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INEQUALITIES OF GRÜSS' TYPE FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we establish some inequalities of Grüss' type for elements in a Hermitian unital Banach $*$-algebra and for positive normalized linear functionals on such algebras. Applications for convex functions of self-adjoint elements that provide reverses of Jensen's inequality and examples for some fundamental convex functions such as the power, logarithmic and exponential functions are given as well.

1. Introduction

In 1935, G. Grüss [11] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [5], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1. *Let $(H,\langle .,.\rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\ \mathbb{C})$ and $e\in H$, $\|e\|=1$. If $\varphi,\ \gamma,\ \Phi,\ \Gamma$ are real or complex numbers and $x,\ y$ are vectors in H such that the conditions*

$$\operatorname{Re}\langle\Phi e-x,x-\varphi e\rangle\geq 0 \text{ and } \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.3}$$

hold, then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\Phi-\varphi||\Gamma-\gamma|. \tag{1.4}$$

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The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller quantity.*

For other results of this type, see the recent monograph [6] and the references therein. For Grüss' type inequalities for positive maps, see [2], [12] and [14].

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [16] (see also [3, Theorem 41.5]), then

$$|a|^2 := a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [15], Tanahashi and Uchiyama [17] proved the following fundamental properties (see also [10]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

Okayasu [15] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

In order to introduce the real power of a positive element, we need the following facts [3, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\, z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \mathrm{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha} \in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^{\alpha}) = (\sigma(a))^{\alpha} = \{z^{\alpha} : z \in \sigma(a)\} \subset (0, \infty).$$

Following [10], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $\left(a^2\right)^{1/2} = a$, [17, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^{\alpha} a^{\beta} = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^{\alpha})^{-1} = \left(a^{-1}\right)^{\alpha} = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha} b^{\beta} = b^{\beta} a^{\alpha}$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [7].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

In the recent paper [8] we established the following McCarthy type inequality:

Theorem 2. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.*

(i) If $p \in (0, 1)$ and $a \geq 0$, then

$$\psi^{p}(a) \geq \psi(a^{p}) \geq 0; \tag{1.5}$$

(ii) If $q \geq 1$ and $b \geq 0$, then

$$\psi(b^{q}) \geq \psi^{q}(b) \geq 0; \tag{1.6}$$

(iii) If $r < 0$, $c > 0$ with $\psi(c) > 0$, then

$$\psi(c^{r}) \geq \psi^{r}(c) > 0. \tag{1.7}$$

In [9] we obtained the following result for analytic convex functions:

Theorem 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex (in the usual sense) on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$, we have*

$$f(\psi(c)) \leq \psi(f(c)) \leq f(\psi(c)) + \psi(cf'(c)) - \psi(c)\psi(f'(c)). \tag{1.8}$$

Motivated by the above results, we establish in this paper some inequalities of Grüss' type for elements in a Hermitian unital Banach $*$-algebra and for positive normalized linear functionals on such algebras. Applications for convex functions of selfadjoint elements that provide reverses of Jensen's inequality and examples for some fundamental convex functions such as the power, logarithmic and exponential functions are given as well.

2. Grüss' Type Inequalities

We have the following facts:

Lemma 2. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. The functional $\langle \cdot, \cdot \rangle_\psi : A \times A \to \mathbb{C}$ defined by*

$$\langle a, b \rangle_\psi = \psi (b^* a) \tag{2.1}$$

satisfies the following properties:

(i) $\|a\|_\psi^2 := \langle a, a \rangle_\psi \geq 0$ for any $a \in A$;

(ii) $\langle \alpha a + \beta c, b \rangle_\psi = \alpha \langle a, b \rangle_\psi + \beta \langle c, b \rangle_\psi$ and $\langle a, \alpha b \rangle_\psi = \overline{\alpha} \langle a, b \rangle_\psi$ for any α, $\beta \in \mathbb{C}$ and a, b, $c \in A$;

(iii) $\langle b, a \rangle_\psi = \overline{\langle a, b \rangle_\psi}$ for any a, $b \in A$;

namely, $\langle \cdot, \cdot \rangle_\psi$ is a positive semi-definite Hermite sesquilinear form.

We have the equality

$$\|\alpha a + b\|_\psi^2 = |\alpha|^2 \|a\|_\psi^2 + 2 \operatorname{Re} \left[\alpha \langle a, b \rangle_\psi \right] + \|b\|_\psi^2 \tag{2.2}$$

for any $\alpha \in \mathbb{C}$ and a, $b \in A$;

We have the Schwarz inequality

$$\left| \langle a, b \rangle_\psi \right|^2 \leq \|a\|_\psi^2 \|b\|_\psi^2 \tag{2.3}$$

and the triangle inequality

$$\|a + b\|_\psi \leq \|a\|_\psi + \|b\|_\psi \tag{2.4}$$

for any a, $b \in A$;

The functional $\|\cdot\|_\psi$ is a seminorm on A;

In particular

$$|\psi (a)|^2 \leq \psi \left(|a|^2 \right) = \|a\|_\psi^2 \tag{2.5}$$

for any $a \in A$.

Proof. For a proof in the case of C^*-algebras and states, see for instance [1, pp. 17-18].

In a similar way, we can give a proof for a Hermitian unital Banach $*$-algebra A and $\psi : A \to \mathbb{C}$ a positive normalized linear functional. This is as follows.

(i) Follows by the positivity of ψ;

(ii) Follows by the linearity of ψ;

(iii) For any $\alpha \in \mathbb{C}$ and a, $b \in A$ we have

$$0 \leq \|\alpha a + b\|_\psi^2 = |\alpha|^2 \|a\|_\psi^2 + \alpha \langle a, b \rangle_\psi + \overline{\alpha} \langle b, a \rangle_\psi + \|b\|_\psi^2 . \tag{2.6}$$

This implies that

$$\operatorname{Im} \left(\alpha \langle a, b \rangle_\psi + \overline{\alpha} \langle b, a \rangle_\psi \right) = 0 \tag{2.7}$$

for any $\alpha \in \mathbb{C}$.

If we take in (2.7) $\alpha = 1$, then we get $\operatorname{Im} \langle b, a\rangle_\psi = -\operatorname{Im} \langle a, b\rangle_\psi$. If we take in (2.7) $\alpha = i$, then we get $\operatorname{Re} \langle b, a\rangle_\psi = \operatorname{Re} \langle a, b\rangle_\psi$, which implies that (iii) is valid.

The equality (2.2) follows by (2.6).

Assume that $\|a\|_\psi^2 = 0$. If we assume that $\langle a, b\rangle_\psi \neq 0$ then by taking $\alpha = t\frac{\left|\langle a,b\rangle_\psi\right|}{\langle a,b\rangle_\psi}$ with $t \in \mathbb{R}$ in (2.2) we get

$$0 \leq 2t\left|\langle a, b\rangle_\psi\right| + \|b\|_\psi^2 \tag{2.8}$$

for any $t \in \mathbb{R}$. By taking negative t with large $|t|$ in (2.8) we obtain a contradiction. Therefore we conclude that $\langle a, b\rangle_\psi = 0$ and the inequality (2.3) is valid with equality.

If we assume that $\|a\|_\psi^2 \neq 0$ and take in (2.2) $\alpha = -\frac{\overline{\langle a,b\rangle_\psi}}{\|a\|_\psi^2}$, then we get

$$\begin{aligned} \left\|b - \frac{\overline{\langle a, b\rangle_\psi}}{\|a\|_\psi^2}a\right\|_\psi^2 &= \frac{\left|\langle a, b\rangle_\psi\right|^2}{\|a\|_\psi^2} - 2\frac{\left|\langle a, b\rangle_\psi\right|^2}{\|a\|_\psi^2} + \|b\|_\psi^2 \\ &= \frac{\|a\|_\psi^2 \|b\|_\psi^2 - \left|\langle a, b\rangle_\psi\right|^2}{\|a\|_\psi^2} \end{aligned} \tag{2.9}$$

and the inequality (2.3) is thus satisfied.

By (2.2) we have

$$\begin{aligned} \|a+b\|_\psi^2 &= \|a\|_\psi^2 + 2\operatorname{Re}\left[\langle a, b\rangle_\psi\right] + \|b\|_\psi^2 \\ &\leq \|a\|_\psi^2 + 2\left|\langle a, b\rangle_\psi\right| + \|b\|_\psi^2 \\ &\leq \|a\|_\psi^2 + 2\|a\|_\psi \|b\|_\psi + \|b\|_\psi^2 \leq \left(\|a\|_\psi + \|b\|_\psi\right)^2 \end{aligned}$$

implying the desired result (2.4).

The inequality (2.5) follows by (2.3) for $b = 1$. □

For an element $c \in A$ we define the selfadjoint elements

$$\operatorname{Re}(c) := \frac{1}{2}(c^* + c) \text{ and } \operatorname{Im}(c) := \frac{1}{2i}(c^* - c).$$

We have the *Cartesian* decomposition $c = \operatorname{Re}(c) + i\operatorname{Im}(c)$ and $c^* = \operatorname{Re}(c) - i\operatorname{Im}(c)$.

Lemma 3. *Assume that A is a Hermitian Banach $*$-algebra.*

If a, $b \in A$, then

$$\operatorname{Re}(a^*b) = \frac{1}{4}\left[|a+b|^2 - |a-b|^2\right]. \tag{2.10}$$

If c, $d \in A$ and γ, $\Gamma \in \mathbb{C}$ then

$$\operatorname{Re}\left((c^* - \overline{\gamma}d^*)(\Gamma d - c)\right) = \frac{1}{4}|\Gamma - \gamma|^2 |d|^2 - \left|c - \frac{\Gamma+\gamma}{2}d\right|^2. \tag{2.11}$$

If c, $d \in A$, and γ, $\Gamma \in \mathbb{C}$ then, the following inequalities are equivalent

$$\left|c - \frac{\Gamma+\gamma}{2}d\right|^2 \leq \frac{1}{4}|\Gamma - \gamma|^2 |d|^2 \tag{2.12}$$

and

$$\operatorname{Re}\left(\left(c^{*}-\overline{\gamma}d^{*}\right)\left(\Gamma d-c\right)\right)\geq 0. \tag{2.13}$$

Proof. The identity (2.10) follows by the equalities

$$|a\pm b|^{2}=|a|^{2}\pm a^{*}b\pm b^{*}a+|b|^{2}$$

that hold for any $a,\ b\in A$.

The equality (2.11) follow by (2.10) for $a=c-\gamma d$ and $b=\Gamma d-c$.

The equivalence of the inequalities (2.12) and (2.13) follow by the identity (2.11). □

We have the following result:

Theorem 4. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi:A\to\mathbb{C}$ a positive normalized linear functional on A. If $c,\ d\in A$, then*

$$0\leq\psi\left(|c|^{2}\right)\psi\left(|d|^{2}\right)-|\psi\left(d^{*}c\right)|^{2}\leq\psi\left(|d|^{2}\right)\psi\left(|c-\alpha d|^{2}\right) \tag{2.14}$$

for any $\alpha\in C$.

Proof. If $\psi\left(|d|^{2}\right)=0$ the inequality is trivial.

If $\psi\left(|d|^{2}\right)>0$, then by the properties of the functional ψ, we have for any complex number $\alpha\in\mathbb{C}$ that

$$\begin{aligned}
&\psi\left(\left(c^{*}-\frac{\overline{\psi\left(d^{*}c\right)}}{\psi\left(|d|^{2}\right)}d^{*}\right)\left(c-\alpha d\right)\right) \\
&=\psi\left(|c|^{2}-\frac{\overline{\psi\left(d^{*}c\right)}}{\psi\left(|d|^{2}\right)}d^{*}c-\alpha c^{*}d+\alpha\frac{\overline{\psi\left(d^{*}c\right)}}{\psi\left(|d|^{2}\right)}|d|^{2}\right) \\
&=\psi\left(|c|^{2}\right)-\frac{\overline{\psi\left(d^{*}c\right)}}{\psi\left(|d|^{2}\right)}\psi\left(d^{*}c\right)-\alpha\psi\left(c^{*}d\right)+\alpha\frac{\overline{\psi\left(d^{*}c\right)}}{\psi\left(|d|^{2}\right)}\psi\left(|d|^{2}\right) \\
&=\psi\left(|c|^{2}\right)-\frac{|\psi\left(d^{*}c\right)|^{2}}{\psi\left(|d|^{2}\right)}-\alpha\psi\left(c^{*}d\right)+\alpha\overline{\psi\left(d^{*}c\right)} \\
&=\psi\left(|c|^{2}\right)-\frac{|\psi\left(d^{*}c\right)|^{2}}{\psi\left(|d|^{2}\right)}-\alpha\psi\left(c^{*}d\right)+\alpha\psi\left(c^{*}d\right) \\
&=\psi\left(|c|^{2}\right)-\frac{|\psi\left(d^{*}c\right)|^{2}}{\psi\left(|d|^{2}\right)}=\frac{\psi\left(|c|^{2}\right)\psi\left(|d|^{2}\right)-|\psi\left(d^{*}c\right)|^{2}}{\psi\left(|d|^{2}\right)}.
\end{aligned} \tag{2.15}$$

Using the Schwarz inequality we have

$$\begin{aligned}(2.16)\qquad & \left|\psi\left(\left(c^* - \frac{\overline{\psi(d^*c)}}{\psi\left(|d|^2\right)}d^*\right)(c-\alpha d)\right)\right| \\ & = \left|\psi\left(\left(c - \frac{\psi(d^*c)}{\psi\left(|d|^2\right)}d\right)^*(c-\alpha d)\right)\right| \\ & \le \psi^{1/2}\left(\left|c - \frac{\psi(d^*c)}{\psi\left(|d|^2\right)}d\right|^2\right)\psi^{1/2}\left(|c-\alpha d|^2\right),\end{aligned}$$

for any $\alpha \in C$.

If we take in (2.15) $\alpha = \frac{\psi(d^*c)}{\psi\left(|d|^2\right)}$ we also have

$$(2.17)\qquad \psi\left(\left|c - \frac{\psi(d^*c)}{\psi\left(|d|^2\right)}d\right|^2\right) = \frac{\psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2}{\psi\left(|d|^2\right)}.$$

By (2.15)-(2.17) we get

$$\begin{aligned}(2.18)\qquad & \frac{\psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2}{\psi\left(|d|^2\right)} \\ & \le \left(\frac{\psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2}{\psi\left(|d|^2\right)}\right)^{1/2}\psi^{1/2}\left(|c-\alpha d|^2\right),\end{aligned}$$

which implies that

$$\left(\frac{\psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2}{\psi\left(|d|^2\right)}\right)^{1/2} \le \psi^{1/2}\left(|c-\alpha d|^2\right),$$

namely

$$0 \le \psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2 \le \psi\left(|d|^2\right)\psi\left(|c-\alpha d|^2\right)$$

for any $\alpha \in C$. □

Corollary 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. If c, $d \in A$, and γ, $\Gamma \in \mathbb{C}$ are such that, either of the conditions (2.12) or (2.13) is valid, then*

$$(2.19)\qquad 0 \le \psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2 \le \frac{1}{4}|\Gamma-\gamma|^2\psi^2\left(|d|^2\right).$$

Proof. If we take in (2.14) $\alpha = \frac{\Gamma+\gamma}{2}$, then we have

$$(2.20)\qquad 0 \le \psi\left(|c|^2\right)\psi\left(|d|^2\right) - |\psi(d^*c)|^2 \le \psi\left(|d|^2\right)\psi\left(\left|c - \frac{\Gamma+\gamma}{2}d\right|^2\right).$$

By taking the functional ψ in (2.12) we get

$$\psi\left(\left|c-\frac{\Gamma+\gamma}{2}d\right|^{2}\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\psi\left(\left|d\right|^{2}\right). \tag{2.21}$$

Utilising (2.20) and (2.21) we get (2.19). □

Corollary 2. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. If $c \in A$ and γ, $\Gamma \in \mathbb{C}$ are such that, either of the conditions*

$$\left|c-\frac{\Gamma+\gamma}{2}\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2} \tag{2.22}$$

or

$$\operatorname{Re}\left(\left(c^{*}-\overline{\gamma}\right)\left(\Gamma-c\right)\right)\geq0 \tag{2.23}$$

is valid, then

$$0\leq\psi\left(\left|c\right|^{2}\right)-\left|\psi\left(c\right)\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{2.24}$$

We say that the element $c \in A$ satisfies the *accretive property* (γ, Γ) for some γ, $\Gamma \in \mathbb{C}$ if either of the conditions (2.22) or (2.23) is valid.

We have:

Corollary 3. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. If $c \in A$ is a selfadjoint element and m, M are real numbers such that*

$$m\leq c\leq M, \tag{2.25}$$

then

$$0\leq\psi\left(c^{2}\right)-\psi^{2}\left(c\right)\leq\frac{1}{4}\left(M-m\right)^{2}. \tag{2.26}$$

The proof follows by Corollary 2 on observing that the condition (2.25) implies the fact that

$$\left(c-\frac{m+M}{2}\right)^{2}\leq\frac{1}{2}\left(M-m\right)^{2}. \tag{2.27}$$

Indeed, if $z \in [m, M]$ then $\left(z-\frac{m+M}{2}\right)^{2}\leq\frac{1}{2}\left(M-m\right)^{2}$ and by using Lemma 1 for the selfadjoint element c with $\sigma(c) \subseteq [m, M]$ we obtain (2.27).

Remark 1. *Let $\psi : A \to \mathbb{C}$ be a positive normalized linear functional on A. If we take the functional ψ in the equality (2.11), then we get*

$$\psi\left(\left|c-\frac{\Gamma+\gamma}{2}d\right|^{2}\right)=\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\psi\left(\left|d\right|^{2}\right)-\psi\left[\operatorname{Re}\left(\left(c^{*}-\overline{\gamma}d^{*}\right)\left(\Gamma d-c\right)\right)\right]. \tag{2.28}$$

By utilising (2.20) we get the inequality

$$\begin{aligned}0\leq\psi\left(\left|c\right|^{2}\right)\psi\left(\left|d\right|^{2}\right)-\left|\psi\left(d^{*}c\right)\right|^{2}&\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\psi^{2}\left(\left|d\right|^{2}\right)\\&-\psi\left(\left|d\right|^{2}\right)\psi\left[\operatorname{Re}\left(\left(c^{*}-\overline{\gamma}d^{*}\right)\left(\Gamma d-c\right)\right)\right],\end{aligned} \tag{2.29}$$

for any γ, $\Gamma \in \mathbb{C}$.

If

$$\psi\left[\operatorname{Re}\left(\left(c^*-\overline{\gamma}d^*\right)\left(\Gamma d-c\right)\right)\right]\geq 0, \tag{2.30}$$

which is a weaker condition than (2.13, then the inequality (2.19) also holds.

In particular, for $d=1$ we get

$$0\leq\psi\left(|c|^2\right)-|\psi\left(c\right)|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2-\psi\left[\operatorname{Re}\left(\left(c^*-\overline{\gamma}\right)\left(\Gamma-c\right)\right)\right] \tag{2.31}$$

for any γ, $\Gamma\in\mathbb{C}$.

Therefore a weaker condition for the inequality (2.24) to hold is that

$$\psi\left[\operatorname{Re}\left(\left(c^*-\overline{\gamma}\right)\left(\Gamma-c\right)\right)\right]\geq 0.$$

We have the following Grüss' type inequalities:

Theorem 5. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi:A\to\mathbb{C}$ a positive normalized linear functional on A. Then for any c, $d\in A$ and $\alpha\in\mathbb{C}$ we have*

$$\left|\psi\left(d^*c\right)-\psi\left(d^*\right)\psi\left(c\right)\right|\leq\psi^{1/2}\left(|d-\alpha|^2\right)\left(\psi\left(|c|^2\right)-|\psi\left(c\right)|^2\right)^{1/2}. \tag{2.32}$$

In particular, for $\alpha=\psi\left(d\right)$ we have

$$\left|\psi\left(d^*c\right)-\psi\left(d^*\right)\psi\left(c\right)\right|\leq\left(\psi\left(|d|^2\right)-|\psi\left(d\right)|^2\right)^{1/2}\left(\psi\left(|c|^2\right)-|\psi\left(c\right)|^2\right)^{1/2}. \tag{2.33}$$

Proof. For any $\alpha\in\mathbb{C}$ we have

$$\begin{aligned}\psi\left(\left(d-\alpha\right)^*\left(c-\psi\left(c\right)\right)\right)&=\psi\left(\left(d^*-\overline{\alpha}\right)\left(c-\psi\left(c\right)\right)\right)\\&=\psi\left(d^*c-d^*\psi\left(c\right)\right)-\overline{\alpha}\psi\left(c-\psi\left(c\right)\right)\\&=\psi\left(d^*c\right)-\psi\left(d^*\right)\psi\left(c\right).\end{aligned} \tag{2.34}$$

Using Schwarz's inequality we have

$$\begin{aligned}\left|\psi\left(d^*c\right)-\psi\left(d^*\right)\psi\left(c\right)\right|&=\left|\psi\left(\left(d-\alpha\right)^*\left(c-\psi\left(c\right)\right)\right)\right|\\&\leq\psi^{1/2}\left(|d-\alpha|^2\right)\psi^{1/2}\left(|c-\psi\left(c\right)|^2\right)\\&=\psi^{1/2}\left(|d-\alpha|^2\right)\left(\psi\left(\left(c-\psi\left(c\right)\right)^*\left(c-\psi\left(c\right)\right)\right)\right)^{1/2}\\&=\psi^{1/2}\left(|d-\alpha|^2\right)\left(\psi\left(\left(c^*-\overline{\psi\left(c\right)}\right)\left(c-\psi\left(c\right)\right)\right)\right)^{1/2}\\&=\psi^{1/2}\left(|d-\alpha|^2\right)\left(\psi\left(c^*c-\overline{\psi\left(c\right)}c-\psi\left(c\right)c^*+|\psi\left(c\right)|\right)\right)^{1/2}\\&=\psi^{1/2}\left(|d-\alpha|^2\right)\\&\times\left(\psi\left(|c|^2\right)-\overline{\psi\left(c\right)}\psi\left(c\right)-\psi\left(c\right)\psi\left(c^*\right)+|\psi\left(c\right)|\right)^{1/2}\\&=\psi^{1/2}\left(|d-\alpha|^2\right)\left(\psi\left(|c|^2\right)-|\psi\left(c\right)|\right)^{1/2}\end{aligned}$$

for any $\alpha\in\mathbb{C}$. This proves (2.32). □

Corollary 4. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. Then for any c, $d \in A$ and γ, Γ, δ, $\Delta \in \mathbb{C}$ we have*

$$\begin{aligned} |\psi(d^*c) - \psi(d^*)\psi(c)| &\le \left(\psi\left(|c|^2\right) - |\psi(c)|^2\right)^{1/2} \left(\psi\left(|d|^2\right) - |\psi(d)|^2\right)^{1/2} \\ &\le \left(\frac{1}{4}|\Gamma-\gamma|^2 - \psi\left[\operatorname{Re}\left((c^* - \overline{\gamma})(\Gamma - c)\right)\right]\right) \\ &\times \left(\frac{1}{4}|\Delta-\delta|^2 - \psi\left[\operatorname{Re}\left(\left(d^* - \overline{\delta}\right)(\Delta - d)\right)\right]\right). \end{aligned} \tag{2.35}$$

The proof follows by (2.33) and (2.31).

Corollary 5. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. If the element $c \in A$ satisfies the accretive property (γ, Γ) for some $\gamma, \Gamma \in \mathbb{C}$ and $d \in A$ satisfies the accretive property (δ, Δ) for some δ, $\Delta \in \mathbb{C}$ then we have*

$$\begin{aligned} &|\psi(d^*c) - \psi(d^*)\psi(c)| \\ &\le \left(\psi\left(|c|^2\right) - |\psi(c)|^2\right)^{1/2} \left(\psi\left(|d|^2\right) - |\psi(d)|^2\right)^{1/2} \\ &\le \left(\frac{1}{4}|\Gamma-\gamma|^2 - \psi\left[\operatorname{Re}\left((c^* - \overline{\gamma})(\Gamma - c)\right)\right]\right) \\ &\times \left(\frac{1}{4}|\Delta-\delta|^2 - \psi\left[\operatorname{Re}\left(\left(d^* - \overline{\delta}\right)(\Delta - d)\right)\right]\right) \\ &\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta| \\ &- \psi^{1/2}\left[\operatorname{Re}\left((c^* - \overline{\gamma})(\Gamma - c)\right)\right] \psi^{1/2}\left[\operatorname{Re}\left(\left(d^* - \overline{\delta}\right)(\Delta - d)\right)\right] \\ &\le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|. \end{aligned} \tag{2.36}$$

Proof. We must prove only the third inequality. This follows by the elementary inequality

$$\left(m^2 - n^2\right)\left(p^2 - q^2\right) \le mp - nq$$

where $m \ge n \ge 0$ and $p \ge q \ge 0$ on choosing

$$m = \frac{1}{2}|\Gamma-\gamma|, \ n = \psi^{1/2}\left[\operatorname{Re}\left((c^* - \overline{\gamma})(\Gamma - c)\right)\right]$$

and

$$p = \frac{1}{2}|\Delta-\delta|, \ q = \psi^{1/2}\left[\operatorname{Re}\left(\left(d^* - \overline{\delta}\right)(\Delta - d)\right)\right].$$

□

We have:

Corollary 6. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A. If c, $d \in A$ are selfadjoint elements and m, M, n, N are real numbers such that*

$$m \le c \le M \text{ and } n \le d \le N, \tag{2.37}$$

then

$$
\begin{aligned}
(2.38)\qquad |\psi(dc)-\psi(d)\psi(c)| &\leq \left(\psi\left(c^2\right)-\psi^2(c)\right)^{1/2}\left(\psi\left(d^2\right)-\psi^2(d)\right)^{1/2} \\
&\leq \left(\frac{1}{4}(M-m)^2-\psi((c-m)(M-c))\right) \\
&\times \left(\frac{1}{4}(N-n)^2-\psi((d-n)(N-d))\right) \\
&\leq \frac{1}{4}(M-m)(N-n) \\
&-\psi^{1/2}((c-m)(M-c))\,\psi^{1/2}((d-n)(N-d)) \\
&\leq \frac{1}{4}(M-m)(N-n).
\end{aligned}
$$

Remark 2. *For applications, the following inequalities may be useful*

$$
(2.39)\qquad |\psi(dc)-\psi(d)\psi(c)| \leq \begin{cases} \frac{1}{2}(M-m)\left(\psi\left(d^2\right)-\psi^2(d)\right)^{1/2} \\ \\ \frac{1}{2}(N-n)\left(\psi\left(c^2\right)-\psi^2(c)\right)^{1/2} \end{cases}
\leq \frac{1}{4}(M-m)(N-n),
$$

provided that $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, c, $d \in A$ are selfadjoint elements and m, M, n, N are real numbers such that the condition (2.37) holds.

3. Applications for Convex Functions

Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subset I$, we have (see Theorem 3)

$$
(3.1)\qquad 0 \leq \psi(f(c)) - f(\psi(c)) \leq \psi(cf'(c)) - \psi(c)\psi(f'(c)).
$$

If bounds for the spectrum $\sigma(c)$ are known, then further simpler bounds may be provided.

Theorem 6. *With the assumptions of Theorem 3, then for any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m, M] \subset I$ for some real numbers $m < M$,*

$$
(3.2)\qquad 0 \leq \psi(f(c)) - f(\psi(c)) \leq \psi(cf'(c)) - \psi(c)\psi(f'(c))
$$

$$
\leq \begin{cases} \frac{1}{2}(M-m)\left[\psi\left([f'(c)]^2\right)-\psi^2(f'(c))\right]^{1/2} \\ \\ \frac{1}{2}[f'(M)-f'(m)]\left(\psi\left(c^2\right)-\psi^2(c)\right)^{1/2} \end{cases}
$$

$$
\leq \frac{1}{4}(M-m)[f'(M)-f'(m)].
$$

The proof follows by (3.1) and the Grüss' type inequality (2.39).

Using (3.2) we can provide the following reverses of McCarthy inequalities from Theorem 2.

Assume that the selfadjoint element $c \in A$ satisfies the condition $\sigma(c) \subseteq [m, M] \subset [0, \infty)$. Then for any $q \geq 1$ we have

$$0 \leq \psi(b^q) - \psi^q(b) \leq q\left[\psi(c^q) - \psi(c)\psi\left(c^{q-1}\right)\right] \tag{3.3}$$

$$\leq \begin{cases} \frac{1}{2}q(M-m)\left[\psi\left(c^{2(q-1)}\right) - \psi^2\left(c^{q-1}\right)\right]^{1/2} \\ \\ \frac{1}{2}q\left(M^{q-1} - m^{q-1}\right)\left(\psi\left(c^2\right) - \psi^2(c)\right)^{1/2} \end{cases}$$

$$\leq \frac{1}{4}q(M-m)\left(M^{q-1} - m^{q-1}\right).$$

For any $p \in (0,1)$ and $\sigma(c) \subseteq [m, M] \subset (0, \infty)$ we have

$$0 \leq \psi^p(c) - \psi(c^p) \leq p\left[\psi(c)\psi\left(c^{p-1}\right) - \psi(c^p)\right] \tag{3.4}$$

$$\leq \begin{cases} \frac{1}{2}p(M-m)\left[\psi\left(c^{2(p-1)}\right) - \psi^2\left(c^{p-1}\right)\right]^{1/2} \\ \\ \frac{1}{2}p\frac{M^{1-p}-m^{1-p}}{m^{1-p}M^{1-p}}\left(\psi\left(c^2\right) - \psi^2(c)\right)^{1/2} \end{cases}$$

$$\leq \frac{1}{4}p(M-m)\frac{M^{1-p} - m^{1-p}}{m^{1-p}M^{1-p}}.$$

If we take $p = 1/2$ in (3.4) we get

$$0 \leq \psi^{1/2}(c) - \psi\left(c^{1/2}\right) \leq \frac{1}{2}\left[\psi(c)\psi\left(c^{-1/2}\right) - \psi\left(c^{1/2}\right)\right] \tag{3.5}$$

$$\leq \begin{cases} \frac{1}{4}(M-m)\left[\psi\left(c^{-1}\right) - \psi^2\left(c^{-1/2}\right)\right]^{1/2} \\ \\ \frac{1}{4}\frac{M^{1/2}-m^{1/2}}{m^{1/2}M^{1/2}}\left(\psi\left(c^2\right) - \psi^2(c)\right)^{1/2} \end{cases}$$

$$\leq \frac{1}{8}(M-m)\frac{M^{1/2} - m^{1/2}}{m^{1/2}M^{1/2}}.$$

For $r<0$, $\sigma(c)\subseteq[m,M]\subset(0,\infty)$ then

$$0\leq\psi\left(c^{r}\right)-\psi^{r}\left(c\right)\leq r\left(\psi\left(c^{r}\right)-\psi\left(c\right)\psi\left(c^{r-1}\right)\right) \tag{3.6}$$

$$\leq\begin{cases}\frac{1}{2}\left(M-m\right)\left|r\right|\left[\psi\left(c^{2(r-1)}\right)-\psi^{2}\left(c^{r-1}\right)\right]^{1/2}\\ \\ \frac{1}{2}r\left(M^{r-1}-m^{r-1}\right)\left(\psi\left(c^{2}\right)-\psi^{2}\left(c\right)\right)^{1/2}\end{cases}$$

$$\leq\frac{1}{4}r\left(M-m\right)\left(M^{r-1}-m^{r-1}\right).$$

If we take $r=-1$ then we get

$$0\leq\psi\left(c^{-1}\right)-\psi^{-1}\left(c\right)\leq\psi\left(c\right)\psi\left(c^{-2}\right)-\psi\left(c^{-1}\right) \tag{3.7}$$

$$\leq\begin{cases}\frac{1}{2}\left(M-m\right)\left[\psi\left(c^{-4}\right)-\psi^{2}\left(c^{-2}\right)\right]^{1/2}\\ \\ \frac{1}{2}\frac{M^{2}-m^{2}}{M^{2}m^{2}}\left(\psi\left(c^{2}\right)-\psi^{2}\left(c\right)\right)^{1/2}\end{cases}$$

$$\leq\frac{1}{4}\left(M-m\right)^{2}\frac{M+m}{M^{2}m^{2}}.$$

Assume that $\sigma(c)\subseteq[m,M]\subset(0,\infty)$ and $\psi:A\to\mathbb{C}$ is a positive normalized linear functional on A. By applying the inequality (3.2) for the convex function $f(t)=-\ln t$, we get

$$0\leq\ln\left(\psi\left(c\right)\right)-\psi\left(\ln c\right)\leq\psi\left(c\right)\psi\left(c^{-1}\right)-1 \tag{3.8}$$

$$\leq\begin{cases}\frac{1}{2}\left(M-m\right)\left[\psi\left(c^{-2}\right)-\psi^{2}\left(c^{-1}\right)\right]^{1/2}\\ \\ \frac{1}{2}\frac{M-m}{mM}\left(\psi\left(c^{2}\right)-\psi^{2}\left(c\right)\right)^{1/2}\end{cases}$$

$$\leq\frac{1}{4}\frac{\left(M-m\right)^{2}}{mM}.$$

Assume that $\sigma(c)\subseteq[m,M]\subset\mathbb{R}$ and $\psi:A\to\mathbb{C}$ is a positive normalized linear functional on A. By applying the inequality (3.2) for the convex function $f(t)=\exp(\tau t)$, with $\tau\in\mathbb{R}$ and $\tau\neq0$, we get

$$0\leq\psi\left(\exp\left(\tau c\right)\right)-\exp\left(\tau\psi\left(c\right)\right)\leq\tau\left[\psi\left(c\exp\left(\tau c\right)\right)-\psi\left(c\right)\psi\left(\exp\left(\tau c\right)\right)\right] \tag{3.9}$$

$$\leq\begin{cases}\frac{1}{2}\left(M-m\right)\left|\tau\right|\left[\psi\left(\exp\left(2\tau c\right)\right)-\psi^{2}\left(\exp\left(\tau c\right)\right)\right]^{1/2}\\ \\ \frac{1}{2}\tau\left[\exp\left(\tau M\right)-\exp\left(\tau m\right)\right]\left(\psi\left(c^{2}\right)-\psi^{2}\left(c\right)\right)^{1/2}\end{cases}$$

$$\leq\frac{1}{4}\tau\left(M-m\right)\left[\exp\left(\tau M\right)-\exp\left(\tau m\right)\right].$$

References

[1] C. Bar and C. Becker, C^*-algebras, *Lect. Notes Phys.* **786** (2009), 1-37.

[2] S. Banić, D. Ilišević and S. Varošanec, Bessel- and Grüss-type inequalities in inner product modules. *Proc. Edinb. Math. Soc.* (2) 50 (2007), no. 1, 23–36.

[3] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.

[4] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[5] S. S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[6] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers Inc, New York, 2005, x+249 pp.

[7] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach $*$-algebras, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 162. [http://rgmia.org/papers/v19/v19a162.pdf].

[8] S. S. Dragomir, Inequalities of McCarthy's type in Hermitian unital Banach $*$-algebras, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 171. [http://rgmia.org/papers/v19/v19a171.pdf].

[9] S. S. Dragomir, Inequalities of Jensen's type for positive linear functionals on Hermitian unital Banach $*$-Algebras, *RGMIA Res. Rep. Coll.* **19** (2016), Art. 172. [http://rgmia.org/papers/v19/v19a172.pdf].

[10] B. Q. Feng, The geometric means in Banach $*$-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.

[11] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[12] J. S. Matharu and M. S. Moslehian, Grüss inequality for some types of positive linear maps. *J. Operator Theory* **73** (2015), no. 1, 265–278.

[13] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, 1990.

[14] L. Nikolova and S. Varošanec, Chebyshev and Grüss type inequalities involving two linear functionals and applications. *Math. Inequal. Appl.* **19** (2016), no. 1, 127–143.

[15] T. Okayasu, The Löwner-Heinz inequality in Banach $*$-algebra, *Glasgow Math. J.* **42** (2000), 243-246.

[16] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.

[17] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach $*$-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

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