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INEQUALITIES OF JENSEN'S TYPE FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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ABSTRACT. We establish in this paper some inequalities of Jensen's and Slater's type in the general setting of Hermitian unital Banach $*$-algebra, analytic convex functions and positive normalized linear functionals.

1. INTRODUCTION

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [8] (see also [1, Theorem 41.5]), then

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [7], Tanahashi and Uchiyama [9] proved the following fundamental properties (see also [5]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [7] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

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In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [5], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [9, Lemma 6];

(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;

(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;

(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [3].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

In the recent paper [4] we established the following McCarthy type inequality:

Theorem 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.*

(i) If $p \in (0,1)$ and $a \geq 0$, then

$$\psi^p(a) \geq \psi(a^p) \geq 0; \tag{1.1}$$

(ii) If $q \geq 1$ and $b \geq 0$, then

$$\psi(b^q) \geq \psi^q(b) \geq 0. \tag{1.2}$$

(iii) If $r < 0$, $c > 0$ with $\psi(c) > 0$, then

$$\psi(c^r) \geq \psi^r(c) > 0. \tag{1.3}$$

Motivated by these results we establish in this paper some inequalities for analytic and convex functions on an open interval and positive normalized functionals defined on a Hermitian unital Banach $*$-algebra. Versions of Jensen's and Slater's inequalities are provided. Some examples for particular convex functions of interest are given as well.

2. Jensen's Type Inequalities

We have the following result:

Theorem 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex (in the usual sense) on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$,*

$$\psi(f(c)) \geq f(s) + f'(s)(\psi(c) - s) \tag{2.1}$$

for any $s \in I$.

In particular, we have the Jensen inequality

$$\psi(f(c)) \geq f(\psi(c)). \tag{2.2}$$

Moreover, if $\sigma(c) \subseteq [m, M] \subset I$, then

$$\psi(f(c)) \geq f\left(\frac{m+M}{2}\right) + f'\left(\frac{m+M}{2}\right)\left(\psi(c) - \frac{m+M}{2}\right) \tag{2.3}$$

and

$$\begin{aligned}\frac{1}{2}\left[\psi(f(c)) + \frac{f(M)(M-\psi(c)) + f(m)(\psi(c)-m)}{M-m}\right] \\ \geq \frac{1}{M-m}\int_m^M f(s)\,ds.\end{aligned} \tag{2.4}$$

Proof. Since f is differentiable and convex on I we have by the gradient inequality that

$$f(t) \geq f(s) + (t-s) f'(s)$$

for any t, $s \in I$.

Fix $s \in I$ and apply Lemma 1 for the analytic functions $f(z)$ and $g_s(z) := f(s) + f'(s)(z-s)$ to get for $c \in A$ with $\sigma(c) \subset I$ that the following inequality holds

$$f(c) \geq f(s) + f'(s)(c-s) \tag{2.5}$$

in the order of A and for any $s \in I$.

If we take the functional ψ on (2.5) we get

$$\begin{aligned}\psi(f(c)) &\geq \psi[f(s)+f'(s)(c-s)]\\ &= f(s)\psi(1)+f'(s)(\psi(c)-s\psi(1))\\ &= f(s)\psi+f'(s)(\psi(c)-s)\end{aligned}$$

and the inequality (2.1) is proved.

Since $\sigma(c)$ is compact and $\sigma(c) \subset I$, then there exists the real numbers m, M with $\sigma(c) \subseteq [m,M] \subset I$. This means that we have $m \leq c \leq M$ in the order of A and by taking the functional ψ, we have $m \leq \psi(c) \leq M$, meaning that $\psi(c) \in [m,M] \subset I$. Therefore, by taking $s = \psi(c) \in [m,M]$ in (2.1) we get (2.2).

If we take $s = \frac{m+M}{2}$ in (2.1), then we get (2.3).

Now, if we take the integral mean $\frac{1}{M-m}\int_m^M$ in (2.1), then we get

$$\begin{aligned}\psi(f(c)) \geq{}& \frac{1}{M-m}\int_m^M f(s)\,ds+\psi(c)\frac{1}{M-m}\int_m^M f'(s)\,ds\\ &-\frac{1}{M-m}\int_m^M f'(s)\,sds.\end{aligned} \tag{2.6}$$

Since

$$\frac{1}{M-m}\int_m^M f'(s)\,ds = \frac{f(M)-f(m)}{M-m}$$

and

$$\begin{aligned}\frac{1}{M-m}\int_m^M f'(s)\,sds &= \frac{1}{M-m}\left[sf(s)|_m^M - \int_m^M f(t)\,dt\right]\\ &= \frac{Mf(M)-mf(m)}{M-m}-\frac{1}{M-m}\int_m^M f(s)\,ds\end{aligned}$$

hence by (2.6) we have

$$\begin{aligned}\psi(f(c)) \geq{}& \frac{f(M)-f(m)}{M-m}\psi(c)+\frac{1}{M-m}\int_m^M f(s)\,ds\\ &-\left(\frac{Mf(M)-mf(m)}{M-m}-\frac{1}{M-m}\int_m^M f(s)\,ds\right)\\ ={}& \frac{2}{M-m}\int_m^M f(s)\,ds-\frac{f(M)(M-\psi(c))+f(m)(\psi(c)-m)}{M-m}\end{aligned}$$

that is equivalent to the second inequality in (2.4). □

We also have:

Theorem 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$,*

$$\begin{aligned}\psi(f(c)) &\leq f(s)-s\psi(f'(c))+\psi(cf'(c))\\ &\leq f(s)+f'(s)(\psi(c)-s)+\sup_{t\in I}[(f'(t)-f'(s))(t-s)]\end{aligned} \tag{2.7}$$

for any $s \in I$.

In particular, we have the reverse of Jensen inequality

$$\begin{aligned}\psi\left(f\left(c\right)\right) &\leq f\left(\psi\left(c\right)\right)+\psi\left(cf'\left(c\right)\right)-\psi\left(c\right)\psi\left(f'\left(c\right)\right) \\ &\leq f\left(\psi\left(c\right)\right)+\sup_{t\in I}\left[\left(f'\left(t\right)-f'\left(\psi\left(c\right)\right)\right)\left(t-\psi\left(c\right)\right)\right].\end{aligned}\tag{2.8}$$

Moreover, if $\sigma\left(c\right)\subseteq\left[m,M\right]\subset I$*, then*

$$\begin{aligned}\psi\left(f\left(c\right)\right) &\leq f\left(s\right)-s\psi\left(f'\left(c\right)\right)+\psi\left(cf'\left(c\right)\right) \\ &\leq f\left(s\right)+f'\left(s\right)\left(\psi\left(c\right)-s\right) \\ &+\max\left\{\left(f'\left(s\right)-f'\left(m\right)\right)\left(s-m\right),\left(f'\left(M\right)-f'\left(s\right)\right)\left(M-s\right)\right\},\end{aligned}\tag{2.9}$$

for any $s\in I$.

In particular, we have

$$\begin{aligned}\psi\left(f\left(c\right)\right) &\leq f\left(\frac{m+M}{2}\right)+\psi\left(cf'\left(c\right)\right)-\frac{m+M}{2}\psi\left(f'\left(c\right)\right) \\ &\leq f\left(\frac{m+M}{2}\right)+f'\left(\frac{m+M}{2}\right)\left(\psi\left(c\right)-\frac{m+M}{2}\right) \\ &+\frac{1}{2}\left(M-m\right) \\ &\times\max\left\{f'\left(\frac{m+M}{2}\right)-f'\left(m\right),f'\left(M\right)-f'\left(\frac{m+M}{2}\right)\right\}\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}\psi\left(f\left(c\right)\right) &\leq f\left(\psi\left(c\right)\right)+\psi\left(cf'\left(c\right)\right)-\psi\left(c\right)\psi\left(f'\left(c\right)\right) \\ &\leq f\left(\psi\left(c\right)\right)+\max\left\{\left(f'\left(\psi\left(c\right)\right)-f'\left(m\right)\right)\left(\psi\left(c\right)-m\right),\right. \\ &\left.\left(f'\left(M\right)-f'\left(\psi\left(c\right)\right)\right)\left(M-\psi\left(c\right)\right)\right\}.\end{aligned}\tag{2.11}$$

Proof. Since f is differentiable and convex on I we have by the gradient inequality that

$$\left(t-s\right)f'\left(t\right)+f\left(s\right)\geq f\left(t\right)$$

for any t, $s\in I$.

With a similar approach to the one in the proof of Theorem 2 we obtain that

$$cf'\left(c\right)-sf'\left(c\right)+f\left(s\right)=\left(c-s\right)f'\left(c\right)+f\left(s\right)\geq f\left(c\right)\tag{2.12}$$

for any $s\in I$ and $c\in A$ with $\sigma\left(c\right)\subset I$, in the order of A.

If we take the functional ψ on (2.12) we get

$$\psi\left(cf'\left(c\right)\right)-s\psi\left(f'\left(c\right)\right)+f\left(s\right)\geq\psi\left(f\left(c\right)\right),$$

or any $s\in I$ and $c\in A$ with $\sigma\left(c\right)\subset I$, which proves the first inequality in (2.7).

We also have

$$\begin{aligned}f\left(s\right)+\left(t-s\right)f'\left(t\right) &= f\left(s\right)+f'\left(s\right)\left(t-s\right)+\left(f'\left(t\right)-f'\left(s\right)\right)\left(t-s\right) \\ &\leq f\left(s\right)+f'\left(s\right)\left(t-s\right)+\sup_{t\in I}\left[\left(f'\left(t\right)-f'\left(s\right)\right)\left(t-s\right)\right]\end{aligned}$$

for any t, $s\in I$.

This inequality implies in the order of A that

$$cf'\left(c\right)-sf'\left(c\right)+f\left(s\right)\leq f\left(s\right)+f'\left(s\right)\left(c-s\right)+\sup_{t\in I}\left[\left(f'\left(t\right)-f'\left(s\right)\right)\left(t-s\right)\right]\tag{2.13}$$

for any $s\in I$ and $c\in A$ with $\sigma\left(c\right)\subset I$.

If we apply the functional ψ on (2.13) we get the second inequality in (2.7).

Now, for $s \in [m, M] \subset I$ consider the function $\varphi_s : [m, M] \to \mathbb{R}$ defined by

$$\varphi_s(t) := (f'(t) - f'(s))(t - s).$$

The function φ_s is continuous on $[m, M]$, differentiable on (m, M) and

$$\varphi_s'(t) := f''(t)(t - s) + f'(t) - f'(s).$$

We observe that $\varphi_s'(s) = 0$ and since f is convex on $[m, M]$, it follows that φ_s is nonincreasing on $[m, s]$ and nondecreasing on $[s, M]$. Therefore

$$\begin{aligned}\max_{t\in[m,M]} \varphi_s(t) &= \max\{\varphi_s(m), \varphi_s(M)\}\\ &= \max\{(f'(s) - f'(m))(s - m), (f'(M) - f'(s))(M - s)\}\end{aligned}$$

and the inequality (2.9) is obtained. □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}&\psi(f(c))\\ &\leq \frac{1}{M-m}\int_m^M f(s)\,ds - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c))\\ &\leq \frac{2}{M-m}\int_m^M f(s)\,ds - \frac{f(M)(M-\psi(c)) + f(m)(\psi(c)-m)}{M-m}\\ &+ \frac{1}{M-m}\\ &\times \int_m^M \max\{(f'(s) - f'(m))(s-m), (f'(M) - f'(s))(M-s)\}\,ds\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\psi(f(c)) + \frac{f(M)(M-\psi(c)) + f(m)(\psi(c)-m)}{M-m}\right]\\ &\leq \frac{1}{M-m}\int_m^M f(s)\,ds + \frac{3}{8}(f'(M) - f'(m))(M-m).\end{aligned} \tag{2.15}$$

Proof. If we take the integral mean in (2.9) we get

$$\begin{aligned}&\psi(f(c))\\ &\leq \frac{1}{M-m}\int_m^M f(s)\,ds - \frac{m+M}{2}\psi(f'(c)) + \psi(cf'(c))\\ &\leq \frac{1}{M-m}\int_m^M f(s)\,ds + \frac{1}{M-m}\int_m^M f'(s)(\psi(c)-s)\,ds\\ &+ \frac{1}{M-m}\\ &\times \int_m^M \max\{(f'(s) - f'(m))(s-m), (f'(M) - f'(s))(M-s)\}\,ds,\end{aligned} \tag{2.16}$$

and since

$$\begin{aligned}&\frac{1}{M-m}\int_m^M f'(s)(\psi(c)-s)\,ds\\&=\frac{1}{M-m}\int_m^M f(s)\,ds-\frac{f(M)(M-\psi(c))+f(m)(\psi(c)-m)}{M-m},\end{aligned}$$

hence the second inequality in (2.14) is proved.

Observe that, by the monotonicity of the derivative, we have

$$\begin{aligned}&\max\{(f'(s)-f'(m))(s-m),(f'(M)-f'(s))(M-s)\}\\&\leq(f'(M)-f'(m))\max\{s-m,M-s\}\\&=(f'(M)-f'(m))\left(\frac{1}{2}(M-m)+\left|s-\frac{m+M}{2}\right|\right)\end{aligned}\tag{2.17}$$

and by taking the integral mean we get

$$\begin{aligned}&\frac{1}{M-m}\int_m^M\max\{(f'(s)-f'(m))(s-m),(f'(M)-f'(s))(M-s)\}\,ds\\&\leq(f'(M)-f'(m))\max\{s-m,M-s\}\\&=(f'(M)-f'(m))\left(\frac{1}{2}(M-m)+\frac{1}{M-m}\int_m^M\left|s-\frac{m+M}{2}\right|ds\right)\\&=(f'(M)-f'(m))\left(\frac{1}{2}(M-m)+\frac{1}{4}(M-m)\right)\\&=\frac{3}{4}(f'(M)-f'(m))(M-m).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{2}{M-m}\int_m^M f(s)\,ds-\frac{f(M)(M-\psi(c))+f(m)(\psi(c)-m)}{M-m}\\&+\frac{1}{M-m}\int_m^M\max\{(f'(s)-f'(m))(s-m),(f'(M)-f'(s))(M-s)\}\,ds\\&\leq\frac{2}{M-m}\int_m^M f(s)\,ds-\frac{f(M)(M-\psi(c))+f(m)(\psi(c)-m)}{M-m}\\&+\frac{3}{4}(f'(M)-f'(m))(M-m).\end{aligned}$$

and by (2.14) we get

$$\begin{aligned}&\psi(f(c))\\&\leq\frac{2}{M-m}\int_m^M f(s)\,ds-\frac{f(M)(M-\psi(c))+f(m)(\psi(c)-m)}{M-m}\\&+\frac{3}{4}(f'(M)-f'(m))(M-m)\end{aligned}$$

that is equivalent to the desired result (2.15). □

Corollary 2. *With the assumptions of Theorem 3 and if* $\psi(f'(c))\neq 0$ *and*

$$s=\frac{\psi(cf'(c))}{\psi(f'(c))}\in I,$$

then we have the Slater's type inequality

$$\psi\left(f\left(c\right)\right)\leq f\left(\frac{\psi\left(cf'\left(c\right)\right)}{\psi\left(f'\left(c\right)\right)}\right). \tag{2.18}$$

Remark 1. *By the use of inequalities (2.4) and (2.15) and under the assumptions of Theorem 3 we can get the following upper and lower bounds for* $\psi\left(f\left(c\right)\right)$,

$$\begin{aligned}&\frac{2}{M-m}\int_m^M f\left(s\right)ds-\frac{f\left(M\right)\left(M-\psi\left(c\right)\right)+f\left(m\right)\left(\psi\left(c\right)-m\right)}{M-m}\\&\leq\psi\left(f\left(c\right)\right)\\&\leq\frac{2}{M-m}\int_m^M f\left(s\right)ds-\frac{f\left(M\right)\left(M-\psi\left(c\right)\right)+f\left(m\right)\left(\psi\left(c\right)-m\right)}{M-m}\\&+\frac{3}{4}\left(f'\left(M\right)-f'\left(m\right)\right)\left(M-m\right).\end{aligned} \tag{2.19}$$

We also observe that if $\sigma\left(c\right)\subseteq\left[m,M\right]\subset I,\ f'\left(c\right)>0$ *and* $\psi\left(f'\left(c\right)\right)>0$ *then*

$$m\leq\frac{\psi\left(cf'\left(c\right)\right)}{\psi\left(f'\left(c\right)\right)}\leq M$$

and the inequality (2.18) holds true.

3. Some Examples

Assume that $\psi:A\to\mathbb{C}$ is a positive normalized linear functional on A.

For $p\geq1$, consider the power function $f_p:(0,\infty)\to(0,\infty)$ and $0<c\in A$ which is analytic and convex on $(0,\infty)$. By using the inequalities (2.1) and (2.7) we have

$$s^p+ps^{p-1}\left(\psi\left(c\right)-s\right)\leq\psi\left(c^p\right)\leq s^p-s\left(p-1\right)\psi\left(c^{p-1}\right)+\left(p-1\right)\psi\left(c^p\right) \tag{3.1}$$

for any $s>0$.

The first inequality can be written as

$$\left(1-p\right)s^p+ps^{p-1}\psi\left(c\right)\leq\psi\left(c^p\right) \tag{3.2}$$

while the second as

$$\left(2-p\right)\psi\left(c^p\right)\leq s^p+s\left(1-p\right)\psi\left(c^{p-1}\right) \tag{3.3}$$

for any $s>0$.

If $p\in[1,2)$ then we get the double inequality

$$\left(1-p\right)s^p+ps^{p-1}\psi\left(c\right)\leq\psi\left(c^p\right)\leq\frac{1}{2-p}\left[s^p+s\left(1-p\right)\psi\left(c^{p-1}\right)\right] \tag{3.4}$$

for any $s>0$.

For $p>2$ we get by (3.3) that

$$\max\left\{\left(1-p\right)s^p+ps^{p-1}\psi\left(c\right),\frac{1}{2-p}\left[s^p+s\left(1-p\right)\psi\left(c^{p-1}\right)\right]\right\}\leq\psi\left(c^p\right) \tag{3.5}$$

for any $s>0$.

If $0<c\in A$ and for $p\geq1$ we have $\psi\left(c^{p-1}\right)>0$, then by the inequality (2.2) and (2.18)

$$\psi^p\left(c\right)\leq\psi\left(c^p\right)\leq\frac{\psi^p\left(c^p\right)}{\psi^p\left(c^{p-1}\right)}. \tag{3.6}$$

From this we get

$$\psi^p(c)\,\psi^p\left(c^{p-1}\right) \le \psi(c^p)\,\psi^p\left(c^{p-1}\right) \le \psi^p(c^p),$$

namely

$$\psi(c)\,\psi\left(c^{p-1}\right) \le \psi^{1/p}(c^p)\,\psi\left(c^{p-1}\right) \le \psi(c^p). \tag{3.7}$$

The second inequality in (3.7) is equivalent to

$$\psi^{1/(p-1)}\left(c^{p-1}\right) \le \psi^{1/p}(c^p),\ p > 1. \tag{3.8}$$

Similar results may be stated for $p < 1$ or for $p \in (0,1)$.

Assume that $0 < c \in A$ and there exist the constants $0 < m < M$ such that $\sigma(c) \subseteq [m, M]$. Then by (2.10), we have for $p > 1$ that

$$\begin{aligned}\psi(c^p) \le\ & p\left(\frac{m+M}{2}\right)^{p-1}\psi(c) + (1-p)\left(\frac{m+M}{2}\right)^p \\ & + \frac{1}{2}p(M-m) \\ & \times \max\left\{\left(\frac{m+M}{2}\right)^{p-1} - m^{p-1}, M^{p-1} - \left(\frac{m+M}{2}\right)^{p-1}\right\}\end{aligned} \tag{3.9}$$

and by (2.11) that

$$\begin{aligned}&\psi(c^p) \le \psi^p(c) \\ &\quad + p\max\left(\psi^{p-1}(c) - m^{p-1}\right)(\psi(c) - m), \left(M^{p-1} - \psi^{p-1}(c)\right)(M - \psi(c)).\end{aligned} \tag{3.10}$$

Consider the analytic convex function $f : (0,\infty) \to \mathbb{R},\ f(t) = -\ln t$. By using the inequalities (2.1) and (2.7) we have for $0 < c \in A$ that

$$-s\psi\left(c^{-1}\right) + \ln s + 1 \le \psi(\ln c) \le \psi(c)\,s^{-1} + \ln s - 1 \tag{3.11}$$

for any $s > 0$.

By the inequality (2.2) and (2.18) we have

$$-\ln\left(\psi\left(c^{-1}\right)\right) \le \psi(\ln(c)) \le \ln(\psi(c)), \tag{3.12}$$

provided $0 < c \in A$ and $\psi(c),\ \psi\left(c^{-1}\right) > 0$.

Assume that $0 < c \in A$ and there exist the constants $0 < m < M$ such that $\sigma(c) \subseteq [m, M]$. Then by (2.10), we have

$$\psi(\ln c) \ge \left(\frac{m+M}{2}\right)^{-1}\psi(c) + \ln\left(\frac{m+M}{2}\right) - \frac{1}{2}\frac{(M-m)^2}{m(m+M)} - 1 \tag{3.13}$$

and by (2.11) that

$$\psi(\ln c) \ge \ln(\psi(c)) - \frac{1}{\psi(c)}\max\left\{\frac{(\psi(c) - m)^2}{m}, \frac{\left(M - \psi(c)^2\right)}{M}\right\}. \tag{3.14}$$

For any selfadjoint element $c \in A$ we have

$$\exp(\psi(c)) \le \psi(\exp(c)) \le \exp\left(\frac{\psi(c\exp(c))}{\psi(\exp(c))}\right) \tag{3.15}$$

for any $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.

Assume that $c \in A$ is a selfadjoint element and there exist the real constants $m < M$ such that $\sigma(c) \subseteq [m, M]$. Then by (2.10),

$$\psi(\exp c) \leq \exp\left(\frac{m+M}{2}\right) + \exp\left(\frac{m+M}{2}\right)\left(\psi(c) - \frac{m+M}{2}\right) + \frac{1}{2}(M-m) \times \max\left\{\exp\left(\frac{m+M}{2}\right) - \exp(m), \exp(M) - \exp\left(\frac{m+M}{2}\right)\right\} \tag{3.16}$$

and by (2.11)

$$\psi(\exp(c)) \leq \exp(\psi(c)) + \max\{(\exp(\psi(c)) - \exp(m))(\psi(c) - m), (\exp(M) - \exp(\psi(c)))(M - \psi(c))\}, \tag{3.17}$$

for any $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.
[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.
[3] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach ∗-algebras, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 162. [http://rgmia.org/papers/v19/v19a162.pdf].
[4] S. S. Dragomir, Inequalities of McCarthy's type in Hermitian unital Banach ∗-algebras, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 171. [http://rgmia.org/papers/v19/v19a171.pdf].
[5] B. Q. Feng, The geometric means in Banach ∗-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.
[6] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990.
[7] T. Okayasu, The Löwner-Heinz inequality in Banach ∗-algebra, *Glasgow Math. J.* **42** (2000), 243-246.
[8] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.
[9] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach ∗-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

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