi^{1-p}(a)\psi(a^{p})$, which by dividing with $\psi^{1-p}(a) > 0$ produces

$$\psi^{p}(a) \geq \psi(a^{p}). \tag{3.5}$$

If $\psi(a) = 0$ then by (3.4) we get

$$(1-p)\alpha \geq \alpha^{1-p}\psi(a^{p}) \tag{3.6}$$

for any $\alpha > 0$. Dividing by $\alpha^{1-p} > 0$ we get $(1-p)\alpha^{\alpha} \geq \psi(a^{p})$ for any $\alpha > 0$. By letting $\alpha \to 0+$ in this inequality, we get $0 \geq \psi(a^{p})$ and since $\psi(a^{p}) \geq 0$ we conclude that $\psi(a^{p}) = 0$ and the inequality (3.1) is verified with equality.

Now, if $a \geq 0$ then for $\varepsilon > 0$ we have in the order of A that $a + \varepsilon 1 > 0$ and by (3.5)

$$\psi^{p}(a+\varepsilon 1) \geq \psi((a+\varepsilon 1)^{p}) \geq 0; \tag{3.7}$$

for any for $\varepsilon > 0$.

By taking the limit over $\varepsilon \to 0+$ in (3.7) and using the continuity of the functional ψ and the power function, we recapture (3.1).

(ii) For $q = 1$ we have equality. If $q > 1$, then for $p = \frac{1}{q} \in (0,1)$ and $a = b^{q} \geq 0$ for $b \geq 0$ we get from (3.1) that

$$\psi^{1/q}(b^{q}) \geq \psi\left((b^{q})^{1/q}\right) = \psi(b) \geq 0$$

and by taking the power q in this inequality we get (3.2). □

Remark 1. *From (3.1) we have for $p = 1/2$, that*

$$\psi^{1/2}(a) \geq \psi\left(a^{1/2}\right) \geq 0, \tag{3.8}$$

while from (3.2) for $p = 2$ that

$$\psi\left(b^{2}\right) \geq \psi^{2}(b) \geq 0, \tag{3.9}$$

for any a, $b \geq 0$.

Theorem 2. *If* $r < 0$, $c > 0$ *and* $\psi : A \to \mathbb{C}$ *is a positive normalized linear functional on* A *with* $\psi(c) > 0$, *then*

$$\psi(c^r) \geq \psi^r(c) > 0. \tag{3.10}$$

Proof. Using the gradient inequality for the convex function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$, $r < 0$ we have

$$f(t) - f(s) \geq f'(s)(t - s),$$

for any t, $s > 0$, namely

$$t^r - s^r \geq rs^{r-1}(t - s),$$

for any t, $s > 0$, which, as above, gives the inequality in the order of A that

$$u^r - s^r \geq rs^{r-1}(u - s), \tag{3.11}$$

for any element $u \in A$ with $u > 0$ and positive real number s.

From (3.11) we then get

$$c^r - \psi^r(c) \geq r\psi^{r-1}(c)(c - \psi(c)),$$

and if we take in this inequality the functional ψ we get

$$\psi(c^r) - \psi^r(c) \geq r\psi^{r-1}(c)(\psi(c) - \psi(c)) = 0,$$

which proves the desired result (3.10). □

Remark 2. *The above technique based on the convexity of the power function can be employed to provide another proof of the inequalities (3.1) and (3.2).*

From (3.10) we get for $r = -1$ *that*

$$\psi\left(c^{-1}\right) \geq \psi^{-1}(c) > 0, \tag{3.12}$$

where $c > 0$ *and* $\psi : A \to \mathbb{C}$ *is a positive normalized linear functional on* A *with* $\psi(c) > 0$.

4. Refinements and Reverses

We have the following refinement and reverse of the inequality (3.1):

Theorem 3. *Assume that* A *is a Hermitian unital Banach* $*$*-algebra and* $\psi : A \to \mathbb{C}$ *a positive normalized linear functional on* A. *If* $p \in (0, 1)$ *and* $0 < c \in A$ *with* $\psi(c) > 0$, *then*

$$\begin{aligned} 2r\psi^{p-1/2}(c)\left(\psi^{1/2}(c) - \psi\left(c^{1/2}\right)\right) &\leq \psi^p(c) - \psi(c^p) \\ &\leq 2r\psi^{p-1/2}(c)\left(\psi^{1/2}(c) - \psi\left(c^{1/2}\right)\right), \end{aligned} \tag{4.1}$$

where $r := \min\{1 - p, p\}$ *and* $R := \max\{1 - p, p\}$.

Proof. We use the following double inequality obtained by Kittaneh and Manasrah [11], [12] that provide a refinement and an additive reverse for Young's inequality as follows:

$$r\left(\sqrt{\alpha} - \sqrt{\beta}\right)^2 \leq (1 - p)\alpha + p\beta - \alpha^{1-p}\beta^p \leq R\left(\sqrt{\alpha} - \sqrt{\beta}\right)^2 \tag{4.2}$$

where α, $\beta \geq 0$, $p \in [0, 1]$, $r = \min\{1 - p, p\}$ and $R = \max\{1 - p, p\}$.

This is equivalent to

$$r\left(\alpha-2\sqrt{\alpha}\sqrt{\beta}+\beta\right)\leq(1-p)\,\alpha+p\beta-\alpha^{1-p}\beta^{p}\leq R\left(\alpha-2\sqrt{\alpha}\sqrt{\beta}+\beta\right) \tag{4.3}$$

for any $\alpha,\ \beta\geq 0,\ p\in[0,1]$.

Using Lemma 1 and a similar argument to the one in the proof of Theorem 1 we can state that

$$r\left(\alpha-2\sqrt{\alpha}c^{1/2}+c\right)\leq(1-p)\,\alpha+pc-\alpha^{1-p}c^{p}\leq R\left(\alpha-2\sqrt{\alpha}c^{1/2}+c\right) \tag{4.4}$$

for any $\alpha\geq 0$, $p\in[0,1]$ and $c\in A$ with $c>0$.

If we take the functional ψ in the inequality (4.4), then we get

$$\begin{aligned} r\left(\alpha-2\sqrt{\alpha}\psi\left(c^{1/2}\right)+\psi\left(c\right)\right) &\leq(1-p)\,\alpha+p\psi\left(c\right)-\alpha^{1-p}\psi\left(c^{p}\right)\\ &\leq R\left(\alpha-2\sqrt{\alpha}\psi\left(c^{1/2}\right)+\psi\left(c\right)\right)\end{aligned} \tag{4.5}$$

for any $\alpha\geq 0$, $p\in[0,1]$ and $c\in A$ with $c>0$.

If we take in (4.5) $\alpha=\psi\left(c\right),$ then we get

$$\begin{aligned} 2r\psi^{1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right) &\leq\psi^{1-p}\left(c\right)\left(\psi^{p}\left(c\right)-\psi\left(c^{p}\right)\right)\\ &\leq 2R\psi^{1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right),\end{aligned}$$

which, by division with $\psi^{1-p}\left(c\right)>0$ is equivalent to the desired result (4.1). □

Corollary 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi:A\to\mathbb{C}$ a positive normalized linear functional on A. Suppose also that there exists the constant $m,\ M>0$ and $c\in A$ such that*

$$M\geq c\geq m \tag{4.6}$$

in the order of A.

(i) If $p\in\left(\frac{1}{2},1\right)$, then by (4.1) we have

$$\begin{aligned} &2\left(1-p\right)m^{p-\frac{1}{2}}\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\\ &\leq 2\left(1-p\right)\psi^{p-1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\leq\psi^{p}\left(c\right)-\psi\left(c^{p}\right)\\ &\leq 2p\psi^{p-1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\leq 2pM^{p-\frac{1}{2}}\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right).\end{aligned} \tag{4.7}$$

(ii) If $p\in\left(0,\frac{1}{2}\right)$, then by (4.1) we have

$$\begin{aligned} &2pM^{p-\frac{1}{2}}\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\\ &\leq 2p\psi^{p-1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\leq\psi^{p}\left(c\right)-\psi\left(c^{p}\right)\\ &\leq 2\left(1-p\right)\psi^{p-1/2}\left(c\right)\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right)\\ &\leq 2\left(1-p\right)m^{p-\frac{1}{2}}\left(\psi^{1/2}\left(c\right)-\psi\left(c^{1/2}\right)\right).\end{aligned} \tag{4.8}$$

The proof follows by (4.1) on observing that $M\geq\psi\left(c\right)\geq m>0$.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function S is *decreasing* on $(0,1)$ and *increasing* on $(1,\infty)$.

Tominaga [21] had proved a reverse Young inequality with the Specht's ratio [19] as follows:

$$\left(\alpha^{1-\nu}\beta^{\nu} \le\right)(1-\nu)\alpha + \nu\beta \le S\left(\frac{\alpha}{\beta}\right)\alpha^{1-\nu}\beta^{\nu} \tag{4.9}$$

for any α, $\beta > 0$ and $\nu \in [0,1]$.

The following result also holds:

Theorem 4. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. Suppose also that there exists the constant m, $M > 0$ and $c \in A$ such that the condition (4.6) is valid.*

(i) If $p \in (0,1)$, then

$$\psi^{p}(c) \le S\left(\frac{M}{m}\right)\psi(c^{p}). \tag{4.10}$$

(ii) If $q \in (1,\infty)$, then

$$\psi(c^{q}) \le S^{q}\left(\left(\frac{M}{m}\right)^{q}\right)\psi^{q}(c). \tag{4.11}$$

Proof. (i) Assume that $p \in (0,1)$. Let α, $\beta \in [m,M] \subset (0,\infty)$, then $\frac{m}{M} \le \frac{\alpha}{\beta} \le \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{\alpha}{\beta} \in \left[\frac{m}{M},1\right)$ then $S\left(\frac{\alpha}{\beta}\right) \le S\left(\frac{m}{M}\right) = S\left(\frac{M}{m}\right)$. If $\frac{\alpha}{\beta} \in \left(1,\frac{M}{m}\right]$ then also $S\left(\frac{\alpha}{\beta}\right) \le S\left(\frac{M}{m}\right)$. Therefore for any α, $\beta \in [m,M]$ we have by (4.9) that

$$(1-p)\alpha + p\beta \le S\left(\frac{M}{m}\right)\alpha^{1-p}\beta^{p}. \tag{4.12}$$

Using Lemma 1 , we have from (4.12) for $c \in A$ such that the condition (4.6) is valid, that

$$(1-p)\alpha + pc \le S\left(\frac{M}{m}\right)\alpha^{1-p}c^{p} \tag{4.13}$$

for any $\alpha \in [m,M]$ and $p \in (0,1)$.

If we take the functional ψ in (4.13), then we get

$$(1-p)\alpha + p\psi(c) \le S\left(\frac{M}{m}\right)\alpha^{1-p}\psi(c^{p})$$

which for $\alpha = \psi(c) \in [m,M]$ provides

$$\psi(c) \le S\left(\frac{M}{m}\right)\psi^{1-p}(c)\psi(c^{p}).$$

If we divide this inequality by $\psi^{1-p}(c) > 0$, then we get (4.10).

(ii) From the condition (4.6) we have $0 < m^q \leq c^q \leq M^q$. If we write the inequality (4.10) for $p = \frac{1}{q}$ we have

$$\psi^{1/q}(c^q) \leq S\left(\left(\frac{M}{m}\right)^q\right)\psi(c^{qp}) = S\left(\left(\frac{M}{m}\right)^q\right)\psi(c)$$

and by taking the power q in this inequality we get the desired result (4.11). $\square$

Remark 3. *If we take $p = 1/2$ in (4.10), then we have*

$$\psi^{1/2}(c) \leq S\left(\frac{M}{m}\right)\psi\left(c^{1/2}\right). \tag{4.14}$$

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{4.15}$$

The function K is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$(1-\nu)\alpha + \nu\beta \leq K^R\left(\frac{\alpha}{\beta}\right)\alpha^{1-\nu}\beta^\nu, \tag{4.16}$$

where $\alpha, \beta > 0$, $\nu \in [0,1]$ and $R = \max\{1-\nu, \nu\}$.

This inequality has been obtained by Liao et al. in [13].

Theorem 5. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. Suppose also that there exists the constant m, $M > 0$ and $c \in A$ such that the condition (4.6) is valid.*

(i) If $p \in (0,1)$, then

$$\psi^p(c) \leq K^R\left(\frac{M}{m}\right)\psi(c^p), \tag{4.17}$$

where $R = \max\{p, 1-p\}$.

(ii) If $q \in (1,\infty)$, then

$$\psi(c^q) \leq K^Q\left(\left(\frac{M}{m}\right)^q\right)\psi^q(c), \tag{4.18}$$

where $Q = \max\{q-1, 1\}$.

Proof. (i) Assume that $p \in (0,1)$ and put $R = \max\{1-p, p\}$. Let $\alpha, \beta \in [m,M] \subset (0,\infty)$, then $\frac{m}{M} \leq \frac{\alpha}{\beta} \leq \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{\alpha}{\beta} \in \left[\frac{m}{M}, 1\right)$ then $K^R\left(\frac{\alpha}{\beta}\right) \leq K^R\left(\frac{m}{M}\right) = K^R\left(\frac{M}{m}\right)$. If $\frac{\alpha}{\beta} \in \left(1, \frac{M}{m}\right]$ then also $K^R\left(\frac{\alpha}{\beta}\right) \leq K^R\left(\frac{M}{m}\right)$. Therefore for any $\alpha, \beta \in [m,M]$ we have by (4.16) that

$$(1-p)\alpha + p\beta \leq K^R\left(\frac{M}{m}\right)\alpha^{1-p}\beta^p. \tag{4.19}$$

Now, on making use of a similar argument to the one from (i) in Theorem 4, we get (4.17).

(ii) Let $q \in (1, \infty)$. Then $\frac{1}{q} \in (0,1)$ and $\max\left\{1 - \frac{1}{q}, \frac{1}{q}\right\} = \frac{1}{q} \max\{q-1, 1\} = \frac{1}{q}Q$. Since $m^q I \leq c^q \leq M^q I$ and by applying (4.17) we have

$$\psi^{1/q}(c^q) \leq K^{\frac{1}{q}Q}\left(\left(\frac{M}{m}\right)^q\right)\psi(c). \tag{4.20}$$

Now, by taking the power $q > 1$ in (4.20) we get the desired result (4.18). □

Remark 4. *If we take $p = 1/2$ in (4.17), then we have*

$$\psi^{1/2}(c) \leq K^{1/2}\left(\frac{M}{m}\right)\psi\left(c^{1/2}\right). \tag{4.21}$$

In the recent paper [3] we obtained the following reverses of Young's inequality as well:

$$1 \leq \frac{(1-\nu)\alpha + \nu\beta}{\alpha^{1-\nu}\beta^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{\alpha}{\beta}\right) - 1\right)\right], \tag{4.22}$$

where α, $\beta > 0$, $\nu \in [0,1]$.

By a similar argument employed above, we can state:

Theorem 6. *Under the assumptions of Theorem 5, we have:*

(i) If $p \in (0,1)$, then

$$\psi^p(c) \leq \exp\left[4p(1-p)\left(K\left(\frac{M}{m}\right) - 1\right)\right]\psi(c^p), \tag{4.23}$$

where $R = \max\{p, 1-p\}$.

(ii) If $q \in (1, \infty)$, then

$$\psi(c^q) \leq \exp\left[4\left(\frac{q-1}{q}\right)\left(K\left(\left(\frac{M}{m}\right)^q\right) - 1\right)\right]\psi^q(c). \tag{4.24}$$

Finally, by the use of the inequality [5]

$$\frac{(1-\nu)\alpha + \nu\beta}{\alpha^{1-\nu}\beta^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(\beta-\alpha)^2}{\min^2\{\alpha,\beta\}}\right], \tag{4.25}$$

we also have:

Theorem 7. *Under the assumptions of Theorem 5, we have:*

(i) If $p \in (0,1)$, then

$$\psi^p(c) \leq \exp\left[\frac{1}{2}p(1-p)\left(\frac{M}{m} - 1\right)^2\right]\psi(c^p). \tag{4.26}$$

(ii) If $q \in (1, \infty)$, then

$$\psi(c^q) \leq \exp\left[\frac{q-1}{2q}\left(\left(\frac{M}{m}\right)^q - 1\right)^2\right]\psi^q(c). \tag{4.27}$$

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