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ADDITIVE REFINEMENTS AND REVERSES OF YOUNG AND HÖLDER'S INEQUALITIES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we obtain some additive refinements and reverses of the celebrated Young and Hölder's inequalities in the general setting of Hermitian unital Banach $*$-algebras and for positive linear functionals defined on such algebras.

1. Introduction

The famous *Young inequality* for scalars says that if $a,\, b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

In 1978, Cartwright & Field [3] obtained the following additive refinement and reverse of the arithmetic mean-geometric mean inequality,

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\max\{a,b\}}\leq A_{\nu}(a,b)-G_{\nu}(a,b)\leq\frac{1}{2}\nu(1-\nu)\frac{(b-a)^{2}}{\min\{a,b\}}, \tag{CF}$$

where $A_{\nu}(a,b):=(1-\nu)\,a+\nu b$ and $G_{\nu}(a,b):=a^{1-\nu}b^{\nu}$ if $a,\, b>0$ and $\nu\in[0,1]$.

In 2002, Tominaga [23] obtained a different additive reverse inequality, namely

$$A_{\nu}(a,b)-G_{\nu}(a,b)\leq\mathcal{S}\left(\frac{a}{b}\right)L(a,b), \tag{T}$$

where *Specht's ratio* $\mathcal{S}$, was introduced in 1960 in [21], and is defined by

$$\mathcal{S}(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1\end{cases} \tag{S}$$

and the *logarithmic mean* is defined by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if} \quad a\neq b\\ \\ b & \text{if} \quad a=b.\end{cases} \tag{L}$$

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Recall that $\mathcal{S}$ satisfies the properties

$$\lim_{h\to 1}\mathcal{S}(h)=1,\ \mathcal{S}(h)=\mathcal{S}\left(\frac{1}{h}\right)>1 \tag{1.2}$$

for $h>0$, $h\neq 1$, is *decreasing* on $(0,1)$ and *increasing* on $(1,\infty)$.

In 2010-11, Kittaneh & Manasrah, see [16], [17] obtained the inequality,

$$r\left(\sqrt{a}-\sqrt{b}\right)^2\leq A_\nu(a,b)-G_\nu(a,b)\leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{KM}$$

where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

This is a particular case of following result obtained by Dragomir in 2006, [5], that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right]\\ &\leq\frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)\\ &\leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right], \end{aligned} \tag{1.3}$$

where $\Phi:C\to\mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n=\sum_{j=1}^{n}p_j>0$.

For $n=2$, we deduce from (1.3) that

$$\begin{aligned} &2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]\\ &\leq\nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]\\ &\leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.4}$$

for any x, $y\in\mathbb{R}$ and $\nu\in[0,1]$.

If we take $\Phi(x)=\exp(x)$, then we get from (1.4) that

$$\begin{aligned} &2\min\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right]\\ &\leq\nu\exp(x)+(1-\nu)\exp(y)-\exp[\nu x+(1-\nu)y]\\ &\leq 2\max\{\nu,1-\nu\}\left[\frac{\exp(x)+\exp(y)}{2}-\exp\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.5}$$

for any x, $y\in\mathbb{R}$ and $\nu\in[0,1]$. Further, denote $\exp(x)=a$, $\exp(y)=b$ with a, $b>0$, then from (1.5) we obtain the inequality (KM).

In 2015, Alzer & Fonseca & Kovačec, [1] and Dragomir, [10] obtained independently and by using different techniques the following logarithmic upper and lower bounds for the difference of the arithmetic mean and geometric mean:

$$\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\min\{a,b\}\le A_{\nu}(a,b)-G_{\nu}(a,b)\le\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^{2}\max\{a,b\}. \tag{1.6}$$

A different reverse in terms of the logarithm was also obtained recently in the paper [6]

$$A_{\nu}(a,b)-G_{\nu}(a,b)\le\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

for any a, $b>0$ and $\nu\in[0,1]$.

If upper and lower bounds are assumed for the positive numbers a, b namely a, $b\in[\gamma,\Gamma]\subset(0,\infty)$ and $\nu\in[0,1]$, then [7]

$$A_{\nu}(a,b)-G({}_{\nu}a,b)\le\max\{g_{\gamma,\Gamma}(\nu),g_{\gamma,\Gamma}(1-\nu)\}, \tag{1.8}$$

where

$$g_{\gamma,\Gamma}(\nu):=(1-\nu)\gamma+\nu\Gamma-\gamma^{1-\nu}\Gamma^{\nu}. \tag{1.9}$$

In order to extend these results in the abstract setting of Hermitian unital Banach $*$-algebras and for positive linear functionals we need the following preparation.

2. Some Facts on Hermitian Unital Banach $*$-Algebra

Let A be a unital Banach $*$-algebra with unit 1. An element $a\in A$ is called *selfadjoint* if $a^{*}=a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a)\subset\mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a\ge0$ if $a^{*}=a$ and $\sigma(a)\subset[0,\infty)$. We say that a is *positive* and write $a>0$ if $a\ge0$ and $0\notin\sigma(a)$. Thus $a>0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If a, $b\in\operatorname{Inv}(A)$, then $ab\in\operatorname{Inv}(A)$ and $(ab)^{-1}=b^{-1}a^{-1}$. Also, saying that $a\ge b$ means that $a-b\ge0$ and, similarly $a>b$ means that $a-b>0$.

The *Shirali-Ford theorem* asserts that [20] (see also [2, Theorem 41.5])

$$a^{*}a\ge0\text{ for every }a\in A. \tag{SF}$$

Based on this fact, Okayasu [19], Tanahashi and Uchiyama [22] proved the following fundamental properties (see also [13]):

(i) If a, $b\in A$, then $a\ge0$, $b\ge0$ imply $a+b\ge0$ and $\alpha\ge0$ implies $\alpha a\ge0$;
(ii) If a, $b\in A$, then $a>0$, $b\ge0$ imply $a+b>0$;
(iii) If a, $b\in A$, then either $a\ge b>0$ or $a>b\ge0$ imply $a>0$;
(iv) If $a>0$, then $a^{-1}>0$;
(v) If $c>0$, then $0<b<a$ if and only if $cbc<cac$, also $0<b\le a$ if and only if $cbc\le cac$;
(vi) If $0<a<1$, then $1<a^{-1}$;
(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\le a$, then $0<a^{-1}\le b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re}z>0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [13], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [22, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [19] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

We define the following means for $\nu \in [0,1]$, see also [13] for different notations:

$$a \nabla_\nu b := (1-\nu) a + \nu b, \quad a, \; b \in A \tag{A}$$

the *weighted arithmetic mean* of (a, b),

$$a !_\nu b := \left((1-\nu) a^{-1} + \nu b^{-1}\right)^{-1}, \quad a, \; b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a, b) and

$$a \sharp_\nu b := a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^\upsilon a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a, b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a \nabla b$, $a!b$ and $a \sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [13], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a$, $b \in A$, then $a!b = b!a$ and $a \sharp b = b \sharp a$;
(xiii) If $0 < a$, $b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)c = (c^*ac)!(c^*bc) \text{ and } c^*(a\sharp b)c = (c^*ac)\sharp(c^*bc);$$

(xiv) If $0 < a$, $b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = (a^{-1}) \nabla_\nu (b^{-1}) \text{ and } (a^{-1}) \sharp_\nu (b^{-1}) = (a \sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [13] the following inequality between the weighted means introduced above:

$$a\nabla_{\nu}b \geq a\sharp_{\nu}b \geq a!_{\nu}b \tag{HGA}$$

for any $0 < a,\, b \in A$ and $\nu \in [0,1]$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0,\infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [11].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$. The functional ψ is called faithful if $a \geq 0$ and $\psi(a) = 0$ implies that $a = 0$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$, provided ψ is normalized. If the positive linear functional ψ is faithful and $a > 0$ then $\psi(a) > 0$.

In the following we obtain some additive refinements and reverses of the celebrated Young and Hölder's inequalities in the general setting of Hermitian unital Banach $*$-algebras and for positive linear functionals defined on such algebras.

3. Young and Hölder Type Inequalities

If we use the first inequality in (HGA) we can state the *Young type inequality*

$$x^{p}\sharp_{1/q}y^{q} \leq \frac{1}{p}x^{p} + \frac{1}{q}y^{q} \tag{Y}$$

for any $0 \leq x,\, y \in A$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We have the following *Hölder's type inequality* for positive functionals as well:

Theorem 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a faithful normalized positive linear functional. If $0 \leq a,\, b \in A$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\psi\left(a^{p}\sharp_{1/q}b^{q}\right) \leq \psi^{1/p}(a^{p})\,\psi^{1/q}(b^{q}). \tag{H}$$

In particular,

$$\psi^{2}\left(a^{2}\sharp_{1/2}b^{2}\right) \leq \psi\left(a^{2}\right)\psi\left(b^{2}\right). \tag{Sc}$$

Proof. If $\psi(a^p)=0$, then $a^p=0$ which implies that $a^p\sharp_{1/q}b^q=0$ and $\psi\left(a^p\sharp_{1/q}b^q\right)=0$ showing that the inequality (H) holds with equality. The same if $\psi(b^q)=0$.

Assume that $\psi(a^p),\ \psi(b^q)>0$. Then by Young's inequality for $x=\frac{a}{\psi^{1/p}(a^p)}$ and $y=\frac{b}{\psi^{1/q}(b^q)}$ we have

$$\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\sharp_{1/q}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q\le\frac{1}{p}\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p+\frac{1}{q}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q. \tag{3.1}$$

Observe that

$$\begin{aligned}&\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\sharp_{1/q}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q\\&=\left(\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\right)^{1/2}\\&\left(\left(\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\right)^{-1/2}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q\left(\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\right)^{-1/2}\right)^{1/q}\\&\left(\left(\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p\right)^{1/2}\right)^{1/2}\\&=\frac{a^{p/2}}{\psi^{1/2}(a^p)}\left(\frac{a^{-p/2}}{\psi^{-1/2}(a^p)}\frac{b^q}{\psi(b^q)}\frac{a^{-p/2}}{\psi^{-1/2}(a^p)}\right)^{1/q}\frac{a^{p/2}}{\psi^{1/2}(a^p)}\\&=\frac{1}{\psi(a^p)\psi^{-1/q}(a^p)\psi^{1/q}(b^q)}a^{p/2}\left(a^{-p/2}b^qa^{-p/2}\right)^{1/q}\\&=\frac{a^p\sharp_{1/q}b^q}{\psi^{1/q}(a^p)\psi^{1/q}(b^q)}\end{aligned}$$

and the inequality (3.1) may be written as

$$\frac{a^p\sharp_{1/q}b^q}{\psi^{1/q}(a^p)\psi^{1/q}(b^q)}\le\frac{1}{p\psi(a^p)}a^p+\frac{1}{q\psi(b^q)}b^q. \tag{3.2}$$

If we take in (3.2) the functional ψ then we get

$$\frac{\psi\left(a^p\sharp_{1/q}b^q\right)}{\psi^{1/q}(a^p)\psi^{1/q}(b^q)}\le\frac{1}{p\psi(a^p)}\psi(a^p)+\frac{1}{q\psi(b^q)}\psi(b^q)=1$$

and the inequality (H) is proved. □

Corollary 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi:A\to\mathbb{C}$ a faithful normalized positive linear functional. If $0\le a_i,\ b_i\in A$, $p_i\ge0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\psi\left(\sum_{i=1}^n p_i\left(a_i^p\sharp_{1/q}b_i^q\right)\right)\le\psi^{1/p}\left(\sum_{i=1}^n p_ia_i^p\right)\psi^{1/q}\left(\sum_{i=1}^n p_ib_i^q\right). \tag{3.3}$$

In particular,

$$\psi^2\left(\sum_{i=1}^{n}p_i\left(a_i^2\sharp b_i^2\right)\right)\le\psi\left(\sum_{i=1}^{n}p_ia_i^2\right)\psi\left(\sum_{i=1}^{n}p_ib_i^2\right). \tag{3.4}$$

Proof. Using discrete Hölder's inequality we have

$$\begin{aligned}\psi\left(\sum_{i=1}^{n}p_i\left(a_i^p\sharp_{1/q}b_i^q\right)\right)&=\sum_{i=1}^{n}p_i\psi\left(a_i^p\sharp_{1/q}b_i^q\right)\\&\le\sum_{i=1}^{n}p_i\psi^{1/p}\left(a_i^p\right)\psi^{1/q}\left(b_i^q\right)\text{ by (H)}\\&\le\left(\sum_{i=1}^{n}p_i\left(\psi^{1/p}\left(a_i^p\right)\right)^p\right)^{1/p}\left(\sum_{i=1}^{n}p_i\left(\psi^{1/q}\left(b_i^q\right)\right)^q\right)^{1/q}\\&=\left(\sum_{i=1}^{n}p_i\psi\left(a_i^p\right)\right)^{1/p}\left(\sum_{i=1}^{n}p_i\psi\left(b_i^q\right)\right)^{1/q}\\&=\psi^{1/p}\left(\sum_{i=1}^{n}p_ia_i^p\right)\psi^{1/q}\left(\sum_{i=1}^{n}p_ib_i^q\right)\end{aligned}$$

and the inequality (3.3) is proved. □

4. Additive Refinements and Reverses

We consider the function $f_\nu:[0,\infty)\to[0,\infty)$ defined for $\nu\in(0,1)$ by

$$f_\nu(t)=1-\nu+\nu t-t^\nu=A_\nu(1,t)-G_\nu(1,t),$$

where $A_\nu(\cdot,\cdot)$ and $G_\nu(\cdot,\cdot)$ are the scalar arithmetic and geometric means.

The following lemma holds.

Lemma 2. *For* $[k,K]\subset[0,\infty)$ *we have*

$$\max_{t\in[k,K]}f_\nu(t) \tag{4.1}$$
$$=\Delta_\nu(k,K):=\begin{cases}A_\nu(1,k)-G_\nu(1,k)\ \text{if } K<1,\\ \max\left\{A_\nu(1,k)-G_\nu(1,k),A_\nu(1,K)-G_\nu(1,K)\right\}\\ \text{if } k\le1\le K,\\ A_\nu(1,K)-G_\nu(1,K)\ \text{if } 1<k\end{cases}$$

and

$$\min_{t\in[k,K]}f_\nu(t)=\delta_\nu(k,K):=\begin{cases}A_\nu(1,K)-G_\nu(1,K)\ \text{if } K<1,\\ 0\ \text{if } k\le1\le K,\\ A_\nu(1,k)-G_\nu(1,k)\ \text{if } 1<k.\end{cases} \tag{4.2}$$

Proof. The function f_ν is differentiable and

$$f_\nu'(t)=\nu\left(1-t^{\nu-1}\right)=\nu\frac{t^{1-\nu}-1}{t^{1-\nu}},\ t>0,$$

which shows that the function f_ν is decreasing on $[0,1]$ and increasing on $[1,\infty)$, $f_\nu(0) = 1-\nu$, $f_\nu(1) = 0$, $\lim_{t\to\infty} f_\nu(t) = \infty$ and the equation $f_\nu(t) = 1-\nu$ for $t>0$ has the unique solution $t_\nu = \nu^{\frac{1}{\nu-1}} > 1$.

Therefore, by considering the 3 possible situations for the location of the interval $[k,K]$ and the number 1 we get the desired bounds (4.1) and (4.2). □

Remark 1. *We have the inequalities*

$$0 \le f_\nu(t) \le 1-\nu \text{ for any } t \in \left[0, \nu^{\frac{1}{\nu-1}}\right]$$

and

$$1-\nu \le f_\nu(t) \text{ for any } t \in \left[\nu^{\frac{1}{\nu-1}}, \infty\right).$$

We have the following additive refinement and reverse of Young's inequality:

Theorem 2. *Let $0<x$, $y\in A$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that there exists the constant m, $M>0$ such that*

$$m^q x^p \le y^q \le M^q x^p, \tag{4.3}$$

then

$$\delta_{1/q}(m^q, M^q)\, x^p \le \frac{1}{p}x^p + \frac{1}{q}y^q - x^p \sharp_{1/q} y^q \le \Delta_{1/q}(m^q, M^q)\, x^p \tag{4.4}$$

where the functions δ and Δ are defined by (4.1) and (4.2).

Proof. From the above Lemma 2 we have

$$\delta_\nu(k,K) \le 1-\nu+\nu z - z^\nu \le \Delta_\nu(k,K) \tag{4.5}$$

for any real $z \in [k,K] \subset (0,\infty)$ and for any $\nu \in [0,1]$.

Let $u \in A$ with spectrum $\sigma(u) \subset [k,K] \subset (0,\infty)$. Then by applying Lemma 1 for the corresponding analytic functions in the right half open plane $\{\operatorname{Re} z > 0\}$ involved in the inequality (3.4) we conclude that we have in the order of A that

$$\delta_\nu(k,K) \le 1-\nu+\nu u - u^\nu \le \Delta_\nu(k,K) \tag{4.6}$$

for any $\nu \in [0,1]$.

Since x is invertible, then by multiplying both sides of (4.3) with $x^{-p/2}>0$, we get $m^q \le x^{-p/2}y^q x^{-p/2} \le M^q$ and by taking $\nu = 1/q$, $u = x^{-p/2}y^q x^{-p/2}$, $k = m^q$ and $K = M^q$ we get in the order of A that

$$\begin{aligned}\delta_{1/q}(m^q, M^q) &\le \frac{1}{p} + \frac{1}{q}x^{-p/2}y^q x^{-p/2} - \left(x^{-p/2}y^q x^{-p/2}\right)^{1/q} \\ &\le \Delta_{1/q}(m^q, M^q).\end{aligned} \tag{4.7}$$

If we multiply both sides of (4.7) by $x^{p/2}>0$, then we get

$$\begin{aligned}\delta_{1/q}(m^q, M^q)\, x^p &\le \frac{1}{p}x^p + \frac{1}{q}y^q - x^{p/2}\left(x^{-p/2}y^q x^{-p/2}\right)^{1/q} x^{p/2} \\ &\le \Delta_{1/q}(m^q, M^q)\, x^p\end{aligned} \tag{4.8}$$

and the inequality (4.2) is proved. □

Corollary 2. *Let $0 < x,\ y \in A$. Assume that there exists the constant m, $M > 0$ such that*

$$m^2x^2 \le y^2 \le M^2x^2, \tag{4.9}$$

then

$$\frac{1}{2}\begin{cases} (1-M)^2\,x^2 \ \textit{if}\ M<1, \\ \\ 0\ \textit{if}\ m \le 1 \le M, \\ \\ (m-1)^2\,x^2\ \textit{if}\ 1<m, \end{cases} \tag{4.10}$$
$$\le \frac{x^2+y^2}{2} - x^2\sharp y^2$$
$$\le \frac{1}{2}\begin{cases} (1-m)^2\,x^2 \ \textit{if}\ M<1, \\ \\ \max\left\{(1-m)^2, (M-1)^2\right\}x^2\ \textit{if}\ m \le 1 \le M, \\ \\ (M-1)^2\,x^2\ \textit{if}\ 1<m. \end{cases}$$

Proof. If follows by taking $p = q = 2$ in Theorem 2 and observing that

$$\Delta_{1/2}\left(m^2, M^2\right)$$
$$= \begin{cases} A\left(1,m^2\right) - G\left(1,m^2\right)\ \text{if}\ M<1, \\ \\ \max\left\{A\left(1,m^2\right) - G\left(1,m^2\right), A\left(1,M^2\right) - G\left(1,M^2\right)\right\} \\ \text{if}\ m \le 1 \le M, \\ \\ A\left(1,M^2\right) - G\left(1,M^2\right)\ \text{if}\ 1<m \end{cases}$$
$$= \frac{1}{2}\begin{cases} (1-m)^2\ \text{if}\ M<1, \\ \\ \max\left\{(1-m)^2, (M-1)^2\right\} \\ \text{if}\ m \le 1 \le M, \\ \\ (M-1)^2\ \text{if}\ 1<m \end{cases}$$

and

$$\delta_{1/2}\left(m^2, M^2\right) = \frac{1}{2}\begin{cases} (1-M)^2\ \text{if}\ M<1, \\ \\ 0\ \text{if}\ m \le 1 \le M, \\ \\ (m-1)^2\ \text{if}\ 1<m. \end{cases}$$

□

Remark 2. *Let $0 < x,\ y \in A$. Assume that there exists the constant m_1, M_1, m_2, $M_2 > 0$ such that $m_1 \le x \le M_1$ and $m_2 \le y \le M_2$. Then $m_1^p \le x^p \le M_1^p$ and $m_2^q \le y^q \le M_2^2$ which implies that*

$$\frac{m_2^q}{M_1^p}x^p \le y^q \le \frac{M_2^q}{m_1^p}x^p$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Also,

$$\left(\frac{m_2}{M_1}\right)^2 x^2 \le y^2 \le \left(\frac{M_2}{m_1}\right)^2 x^2.$$

Therefore, by (4.2) we have

$$\delta_{1/q}\left(\frac{m_2^q}{M_1^p},\frac{M_2^q}{m_1^p}\right)x^p \le \frac{1}{p}x^p+\frac{1}{q}y^q - x^p\sharp_{1/q}y^q \le \Delta_{1/q}\left(\frac{m_2^q}{M_1^p},\frac{M_2^q}{m_1^p}\right)x^p \tag{4.11}$$

where the functions δ *and* Δ *are defined by (4.1) and (4.2).*

By (4.10) we also have

$$\begin{aligned}
&\frac{1}{2}\begin{cases}\left(1-\frac{M_2}{m_1}\right)^2 x^2 \text{ if } \frac{M_2}{m_1}<1,\\[2ex] 0 \text{ if } \frac{m_2}{M_1}\le 1\le \frac{M_2}{m_1},\\[2ex] \left(\frac{m_2}{M_1}-1\right)^2 x^2 \text{ if } 1<\frac{m_2}{M_1}\end{cases}\\
&\le \frac{x^2+y^2}{2}-x^2\sharp y^2\\
&\le \frac{1}{2}\begin{cases}\left(1-\frac{m_2}{M_1}\right)^2 x^2 \text{ if } \frac{M_2}{m_1}<1,\\[2ex] \max\left\{\left(1-\frac{m_2}{M_1}\right)^2,\left(\frac{M_2}{m_1}-1\right)^2\right\}x^2 \text{ if } \frac{m_2}{M_1}\le 1\le\frac{M_2}{m_1},\\[2ex] \left(\frac{M_2}{m_1}-1\right)^2 x^2 \text{ if } 1<\frac{m_2}{M_1}.\end{cases}
\end{aligned} \tag{4.12}$$

The following additive reverse of Hölder's inequality holds:

Theorem 3. *Assume that* A *is a Hermitian unital Banach* $*$*-algebra and* $\psi : A \to \mathbb{C}$ *a faithful normalized positive linear functional. If* $0 \le a,\ b \in A$ *such that there exists the constant* $k_1,\ K_1,\ k_2,\ K_2 > 0$ *with* $k_1 \le a \le K_1$ *and* $k_2 \le b \le K_2$ *and* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}
0 &\le 1-\frac{\psi\left(a^p\sharp_{1/q}b^q\right)}{\psi^{1/p}\left(a^p\right)\psi^{1/q}\left(b^q\right)}\\
&\le \max\left\{A_{1/q}\left(1,\left(\frac{k_2}{K_2}\right)^q\left(\frac{k_1}{K_1}\right)^p\right)-G_{1/q}\left(1,\left(\frac{k_2}{K_2}\right)^q\left(\frac{k_1}{K_1}\right)^p\right),\right.\\
&\left.A_{1/q}\left(1,\left(\frac{K_2}{k_2}\right)^q\left(\frac{K_1}{k_1}\right)^p\right)-G_{1/q}\left(1,\left(\frac{K_2}{k_2}\right)^q\left(\frac{K_1}{k_1}\right)^p\right)\right\}.
\end{aligned} \tag{4.13}$$

In particular,

$$0 \le 1-\frac{\psi^2\left(a^2\sharp b^2\right)}{\psi\left(a^2\right)\psi\left(b^2\right)} \le \frac{1}{2}\left(\frac{K_2K_1-k_2k_1}{k_2k_1}\right)^2. \tag{4.14}$$

Proof. We have $0<k_1\le\psi^{1/p}\left(a^p\right)\le K_1$ and $0<k_2\le\psi^{1/q}\left(b^q\right)\le K_2$. These imply that $\frac{k_1}{K_1}\le\frac{a}{\psi^{1/p}(a^p)}\le\frac{K_1}{k_1}$ and $\frac{k_2}{K_2}\le\frac{b}{\psi^{1/q}(b^q)}\le\frac{K_2}{k_2}$ and $\frac{k_1}{K_1},\ \frac{k_2}{K_2}\le 1\le\frac{K_1}{k_1}$,

$\frac{K_2}{k_2}$. Consider $x = \frac{a}{\psi(a)}$, $y = \frac{b}{\psi(b)}$, $m_1 = \frac{k_1}{K_1}$, $M_1 = \frac{K_1}{k_1}$, $m_2 = \frac{k_2}{K_2}$ and $M_2 = \frac{K_2}{k_2}$. Also, observe that

$$\frac{m_2^q}{M_1^p} = \frac{\left(\frac{k_2}{K_2}\right)^q}{\left(\frac{K_1}{k_1}\right)^p} = \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p \le 1$$

and

$$\frac{M_2^q}{m_1^p} = \frac{\left(\frac{K_2}{k_2}\right)^q}{\left(\frac{k_1}{K_1}\right)^p} = \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p \ge 1.$$

Using the inequality (4.11) we have

$$\begin{aligned}
0 &\le \frac{1}{p}\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p + \frac{1}{q}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q - \left(\frac{a}{\psi^{1/p}(a^p)}\right)^p \sharp_{1/q} \left(\frac{b}{\psi^{1/q}(b^q)}\right)^q \\
&\le \Delta_{1/q}\left(\left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right) \left(\frac{a}{\psi^{1/p}(a^p)}\right)^p \\
&= \max\left\{A_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right) - G_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right),\right. \\
&\quad \left. A_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right) - G_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)\right\} \left(\frac{a}{\psi^{1/p}(a^p)}\right)^p
\end{aligned}$$

namely

$$\begin{aligned}
0 &\le \frac{1}{p}\frac{a^p}{\psi(a^p)} + \frac{1}{q}\frac{b^q}{\psi(b^q)} - \frac{a^p \sharp_{1/q} b^q}{\psi^{1/q}(a^p)\,\psi^{1/q}(b^q)} \\
&= \max\left\{A_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right) - G_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right),\right. \\
&\quad \left. A_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right) - G_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)\right\} \frac{a^p}{\psi(a^p)}.
\end{aligned} \tag{4.15}$$

If we take in (4.15) the functional ψ, then we get the desired result (4.13).

We have

$$\frac{m_2}{M_1} = \frac{\frac{k_2}{K_2}}{\frac{K_1}{k_1}} = \frac{k_2 k_1}{K_2 K_1} \text{ and } \frac{M_2}{m_1} = \frac{\frac{K_2}{k_2}}{\frac{k_1}{K_1}} = \frac{K_2 K_1}{k_2 k_1}.$$

Therefore

$$\begin{aligned}
&\max\left\{\left(1 - \frac{m_2}{M_1}\right)^2, \left(\frac{M_2}{m_1} - 1\right)^2\right\} \\
&= \max\left\{\left(1 - \frac{k_2 k_1}{K_2 K_1}\right)^2, \left(\frac{K_2 K_1}{k_2 k_1} - 1\right)^2\right\} \\
&= \max\left\{\left(\frac{K_2 K_1 - k_2 k_1}{K_2 K_1}\right)^2, \left(\frac{K_2 K_1 - k_2 k_1}{k_2 k_1}\right)^2\right\} = \left(\frac{K_2 K_1 - k_2 k_1}{k_2 k_1}\right)^2
\end{aligned}$$

and by (4.12) we get the desired result (4.14). □

5. Further Bounds

By the use of the additive inequalities from the introduction we have further upper and lower bounds for the difference

$$A_{\nu}(1,t)-G_{\nu}(1,t)$$

with $t>0$ and $\nu\in[0,1]$.

Indeed, by (CF), (T), (KM), (1.6) and (1.7) we have the following upper bounds

$$A_{\nu}(1,t)-G_{\nu}(1,t)\leq\begin{cases}\frac{1}{2}\nu(1-\nu)\frac{(t-1)^2}{\min\{t,1\}},\\ \mathcal{S}(t)L(t,1),\\ \max\{\nu,1-\nu\}\left(\sqrt{t}-1\right)^2,\\ \frac{1}{2}\nu(1-\nu)(\ln t)^2\max\{t,1\},\\ \nu(1-\nu)(t-1)\ln t,\end{cases}\tag{5.1}$$

for any $t>0$ and $\nu\in[0,1]$.

By using the inequalities (CF), (KM) and (1.6) we have the lower bounds

$$\left.\begin{array}{r}\frac{1}{2}\nu(1-\nu)\frac{(t-1)^2}{\max\{t,1\}},\\ \min\{\nu,1-\nu\}\left(\sqrt{t}-1\right)^2,\\ \frac{1}{2}\nu(1-\nu)(\ln t)^2\min\{t,1\}\end{array}\right\}\leq A_{\nu}(1,t)-G_{\nu}(1,t)\tag{5.2}$$

for any $t>0$ and $\nu\in[0,1]$.

Observe that for $0<m<M$ and $q>1$, by making use of the definition (4.1) we have

$$\Delta_{1/q}(m^q,M^q)=\begin{cases}A_{1/q}(1,m^q)-G_{1/q}(1,m^q)\ \text{ if } M<1,\\ \\ \max\left\{A_{1/q}(1,m^q)-G_{1/q}(1,m^q),A_{1/q}(1,M^q)-G_{1/q}(1,M^q)\right\}\\ \text{if } m\leq1\leq M,\\ \\ A_{1/q}(1,M^q)-G_{1/q}(1,M^q)\ \text{ if } 1<m.\end{cases}\tag{5.3}$$

Using the inequalities (5.1) we have the following upper bounds for $\Delta_{1/q}(m^q,M^q)$.

If $0<m<M<1$, then

$$\Delta_{1/q}(m^q,M^q)\leq\begin{cases}\frac{1}{2pq}\frac{(m^q-1)^2}{m^q},\\ \mathcal{S}(m^q)L(m^q,1),\\ \max\left\{\frac{1}{p},\frac{1}{q}\right\}\left(m^{q/2}-1\right)^2,\\ \frac{q}{2p}(\ln m)^2,\\ \frac{1}{p}(m^q-1)\ln m.\end{cases}\tag{5.4}$$

If $0<m\leq1\leq M$, then

$$\Delta_{1/q}(m^q,M^q)\leq\begin{cases}\frac{1}{2pq}\max\left\{\frac{(1-m^q)^2}{m^q},(M^q-1)^2\right\},\\ \max\left\{\mathcal{S}(m^q)L(m^q,1),\mathcal{S}(M^q)L(M^q,1)\right\},\\ \max\left\{\frac{1}{p},\frac{1}{q}\right\}\max\left\{\left(1-m^{q/2}\right)^2,\left(M^{q/2}-1\right)^2\right\},\\ \frac{q}{2p}\max\left\{(\ln m)^2,(\ln M)^2M^q\right\},\\ \frac{1}{p}\max\left\{(m^q-1)\ln m,(M^q-1)\ln M\right\}.\end{cases}\tag{5.5}$$

If $1 < m < M$, then

$$\Delta_{1/q}\left(m^q, M^q\right) \leq \begin{cases} \frac{1}{2pq}\left(M^q - 1\right)^2, \\ \mathcal{S}\left(M^q\right) L\left(M^q, 1\right), \\ \max\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(M^{q/2} - 1\right)^2, \\ \frac{q}{2p}\left(\ln M\right)^2 M^q, \\ \frac{1}{p}\left(M^q - 1\right)\ln M. \end{cases} \tag{5.6}$$

Also, we observe that for $0 < m < M$ and $q > 1$, by making use of the definition (4.2) we have that

$$\delta_{1/q}\left(m^q, M^q\right) := \begin{cases} A_{1/q}\left(1, M^q\right) - G_{1/q}\left(1, M^q\right) \text{ if } M < 1, \\ \\ 0 \text{ if } m \leq 1 \leq M, \\ \\ A_{1/q}\left(1, m^q\right) - G_{1/q}\left(1, m^q\right) \text{ if } 1 < m. \end{cases} \tag{5.7}$$

Using the inequalities (5.2) we have the following lower bounds for $\delta_{1/q}\left(m^q, M^q\right)$.

If $0 < m < M < 1$, then

$$\left.\begin{array}{c} \frac{1}{2pq}\left(1 - M^q\right)^2, \\ \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(1 - M^{q/2}\right)^2, \\ \frac{q}{2p}\left(\ln M\right)^2 M^q \end{array}\right\} \leq \delta_{1/q}\left(m^q, M^q\right). \tag{5.8}$$

Finally, if $1 < m < M$, then

$$\left.\begin{array}{c} \frac{1}{2pq}\frac{\left(m^q - 1\right)^2}{m^q}, \\ \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(m^{q/2} - 1\right)^2, \\ \frac{q}{2p}\left(\ln m\right)^2 \end{array}\right\} \leq \delta_{1/q}\left(m^q, M^q\right). \tag{5.9}$$

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