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REVERSES AND REFINEMENTS OF JENSEN'S INEQUALITY FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

S. S. DRAGOMIR[1,2]

Abstract. We establish in this paper some inequalities for analytic and convex functions on an open interval and positive normalized functionals defined on a Hermitian unital Banach $*$-algebra. Reverses and refinements of Jensen's and Slater's type inequalities are provided. Some examples for particular convex functions of interest are given as well.

1. Introduction

We need some preliminary concepts and facts about Banach $*$-algebras.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that if A is a unital Banach $*$-algebra [14] (see also [2, Theorem 41.5]), then

$$|a|^2 := a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [13], Tanahashi and Uchiyama [15] proved the following fundamental properties (see also [9]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to

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be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is well known (see for instance [3, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [9], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [15, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [13] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [5].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$.

In the recent paper [6] we established the following McCarthy type inequality:

Theorem 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.*

(i) If $p \in (0,1)$ and $a \geq 0$, then

$$\psi^p(a) \geq \psi(a^p) \geq 0; \tag{1.1}$$

(ii) If $q \geq 1$ and $b \geq 0$, then

$$\psi(b^q) \geq \psi^q(b) \geq 0; \tag{1.2}$$

(iii) If $r < 0$, $c > 0$ with $\psi(c) > 0$, then

$$\psi(c^r) \geq \psi^r(c) > 0. \tag{1.3}$$

In [7] and [8] we obtained the following result for analytic convex functions:

Theorem 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex (in the usual sense) on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with with $\sigma(c) \subseteq [m,M] \subset I$ for some real numbers $m < M$,*

$$0 \leq \psi(f(c)) - f(\psi(c)) \leq \psi(cf'(c)) - \psi(c)\psi(f'(c)) \tag{1.4}$$

$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\psi\left([f'(c)]^2\right) - \psi^2(f'(c))\right]^{1/2} \\ \\ \frac{1}{2}[f'(M) - f'(m)]\left(\psi(c^2) - \psi^2(c)\right)^{1/2} \end{cases}$$

$$\leq \frac{1}{4}(M-m)[f'(M) - f'(m)].$$

Motivated by these results we establish in this paper some inequalities for analytic and convex functions on an open interval and positive normalized functionals defined on a Hermitian unital Banach $*$-algebra. Reverses and refinements of Jensen's and Slater's type inequalities are provided. Some examples for particular convex functions of interest are given as well.

2. Some Reverses

We have:

Theorem 3. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f is convex on the interval I and $\psi : A \to \mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m,M] \subset I$*

for some real numbers $m < M$,

$$0 \le \psi(f(c)) - f(\psi(c)) \tag{2.1}$$

$$\le \frac{(M - \psi(c))(\psi(c) - m)}{M - m} \sup_{t \in (m,M)} \Theta_f(t; m, M)$$

$$\le \begin{cases} \frac{1}{4}(M - m) \sup_{t \in (m,M)} \Theta_f(t; m, M) \\ \\ (M - \psi(c))(\psi(c) - m) \frac{f'(M) - f'(m)}{M - m} \end{cases}$$

$$\le \frac{1}{4}(M - m)[f'(M) - f'(m)]$$

provided $\psi(c) \in (m, M)$, *where* $\Theta_f(\cdot; m, M) : (m, M) \to \mathbb{R}$ *is defined by*

$$\Theta_f(t; m, M) = \frac{f(M) - f(t)}{M - t} - \frac{f(t) - f(m)}{t - m}.$$

We also have

$$0 \le \psi(f(c)) - f(\psi(c)) \le \frac{1}{4}(M - m)\Theta_f(\psi(c); m, M) \tag{2.2}$$

$$\le \frac{1}{4}(M - m) \sup_{t \in (m,M)} \Theta_f(t; m, M) \le \frac{1}{4}(M - m)[f'(M) - f'(m)],$$

provided $\psi(c) \in (m, M)$.

Proof. By the convexity of f on $[m, M]$ we have for any $z \in [m, M]$ that

$$f(z) \le \frac{z - m}{M - m} f(M) + \frac{M - z}{M - m} f(m). \tag{2.3}$$

Using Lemma 1 we have by (2.3) for any selfadjoint element $c \in A$ with $\sigma(c) \subseteq [m, M]$ that

$$f(c) \le f(M) \frac{c - m}{M - m} + f(m) \frac{M - c}{M - m} \tag{2.4}$$

in the order of A.

If we take in this inequality the functional ψ we get the following reverse of Jensen's inequality

$$\psi(f(c)) \le f(M) \frac{\psi(c) - m}{M - m} + f(m) \frac{M - \psi(c)}{M - m}. \tag{2.5}$$

This generalizes the scalar Lah-Ribarić inequality for convex functions that is well known in the literature, see for instance [10, p. 57] for an extension to selfadjoint operators in Hilbert spaces.

Define

$$\Delta_f(t; m, M) := \frac{(t - m) f(M) + (M - t) f(m)}{M - m} - f(t), \quad t \in [m, M],$$

then we have

$$\begin{aligned}
\Delta_f(t;m,M) &= \frac{(t-m)f(M)+(M-t)f(m)-(M-m)f(t)}{M-m} \\
&= \frac{(t-m)f(M)+(M-t)f(m)-(M-t+t-m)f(t)}{M-m} \\
&= \frac{(t-m)[f(M)-f(t)]-(M-t)[f(t)-f(m)]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Theta_f(t;m,M)
\end{aligned} \tag{2.6}$$

for any $t \in (m, M)$.

From (2.5) we have for $\psi(c) \in (m, M)$ that

$$\begin{aligned}
&\psi(f(c)) - f(\psi(c)) \\
&\le \frac{(\psi(c)-m)f(M)+(M-\psi(c))f(m)}{M-m} - f(\psi(c)) \\
&= \Delta_f(\psi(c);m,M) = \frac{(M-\psi(c))(\psi(c)-m)}{M-m}\Theta_f(\psi(c);m,M) \\
&\le \frac{(M-\psi(c))(\psi(c)-m)}{M-m}\sup_{t\in(m,M)}\Theta_f(t;m,M).
\end{aligned} \tag{2.7}$$

We also have

$$\begin{aligned}
\sup_{t\in(m,M)}\Theta_f(t;m,M) &= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right] \\
&\le \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right] + \sup_{t\in(m,M)}\left[-\frac{f(t)-f(m)}{t-m}\right] \\
&= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right] - \inf_{t\in(m,M)}\left[\frac{\Phi(t)-\Phi(m)}{t-m}\right] \\
&= f'(M) - f'(m)
\end{aligned}$$

and since, obviously

$$\frac{(M-\psi(c))(\psi(c)-m)}{M-m} \le \frac{1}{4}(M-m)$$

we have the desired result (2.1).

From (2.7) we have

$$\begin{aligned}
&\psi(f(c)) - f(\psi(c)) \le \frac{(M-\psi(c))(\psi(c)-m)}{M-m}\Theta_f(\psi(c);m,M) \\
&\le \frac{1}{4}(M-m)\Theta_f(\psi(c);m,M) \le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Theta_f(t;m,M) \\
&\le \frac{1}{4}(M-m)[f'(M)-f'(m)]
\end{aligned}$$

that proves (2.2). □

We also have:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
0 &\le \psi(f(c)) - f(\psi(c)) \\
&\le \left(1 + 2\frac{\left|\psi(c) - \frac{m+M}{2}\right|}{M-m}\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
&\le f(m) + f(M) - 2f\left(\frac{m+M}{2}\right).
\end{aligned} \tag{2.8}
$$

Proof. First of all, we recall the following result obtained by the author in [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
& n \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right] \\
& \le \frac{1}{P_n}\sum_{i=1}^{n}p_i\Phi(x_i) - \Phi\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right) \\
& n \max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n}\Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\right],
\end{aligned} \tag{2.9}
$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.9) that

$$
\begin{aligned}
& 2\min\{t, 1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\
& \le t\Phi(x) + (1-t)\Phi(y) - \Phi(tx + (1-t)y) \\
& \le 2\max\{t, 1-t\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]
\end{aligned} \tag{2.10}
$$

for any $x, y \in C$ and $t \in [0, 1]$.

If we use the second inequality in (2.10) for the convex function $f : I \to \mathbb{R}$ and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset I$, we have for $t = \frac{M-\psi(c)}{M-m}$ that

$$
\begin{aligned}
& \frac{(M-\psi(c))f(m) + (\psi(c)-m)f(M)}{M-m} \\
& - f\left(\frac{m(M-\psi(c)) + M(\psi(c)-m)}{M-m}\right) \\
& \le 2\max\left\{\frac{M-\psi(c)}{M-m}, \frac{\psi(c)-m}{M-m}\right\}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right],
\end{aligned} \tag{2.11}
$$

namely

$$
\begin{aligned}
&\frac{(M-\psi(c))\, f(m)+(\psi(c)-m)\, f(M)}{M-m}-f(\psi(c)) \\
&\leq\left(1+2 \frac{\left|\psi(c)-\frac{m+M}{2}\right|}{M-m}\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned} \tag{2.12}
$$

On making use of the first inequality in (2.7) and (2.12) we get the first part of (2.8).

The last part follows by the fact that $m \leq \psi(c) \leq M$. □

3. Refinements and Reverses

We start with the following result:

Theorem 5. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$, $[m, M] \subset I$ for some real numbers $m<M$, and $\psi: A \rightarrow \mathbb{C}$ is a positive normalized linear functional on A. If there exists the constants $K>k \geq 0$ such that*

$$
K \geq f''(z) \geq k \text{ for any } z \in[m, M], \tag{3.1}
$$

then for any selfadjoint element $c \in A$ with $\sigma(c) \subseteq[m, M] \subset I$,

$$
\frac{1}{2} K \psi\left[(c-t)^{2}\right] \geq \psi(f(c))-f'(t)(\psi(c)-t)-f(t) \geq \frac{1}{2} k \psi\left[(c-t)^{2}\right] \tag{3.2}
$$

and

$$
\frac{1}{2} K \psi\left[(c-t)^{2}\right] \geq \psi\left(c f'(c)\right)-t \psi\left(f'(c)\right)+f(t)-\psi(f(c)) \geq \frac{1}{2} k \psi\left[(c-t)^{2}\right], \tag{3.3}
$$

for any $t \in[m, M]$.

Proof. Using Taylor's representation with the integral remainder we can write the following identity

$$
f(z)=\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(z-t)^{k}+\frac{1}{n!} \int_{t}^{z} f^{(n+1)}(s)(z-s)^{n} d s \tag{3.4}
$$

for any z, $t \in \mathring{I}$, the interior of I.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $s=(1-s) c+s d$, $s \in[0,1]$ that

$$
\int_{c}^{d} h(s)\, d s=(d-c) \int_{0}^{1} h((1-s) c+s d)\, d s .
$$

Therefore,

$$
\begin{aligned}
& \int_{t}^{z} f^{(n+1)}(s)(z-s)^{n} d s \\
& =(z-t) \int_{0}^{1} f^{(n+1)}((1-s) t+s z)(z-(1-s) t-s z)^{n} d s \\
& =(z-t)^{n+1} \int_{0}^{1} f^{(n+1)}((1-s) t+s z)(1-s)^{n} d s .
\end{aligned}
$$

The identity (3.4) can then be written as

$$f(z)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(t)(z-t)^{k} + \frac{1}{n!}(z-t)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)t+sz)(1-s)^{n}ds. \tag{3.5}$$

For $n=1$ we get

$$f(z)=f(t)+(z-t)f'(t)+(z-t)^{2}\int_{0}^{1}f''((1-s)t+sz)(1-s)ds \tag{3.6}$$

for any $z,\, t\in\mathring{I}$.

By the condition (3.1) we have

$$K\int_{0}^{1}(1-s)ds\geq\int_{0}^{1}f''((1-s)t+sz)(1-s)ds\geq k\int_{0}^{1}(1-s)ds,$$

namely

$$\frac{1}{2}K\geq\int_{0}^{1}f''((1-s)t+sz)(1-s)ds\geq\frac{1}{2}k,$$

and by (3.6) we get the double inequality

$$\frac{1}{2}K(z-t)^{2}\geq f(z)-f(t)-(z-t)f'(t)\geq\frac{1}{2}k(z-t)^{2} \tag{3.7}$$

for any $z,\, t\in\mathring{I}$.

Fix $t\in[m,M]$. Using Lemma 1 and the inequality (3.7) we obtain for the element $c\in A$ with $\sigma(c)\subseteq[m,M]\subset I$ the following inequality in the order of A

$$\frac{1}{2}K(c-t)^{2}\geq f(c)-f(t)-(c-t)f'(t)\geq\frac{1}{2}k(c-t)^{2}.$$

If we take in this inequality the functional ψ we get (3.2).

Fix $z\in[m,M]$. Using Lemma 1 and the inequality (3.7) we obtain for the element $c\in A$ with $\sigma(c)\subseteq[m,M]\subset I$ the following inequality in the order of A

$$\frac{1}{2}K(c-z)^{2}\geq f(z)-f(c)-zf'(c)+cf'(c)\geq\frac{1}{2}k(c-z)^{2}. \tag{3.8}$$

If we take in this inequality the functional ψ we get

$$\frac{1}{2}K\psi\left[(c-z)^{2}\right]\geq\psi(cf'(c))-z\psi(f'(c))-\psi(f(c))+f(z) \geq\frac{1}{2}k\psi\left[(c-z)^{2}\right],$$

for any $z\in[m,M]$. If we replace z with t we get the desired result (3.3). □

Corollary 1. *With the assumptions of Theorem 5 we have the Jensen's type inequalities*

$$\frac{1}{2}K\left[\psi\left(c^{2}\right)-\psi^{2}(c)\right]\geq\psi(f(c))-f(\psi(c))\geq\frac{1}{2}k\left[\psi\left(c^{2}\right)-\psi^{2}(c)\right] \tag{3.9}$$

and

$$\frac{1}{2}K\left[\psi\left(c^{2}\right)-\psi^{2}(c)\right]\geq\psi(cf'(c))-\psi(c)\psi(f'(c))+f(\psi(c))-\psi(f(c)) \geq\frac{1}{2}k\left[\psi\left(c^{2}\right)-\psi^{2}(c)\right]. \tag{3.10}$$

Follows by Theorem 5 on choosing $t=\psi(c)\in[m,M]$.

Corollary 2. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}
&\frac{1}{2}K\psi\left[\left(c-\frac{m+M}{2}\right)^{2}\right] \\
&\geq \psi(f(c))-f'\left(\frac{m+M}{2}\right)\left(\psi(c)-\frac{m+M}{2}\right)-f\left(\frac{m+M}{2}\right) \\
&\geq \frac{1}{2}k\psi\left[\left(c-\frac{m+M}{2}\right)^{2}\right]
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&\frac{1}{2}K\psi\left[\left(c-\frac{m+M}{2}\right)^{2}\right] \\
&\geq \psi(cf'(c))-\frac{m+M}{2}\psi(f'(c))+f\left(\frac{m+M}{2}\right)-\psi(f(c)) \\
&\geq \frac{1}{2}k\psi\left[\left(c-\frac{m+M}{2}\right)^{2}\right].
\end{aligned} \tag{3.12}$$

Follows by Theorem 5 on choosing $t=\frac{m+M}{2}$.

Corollary 3. *With the assumptions of Theorem 5 and, if, in addition,* $t=\frac{\psi(cf'(c))}{\psi(f'(c))}\in[m,M]$ *with* $\psi(f'(c))\neq 0$, *then we have the Slater's type inequalities*

$$\begin{aligned}
\frac{1}{2}K\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^{2}\right] &\geq f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)-\psi(f(c)) \\
&\geq \frac{1}{2}k\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^{2}\right],
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\frac{1}{2}K\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^{2}\right] \\
&\geq f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)\frac{\psi(cf'(c))}{\psi(f'(c))}-\psi(c)f'\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right) \\
&-f\left(\frac{\psi(cf'(c))}{\psi(f'(c))}\right)+\psi(f(c)) \\
&\geq \frac{1}{2}k\psi\left[\left(c-\frac{\psi(cf'(c))}{\psi(f'(c))}\right)^{2}\right].
\end{aligned} \tag{3.14}$$

Follows by Follows by Theorem 5 on choosing $t=\frac{\psi(cf'(c))}{\psi(f'(c))}\in[m,M]$. We observe that a sufficient condition for this to happen is that $f'(c)>0$ and $\psi(f'(c))>0$.

Corollary 4. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}(3.15)\qquad &\frac{1}{4}K\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right]\\ &\geq\frac{1}{2}\left[\psi(f(c))+\frac{(M-\psi(c))f(M)+(\psi(c)-m)f(m)}{M-m}\right]\\ &-\frac{1}{M-m}\int_m^M f(t)\,dt\\ &\geq\frac{1}{4}k\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right]\end{aligned}$$

and

$$\begin{aligned}(3.16)\qquad &\frac{1}{2}K\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right]\\ &\geq\frac{1}{M-m}\int_m^M f(z)\,dz-\psi(f(c))-\frac{m+M}{2}\psi(f'(c))-\psi(cf'(c))\\ &\geq\frac{1}{2}k\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right].\end{aligned}$$

Proof. If we take the integral mean over t on $[m,M]$ in the inequality (3.7) we get

$$\begin{aligned}(3.17)\qquad &\frac{1}{2}K\frac{1}{M-m}\int_m^M (z-t)^2\,dt\\ &\geq f(z)-\frac{1}{M-m}\int_m^M f(t)\,dt-\frac{1}{M-m}\int_m^M (z-t)f'(t)\,dt\\ &\geq\frac{1}{2}\frac{1}{M-m}\int_m^M (z-t)^2\,dt\end{aligned}$$

for any $z\in[m,M]$.

Observe that

$$\begin{aligned}\frac{1}{M-m}\int_m^M (z-t)^2 &=\frac{(M-z)^3+(z-m)^3}{3(M-m)}\\ &=\frac{1}{3}\left[(z-m)^2+(M-z)^2-(z-m)(M-z)\right]\\ &=\frac{1}{3}\left[\frac{1}{4}(M-m)^2+3\left(z-\frac{m+M}{2}\right)^2\right]\\ &=\frac{1}{12}(M-m)^2+\left(z-\frac{m+M}{2}\right)^2\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{M-m}\int_m^M (z-t)\,f'(t)\,dt \\
&= \frac{1}{M-m}\left[(z-t)\,f(t)|_m^M + \int_m^M f(t)\,dt\right] \\
&= \frac{1}{M-m}\left[\int_m^M f(t)\,dt - (M-z)\,f(M) - (z-m)\,f(m)\right] \\
&= \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{(M-z)\,f(M) + (z-m)\,f(m)}{M-m}.
\end{aligned}$$

Then by (3.17) we get

$$\begin{aligned}
&\frac{1}{2}K\left[\frac{1}{12}(M-m)^2 + \left(z - \frac{m+M}{2}\right)^2\right] \\
&\geq f(z) - \frac{1}{M-m}\int_m^M f(t)\,dt - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&+ \frac{(M-z)\,f(M) + (z-m)\,f(m)}{M-m} \\
&\geq \frac{1}{2}k\left[\frac{1}{12}(M-m)^2 + \left(z - \frac{m+M}{2}\right)^2\right]
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{4}K\left[\frac{1}{12}(M-m)^2 + \left(z - \frac{m+M}{2}\right)^2\right] \\
&\geq \frac{1}{2}\left[f(z) + \frac{(M-z)\,f(M) + (z-m)\,f(m)}{M-m}\right] - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&\geq \frac{1}{4}k\left[\frac{1}{12}(M-m)^2 + \left(z - \frac{m+M}{2}\right)^2\right]
\end{aligned} \tag{3.18}$$

for any $z \in [m, M]$.

Using Lemma 1 and the inequality (3.18) we obtain for the element $c \in A$ with $\sigma(c) \subseteq [m, M] \subset I$ the following inequality in the order of A

$$\begin{aligned}
&\frac{1}{4}K\left[\frac{1}{12}(M-m)^2 + \left(c - \frac{m+M}{2}\right)^2\right] \\
&\geq \frac{1}{2}\left[f(c) + \frac{(M-c)\,f(M) + (c-m)\,f(m)}{M-m}\right] - \frac{1}{M-m}\int_m^M f(t)\,dt \\
&\geq \frac{1}{4}k\left[\frac{1}{12}(M-m)^2 + \left(c - \frac{m+M}{2}\right)^2\right].
\end{aligned}$$

If we apply to this inequality the functional ψ we get (3.15).

If we take the integral mean over z on $[m, M]$ in the inequality (3.7) we get

$$\begin{aligned}
&\frac{1}{2}K\frac{1}{M-m}\int_m^M (z-t)^2\, dz \\
&\geq \frac{1}{M-m}\int_m^M f(z)\, dz - f(t) - \left(\frac{m+M}{2}-t\right) f'(t) \\
&\geq \frac{1}{2}k\frac{1}{M-m}\int_m^M (z-t)^2\, dz,
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}K\left[\frac{1}{12}(M-m)^2+\left(t-\frac{m+M}{2}\right)^2\right] \\
&\geq \frac{1}{M-m}\int_m^M f(z)\, dz - f(t) - \left(\frac{m+M}{2}-t\right) f'(t) \\
&\geq \frac{1}{2}k\left[\frac{1}{12}(M-m)^2+\left(t-\frac{m+M}{2}\right)^2\right]
\end{aligned} \tag{3.19}$$

for any $t \in [m, M]$.

Using (3.19) and a similar argument as above, we get the desired result (3.16). □

4. Some Examples

Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a positive normalized linear functional on A.

Let $c \in A$ be a selfadjoint element with $\sigma(c) \subseteq [m, M]$ for some real numbers $m < M$. If we take $f(t) = t^2$ and calculate

$$\Theta_f(t; m, M) = \frac{M^2-t^2}{M-t} - \frac{t^2-m^2}{t-m} = M - m$$

then by (2.1) we get

$$0 \leq \psi(c^2) - (\psi(c))^2 \leq (M-\psi(c))(\psi(c)-m) \leq \frac{1}{4}(M-m)^2. \tag{4.1}$$

Consider the convex function $f : [m, M] \subset (0, \infty) \to (0, \infty)$, $f(t) = t^p$, $p > 1$. Using the inequality (2.1) we have

$$\begin{aligned}
0 &\leq \psi(c^p) - (\psi(c))^p \leq p(M-\psi(c))(\psi(c)-m)\frac{M^{p-1}-m^{p-1}}{M-m} \\
&\leq \frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)
\end{aligned} \tag{4.2}$$

for any $c \in A$ a selfadjoint element with $\sigma(c) \subseteq [m, M] \subset (0, \infty)$.

If we use the inequality (2.8) we also get

$$\begin{aligned}
&0 \leq \psi(c^p) - (\psi(c))^p \\
&\leq \left(1 + 2\frac{\left|\psi(c) - \frac{m+M}{2}\right|}{M-m}\right)\left[\frac{m^p+M^p}{2} - \left(\frac{m+M}{2}\right)^p\right] \\
&\leq m^p + M^p - 2^{1-p}(m+M)^p
\end{aligned} \tag{4.3}$$

for any $c \in A$ a selfadjoint element with $\sigma(c) \subseteq [m, M] \subset (0, \infty)$.

Since $f''(t) = p(p-1)t^{p-2}$, $t>0$ then

$$k_p := p(p-1)\begin{cases} M^{p-2} \text{ for } p\in(1,2) \\ \\ m^{p-2} \text{ for } p\in[2,\infty) \end{cases} \tag{4.4}$$

$$\leq f''(t) \leq K_p := p(p-1)\begin{cases} m^{p-2} \text{ for } p\in(1,2) \\ \\ M^{p-2} \text{ for } p\in[2,\infty) \end{cases}$$

for any $t\in[m,M]$.

Using (3.9) and (3.10) we get

$$\frac{1}{2}K_p\left[\psi\left(c^2\right)-\psi^2(c)\right] \geq \psi(c^p)-(\psi(c))^p \geq \frac{1}{2}k_p\left[\psi\left(c^2\right)-\psi^2(c)\right] \tag{4.5}$$

and

$$\begin{aligned}\frac{1}{2}K_p\left[\psi\left(c^2\right)-\psi^2(c)\right] &\geq (p-1)\psi(c^p)+(\psi(c))^p-p\psi(c)\psi\left(c^{p-1}\right) \\ &\geq \frac{1}{2}k_p\left[\psi\left(c^2\right)-\psi^2(c)\right],\end{aligned} \tag{4.6}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Using (3.13) and (3.14) we get

$$\begin{aligned}\frac{1}{2}K_p\psi\left[\left(c-\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right] &\geq \left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^p-\psi(c^p) \\ &\geq \frac{1}{2}k_p\psi\left[\left(c-\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right],\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\frac{1}{2}K_p\psi\left[\left(c-\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right] \\ &\geq p\left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^{p-1}\left(\frac{\psi(c^p)}{\psi(c^{p-1})}-\psi(c)\right)-\left(\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^p+\psi(c^p) \\ &\geq \frac{1}{2}k_p\psi\left[\left(c-\frac{\psi(c^p)}{\psi(c^{p-1})}\right)^2\right]\end{aligned} \tag{4.8}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Using (3.15) and (3.16) we also have

$$\begin{aligned}&\frac{1}{4}K\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right] \\ &\geq \frac{1}{2}\left[\psi(c^p)+\frac{(M-\psi(c))M^p+(\psi(c)-m)m^p}{M-m}\right] \\ &-\frac{M^{p+1}-m^{p+1}}{(p+1)(M-m)} \\ &\geq \frac{1}{4}k\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right]\end{aligned} \tag{4.9}$$

and

$$\frac{1}{2}K_p\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right] \tag{4.10}$$
$$\geq \frac{M^{p+1}-m^{p+1}}{(p+1)(M-m)}-p\frac{m+M}{2}\psi\left(c^{p-1}\right)-(p+1)\psi\left(c^p\right)$$
$$\geq \frac{1}{2}k_p\left[\frac{1}{12}(M-m)^2+\psi\left[\left(c-\frac{m+M}{2}\right)^2\right]\right]$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Consider the convex function $f:[m,M]\subset(0,\infty)\to(0,\infty)$, $f(t)=\frac{1}{t}$. We have

$$\Theta_f(t;m,M)=\frac{\frac{1}{M}-\frac{1}{t}}{M-t}-\frac{\frac{1}{t}-\frac{1}{m}}{t-m}=\frac{M-m}{tmM},$$

which implies that

$$\sup_{t\in(m,M)}\Theta_f(t;m,M)=\frac{M-m}{m^2M}.$$

From (2.1) we get

$$0\leq\psi\left(c^{-1}\right)-\psi^{-1}(c)\leq\frac{(M-\psi(c))(\psi(c)-m)}{m^2M} \tag{4.11}$$
$$\leq\begin{cases}\frac{1}{4m^2M}(M-m)^2\\ \\ (M-\psi(c))(\psi(c)-m)\frac{M+m}{m^2M^2}\end{cases}\leq\frac{1}{4}(M-m)^2\frac{M+m}{M^2m^2}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

From (2.2) we have

$$0\leq\psi\left(c^{-1}\right)-\psi^{-1}(c)\leq\frac{1}{4}\frac{(M-m)^2}{mM}\psi^{-1}(c)\leq\frac{1}{4m^2M}(M-m)^2 \tag{4.12}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

From (2.8) we also have

$$0\leq\psi\left(c^{-1}\right)-\psi^{-1}(c)\leq\frac{(M-m)^2}{2mM(m+M)}\left(1+2\frac{\left|\psi(c)-\frac{m+M}{2}\right|}{M-m}\right) \tag{4.13}$$
$$\leq\frac{(M-m)^2}{mM(m+M)}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Since $f''(t)=\frac{2}{t^3}$, $t>0$, then $\frac{2}{m^3}\geq f''(t)\geq\frac{2}{M^3}$ and by (3.9) and (3.10) we get

$$\frac{1}{m^3}\left[\psi\left(c^2\right)-\psi^2(c)\right]\geq\psi\left(c^{-1}\right)-\psi^{-1}(c)\geq\frac{1}{M^3}\left[\psi\left(c^2\right)-\psi^2(c)\right] \tag{4.14}$$

and

$$\frac{1}{m^3}\left[\psi\left(c^2\right)-\psi^2(c)\right]\geq\frac{1}{2}\left[\psi(c)\psi\left(c^{-2}\right)+\psi^{-1}(c)\right]-\psi\left(c^{-1}\right) \tag{4.15}$$
$$\geq\frac{1}{M^3}\left[\psi\left(c^2\right)-\psi^2(c)\right],$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

From (3.13) and (3.14) we also have

$$\frac{1}{m^3}\psi\left[\left(c-\frac{\psi\left(c^{-1}\right)}{\psi\left(c^{-2}\right)}\right)^2\right] \geq \frac{\psi\left(c^{-2}\right)}{\psi\left(c^{-1}\right)}-\psi\left(c^{-1}\right) \geq \frac{1}{M^3}\psi\left[\left(c-\frac{\psi\left(c^{-1}\right)}{\psi\left(c^{-2}\right)}\right)^2\right], \tag{4.16}$$

and

$$\begin{aligned}&\frac{1}{m^3}\psi\left[\left(c-\frac{\psi\left(c^{-1}\right)}{\psi\left(c^{-2}\right)}\right)^2\right] \\ &\geq \psi\left(c^{-1}\right)-\frac{\psi\left(c^{-1}\right)}{\psi\left(c^{-2}\right)}+\psi\left(c\right)\frac{\psi^2\left(c^{-2}\right)}{\psi^2\left(c^{-1}\right)}-\frac{\psi\left(c^{-2}\right)}{\psi\left(c^{-1}\right)} \\ &\geq \frac{1}{M^3}\psi\left[\left(c-\frac{\psi\left(c^{-1}\right)}{\psi\left(c^{-2}\right)}\right)^2\right]\end{aligned} \tag{4.17}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Similar results may be stated for the convex functions $f(t)=t^r$, $r<0$ and $f(t)=-t^q$, $q\in(0,1)$.

The case of logarithmic function is also of interest. If we take the function $f(t)=-\ln t$ in (2.1), then we get

$$0\leq \ln\left(\psi\left(c\right)\right)-\psi\left(\ln c\right)\leq\frac{\left(M-\psi\left(c\right)\right)\left(\psi\left(c\right)-m\right)}{mM}\leq\frac{1}{4}\frac{\left(M-m\right)^2}{mM} \tag{4.18}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

From (2.8) we have

$$\begin{aligned}0&\leq \ln\left(\psi\left(c\right)\right)-\psi\left(\ln c\right)\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(1+2\frac{\left|\psi\left(c\right)-\frac{m+M}{2}\right|}{M-m}\right) \\ &\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^2\end{aligned} \tag{4.19}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Since $f''(t)=\frac{1}{t^2}$ and $\frac{1}{m^2}\geq f''(t)\geq\frac{1}{M^2}$ for any $t\in[m,M]\subset(0,\infty)$, then by (3.9) and (3.10) we have

$$\frac{1}{2m^2}\left[\psi\left(c^2\right)-\psi^2\left(c\right)\right]\geq \ln\left(\psi\left(c\right)\right)-\psi\left(\ln c\right)\geq\frac{1}{2M^2}\left[\psi\left(c^2\right)-\psi^2\left(c\right)\right] \tag{4.20}$$

and

$$\frac{1}{2m^2}\left[\psi\left(c^2\right)-\psi^2\left(c\right)\right]\geq \psi\left(\ln c\right)-\ln\left(\psi\left(c\right)\right)+\psi\left(c\right)\psi\left(c^{-1}\right)-1 \geq\frac{1}{2M^2}\left[\psi\left(c^2\right)-\psi^2\left(c\right)\right], \tag{4.21}$$

for any $c\in A$ a selfadjoint element with $\sigma(c)\subseteq[m,M]\subset(0,\infty)$.

Finally, by making use of (3.13) and (3.14) we have

$$\frac{1}{2m^2}\psi\left[\left(c-\psi^{-1}\left(c^{-1}\right)\right)^2\right] \geq \psi(\ln c) - \ln\left(\psi^{-1}\left(c^{-1}\right)\right) \geq \frac{1}{2M^2}\psi\left[\left(c-\psi^{-1}\left(c^{-1}\right)\right)^2\right], \tag{4.22}$$

and

$$\frac{1}{2m^2}\psi\left[\left(c-\psi^{-1}\left(c^{-1}\right)\right)^2\right] \geq \psi(c)\,\psi\left(c^{-1}\right) - 1 - \psi(\ln c) + \ln\left(\psi^{-1}\left(c^{-1}\right)\right) \geq \frac{1}{2M^2}\psi\left[\left(c-\psi^{-1}\left(c^{-1}\right)\right)^2\right] \tag{4.23}$$

for any $c \in A$ a selfadjoint element with $\sigma(c) \subseteq [m, M] \subset (0, \infty)$.

The interested reader may obtain other similar inequalities by using the convex functions $f(t) = t \ln t$, $t > 0$ and $f(t) = \exp(\alpha t)$, t, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa