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SOME GENERALIZED OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE CONVEX AND APPLICATIONS

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Abstract. We first establish some Ostrowski type inequalities for mappings whose second derivatives absolute values are convex and then we give some special cases of these inequalities which provide extensions of those given in earlier works. Finally, some applications of these inequalities for special means are also provided.

1. Introduction

The study of various types of integral inequalites has been the focus of great attention for well over a century by a number of scientists, interested both in pure and applied mathematics. One of the many fundemental mathematical discoveries of A. M. Ostrowski [13] is the following classical integral inequality associated with the differentiable mappings:

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality (1.1) has applications in numerical analysis, probability and optimization theory, stochastic, statistics, information and integral operator theory, see for example ([1]- [12],[14]- [20])

The remainder of this work is organized as follows: In this section, we present definition of convex function and give an important ideditity which will be used to establish our main results. In Section 2, some new Ostrowski type integral inequalities are proved for function whose second derivatives absolute values are convex. These inequalities are provided for special means in Section 3. At the end some conclusions of research are discussed in Section 4.

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

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In [10], Erden et al. gave the following important inequality for twice differentiable function:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, the interior of the interval I, where $a, b \in I^{\circ}$ with $a < b$. Then the following identity holds:*

$$
\begin{aligned}
(1.2)\qquad & \frac{1}{2(b-a)}\int_a^b P_h(x,t) f''(t)\,dt \\
&= \frac{h-2}{2}\left(x-\frac{a+b}{2}\right)f'(x)+f(x)-\frac{f(b)-f(a)}{2(b-a)}m_h(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&= : S_{x,h}(f)
\end{aligned}
$$

for

$$
P_h(x,t) := \begin{cases} (a-t)(t-a-m_h(x)) & , a \le t < x \\ (b-t)(t-b-m_h(x)) & , x \le t \le b \end{cases}
$$

where $m_h(x) = h\left(x-\frac{a+b}{2}\right)$, $h \in [0,2]$ *and* $x \in [a,b]$.

In this study, we establish some Ostrowski type inequalities using the identity (1.2).

2. Main Results

Now, we establish our main theorems and also give some results related to these theorems.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, the interior of the interval I, where $a, b \in I^{\circ}$ with $a < b$. If $|f''|$ is a convex mapping on $[a,b]$, then the following inequalities hold:*

$$
\begin{aligned}
(2.1)\ & |S_{x,h}(f)| \\
\le\ & \frac{1}{2(b-a)^2}\Bigg\{|f''(a)|\Bigg[\frac{(b-x)^4-(x-a)^4}{4}+m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\
& +(b-a)\frac{(x-a)^3}{3}-(b-a)m_h(x)\frac{(x-a)^2}{2}+\frac{[m_h(x)]^4}{6}\Bigg] \\
& +|f''(b)|\Bigg[\frac{(x-a)^4-(b-x)^4}{4}-m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \\
& +(b-a)\frac{(b-x)^3}{3}+(b-a)m_h(x)\frac{(b-x)^2}{2}-\frac{[m_h(x)]^4}{6}-(b-a)\frac{[m_h(x)]^3}{3}\Bigg]\Bigg\}
\end{aligned}
$$

for all $a \leq x \leq \frac{a+b}{2}$ *with* $h \in [0,2]$ *and*

$$
\begin{aligned}
(2.2)\quad & |S_{x,h}(f)| \\
\leq\quad & \frac{1}{2(b-a)^2}\left\{ |f''(a)| \left[\frac{(b-x)^4-(x-a)^4}{4} + m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \right.\right. \\
& \left. + (b-a)\frac{(x-a)^3}{3} - (b-a)\,m_h(x)\frac{(x-a)^2}{2} - \frac{[m_h(x)]^4}{6} + (b-a)\frac{[m_h(x)]^3}{3} \right] \\
& + |f''(b)| \left[\frac{(x-a)^4-(b-x)^4}{4} - m_h(x)\frac{(x-a)^3+(b-x)^3}{3} \right. \\
& \left.\left. + (b-a)\frac{(b-x)^3}{3} + (b-a)\,m_h(x)\frac{(b-x)^2}{2} + \frac{[m_h(x)]^4}{6} \right] \right\}
\end{aligned}
$$

for all $\frac{a+b}{2} \leq x \leq b$ *with* $h \in [0,2]$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$.

Proof. Takink modulus in (1.2), we find that

$$
\begin{aligned}
& |S_{x,h}(f)| \\
= \quad & \frac{1}{2(b-a)} \left| \int_a^b P_h(x,t)\, f''(t)\, dt \right| \\
\leq \quad & \frac{1}{2(b-a)} \int_a^b |P_h(x,t)|\, |f''(t)|\, dt \\
= \quad & \frac{1}{2(b-a)} \left[\int_a^x |a-t|\, |t-a-m_h(x)|\, |f''(t)|\, dt \right. \\
& \left. + \int_x^b |b-t|\, |t-b-m_h(x)|\, |f''(t)|\, dt \right].
\end{aligned}
$$

Since $|f''|$ is a convex mapping on $[a,b]$,we get

$$
(2.3)\qquad |f''(t)| = \left| f''\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b \right) \right| \leq \frac{b-t}{b-a}\,|f''(a)| + \frac{t-a}{b-a}\,|f''(b)|.
$$

Using (2.3), we have

$$
\begin{aligned}
(2.4)\qquad & \left|S_{x,h}\left(f\right)\right| \\
\leq \quad & \frac{1}{2\left(b-a\right)^{2}}\left[\int_{a}^{x}\left|a-t\right|\left|t-a-m_{h}(x)\right|\left[\left(b-t\right)\left|f''\left(a\right)\right|+\left(t-a\right)\left|f''\left(b\right)\right|\right]dt\right. \\
& \left.+\int_{x}^{b}\left|b-t\right|\left|t-b-m_{h}(x)\right|\left[\left(b-t\right)\left|f''\left(a\right)\right|+\left(t-a\right)\left|f''\left(b\right)\right|\right]dt\right] \\
= \quad & \frac{1}{2\left(b-a\right)^{2}}\left\{\left|f''\left(a\right)\right|\left[\int_{a}^{x}\left|a-t\right|\left|t-a-m_{h}(x)\right|\left(b-t\right)dt\right.\right. \\
& \left.+\int_{x}^{b}\left|b-t\right|\left|t-b-m_{h}(x)\right|\left(b-t\right)dt\right] \\
& +\left|f''\left(b\right)\right|\left[\int_{a}^{x}\left|a-t\right|\left|t-a-m_{h}(x)\right|\left(t-a\right)dt\right\} \\
& \left.+\int_{x}^{b}\left|b-t\right|\left|t-b-m_{h}(x)\right|\left(t-a\right)dt\right] \\
= \quad & \frac{1}{2\left(b-a\right)^{2}}\left[\left|f''\left(a\right)\right|\left(I_{1}+I_{2}\right)+\left|f''\left(b\right)\right|\left(I_{3}+I_{4}\right)\right].
\end{aligned}
$$

We calculate integrals I_i, $i=1,...,4$, for the cases $a\leq x\leq \frac{a+b}{2}$ and $\frac{a+b}{2}\leq x\leq b$;
Suppose that $a\leq x\leq \frac{a+b}{2}$. Using the fact that $m_h(x)\leq 0$ for $x\in\left[a,\frac{a+b}{2}\right]$, we get

(2.5)

$$
\begin{aligned}
I_{1} &= \int_{a}^{x}\left(t-a\right)\left(t-a-m_{h}(x)\right)\left(b-t\right)dt \\
&= \left(b-a\right)\int_{a}^{x}\left(t-a\right)\left(t-a-m_{h}(x)\right)dt-\int_{a}^{x}\left(t-a\right)^{2}\left(t-a-m_{h}(x)\right)dt \\
&= \left(b-a\right)\frac{\left(x-a\right)^{3}}{3}-\left(b-a\right)m_{h}(x)\frac{\left(x-a\right)^{2}}{2}-\frac{\left(x-a\right)^{4}}{4}+m_{h}(x)\frac{\left(x-a\right)^{3}}{3},
\end{aligned}
$$

(2.6)

$$\begin{aligned}
I_2 &= \int_x^b (b-t)^2 \left| t-b-m_h(x) \right| dt \\
&= \int_x^{b+m_h(x)} (b-t)^2 \left(m_h(x)+b-t \right) dt + \int_{b+m_h(x)}^b (b-t)^2 \left(t-b-m_h(x) \right) dt \\
&= \frac{\left[m_h(x) \right]^4}{6} + m_h(x) \frac{(b-x)^3}{3} + \frac{(b-x)^4}{4},
\end{aligned}$$

$$I_3 = \int_a^x (t-a)^2 \left(t-a-m_h(x) \right) dt = \frac{(x-a)^4}{4} - m_h(x) \frac{(x-a)^3}{3} \tag{2.7}$$

and

$$\begin{aligned}
I_4 &= \int_x^b (b-t) \left| t-b-m_h(x) \right| (t-a)\, dt \\
&= \int_x^{b+m_h(x)} (b-t) \left(m_h(x)+b-t \right) (t-a)\, dt \\
&\quad + \int_x^b (b-t) \left(t-b-m_h(x) \right) (t-a)\, dt \\
&= -\frac{\left[m_h(x) \right]^4}{6} - (b-a) \frac{\left[m_h(x) \right]^3}{3} - \frac{(b-x)^4}{4} - m_h(x) \frac{(b-x)^3}{3} \\
&\quad + (b-a)\, m_h(x) \frac{(b-x)^2}{2} + (b-a) \frac{(b-x)^3}{3}.
\end{aligned} \tag{2.8}$$

If we subsitute the equalities (2.5)-(2.6) in (2.4), then we obtain the required inequality (2.1).

Suppose that $\frac{a+b}{2} \leq x \leq b$. Using the fact that $m_h(x) \geq 0$ for $x \in \left[\frac{a+b}{2}, b \right]$, we get

(2.9)

$$\begin{aligned}
I_1 &= \int_a^x (t-a) \left| t-a-m_h(x) \right| (b-t)\, dt \\
&= -\frac{\left[m_h(x) \right]^4}{6} + (b-a) \frac{\left[m_h(x) \right]^3}{3} - \frac{(x-a)^4}{4} + m_h(x) \frac{(x-a)^3}{3} \\
&\quad - (b-a)\, m_h(x) \frac{(x-a)^2}{2} + (b-a) \frac{(x-a)^3}{3}
\end{aligned}$$

$$I_2 = \int_x^b (b-t)^2 (m_h(x)+b-t)\,dt = m_h(x)\frac{(b-x)^3}{3} + \frac{(b-x)^4}{4}, \tag{2.10}$$

$$I_3 = \int_a^x (t-a)^2 |t-a-m_h(x)|\,dt = \frac{[m_h(x)]^4}{6} + \frac{(x-a)^4}{4} - m_h(x)\frac{(x-a)^3}{3} \tag{2.11}$$

and

$$\begin{aligned} I_4 &= \int_x^b (b-t)(m_h(x)+b-t)(t-a)\,dt \\ &= -m_h(x)\frac{(b-x)^3}{3} - \frac{(b-x)^4}{4} + (b-a)\,m_h(x)\frac{(b-x)^2}{2} + (b-a)\,\frac{(b-x)^3}{3}. \end{aligned} \tag{2.12}$$

If we subsitute the equalities (2.9)-(2.12) in (2.4), then we obtain the desired inequality (2.2). □

Remark 1. *If we choose $x = \frac{a+b}{2}$ in Theorem 1, then the inequalities (2.1) and (2.2) reduce to the inequality (**??**).*

Remark 2. *If we choose $h = 0$ in the inequalities (2.1) and (2.2), then we have the following inequality*

$$\begin{aligned} &\left| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \leq\ & \frac{b-a}{2}\left\{ |f''(a)| \left[\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \left(\frac{a+b}{2} - x \right) + \frac{(x-a)^3}{3(b-a)^2} \right] \right. \\ & \left. + |f''(b)| \left[\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \left(x - \frac{a+b}{2} \right) + \frac{(b-x)^3}{3(b-a)^2} \right] \right\} \end{aligned}$$

for $x \in [a,b]$.

Corollary 1. *Let us subsitute $x = a$ and $x = b$ inTheorem 1. Subsequently, if we add the obtained rersult and use the triangle inequality for the modulus, we get the inequality*

$$\begin{aligned} &\left| \frac{h-2}{2}\frac{b-a}{4}\left(f'(b) - f'(a)\right) + \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ \leq\ & \frac{(b-a)^2}{8}\left[\frac{2}{3} - \frac{h}{2} + \frac{h^3}{12} \right] \left[|f''(a)| + |f''(b)| \right]. \end{aligned}$$

Remark 3. *If we take $h=0$ in Corollary 1, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{4}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^2}{6}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]. \tag{2.13}$$

Particularly, if $|f(x)|<M$, $x\in[a,b]$, then the inequality reduces the inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{b-a}{4}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{M(b-a)^2}{6}$$

which was given by Liu in [12]

Remark 4. *If we take $h=2$ in Corollary 1, then we have the trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{(b-a)^2}{12}\left[\frac{|f''(a)|+|f''(b)|}{2}\right] \tag{2.14}$$

which was given Kiris and Sarikaya in [11].

Theorem 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I°, the interior of the interval I, where $a,b\in I^\circ$ with $a<b$. If $|f''|^q$,$q>1$, is a convex mapping on $[a,b]$, then the following inequalities hold:*

$$\begin{aligned}
&|S_{x,h}(f)| \\
\leq\;&\frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg\{\left((x-a-m_h(x))^{p+1}+(-1)^p\left[m_h(x)\right]^{p+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{(b-a)(x-a)^{q+1}}{q+1}-\frac{(x-a)^{q+2}}{q+2}\right)|f''(a)|^q+\frac{(x-a)^{q+2}}{q+2}|f''(b)|^q\right]^{\frac{1}{q}} \\
&+\left((m_h(x)+b-x)^{p+1}+(-1)^{p+1}\left[m_h(x)\right]^{p+1}\right)^{\frac{1}{p}} \\
&\times\left[\frac{(b-x)^{q+2}}{q+2}|f''(a)|^q+\left(\frac{(b-a)(b-x)^{q+1}}{q+1}-\frac{(b-x)^{q+2}}{q+2}\right)|f''(b)|^q\right]^{\frac{1}{q}}\Bigg\}
\end{aligned} \tag{2.15}$$

for $a \leq x \leq \frac{a+b}{2}$, *and*

$$
\begin{aligned}
&|S_{x,h}(f)| \\
\leq\ & \frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left([m_h(x)]^{p+1}+(x-a-m_h(x))^{p+1}\right)^{\frac{1}{p}}\right. \\
&\times\left[\left(\frac{(b-a)(x-a)^{q+1}}{q+1}-\frac{(x-a)^{q+2}}{q+2}\right)|f''(a)|^q+\frac{(x-a)^{q+2}}{q+2}|f''(b)|^q\right]^{\frac{1}{q}} \\
&+\left((m_h(x)+b-x)^{p+1}-[m_h(x)]^{p+1}\right)^{\frac{1}{p}} \\
&\left.\times\left[\frac{(b-x)^{q+2}}{q+2}|f''(a)|^q+\left(\frac{(b-a)(b-x)^{q+1}}{q+1}-\frac{(b-x)^{q+2}}{q+2}\right)|f''(b)|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}
\tag{2.16}
$$

for $\frac{a+b}{2} \leq x \leq b$ *with* $h \in [0,2]$, *where* $m_h(x) = h\left(x-\frac{a+b}{2}\right)$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking the modulus in Lemma 1 and using the well-known Hölder's ineqıality, we have

$$
\begin{aligned}
&|S_{x,h}(f)| \\
\leq\ & \frac{1}{2(b-a)}\int_a^b |P_h(x,t)|\,|f''(t)|\,dt \\
=\ & \frac{1}{2(b-a)}\left[\int_a^x |a-t|\,|t-a-m_h(x)|\,|f''(t)|\,dt\right. \\
&\left.+\int_x^b |b-t|\,|t-b-m_h(x)|\,|f''(t)|\,dt\right] \\
\leq\ & \frac{1}{2(b-a)}\left[\left(\int_a^x |t-a-m_h(x)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^x (t-a)^q\,|f''(t)|^q\,dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_x^b |t-b-m_h(x)|^p\,dt\right)^{\frac{1}{p}}\left(\int_x^b (b-t)^q\,|f''(t)|^q\,dt\right)^{\frac{1}{p}}\right].
\end{aligned}
\tag{2.17}
$$

Since $|f''|^q$ is a convex mapping on $[a,b]$,we get

$$
|f''(t)|^q = \left|f''\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q \leq \frac{b-t}{b-a}|f''(a)|^q+\frac{t-a}{b-a}|f''(b)|^q. \tag{2.18}
$$

Using (2.18), we have

$$\int_a^x (t-a)^q |f''(t)|^q dt \tag{2.19}$$

$$\leq \frac{1}{b-a}\int_a^x (t-a)^q \left[(b-t)|f''(a)|^q + (t-a)|f''(b)|^q\right]$$

$$= \frac{1}{b-a}\left\{\left[\frac{(b-a)(x-a)^{q+1}}{q+1} - \frac{(x-a)^{q+2}}{q+2}\right]|f''(a)|^q + \frac{(x-a)^{q+2}}{q+2}|f''(b)|^q\right\}$$

and similarly,

$$\int_x^b (b-t)^q |f''(t)|^q dt \tag{2.20}$$

$$\leq \frac{1}{b-a}\int_x^b (b-t)^q \left[(b-t)|f''(a)|^q + (t-a)|f''(b)|^q\right]$$

$$= \frac{1}{b-a}\left\{\frac{(b-x)^{q+2}}{q+2}|f''(a)|^q + \left[\frac{(b-a)(b-x)^{q+1}}{q+1} - \frac{(b-x)^{q+2}}{q+2}\right]|f''(b)|^q\right\}.$$

Moreover, we obtain

$$\int_a^x |t-a-m_h(x)|^p dt = \frac{(x-a-m_h(x))^{p+1} + (-1)^p [m_h(x)]^{p+1}}{p+1} \tag{2.21}$$

for $a \leq x \leq \frac{a+b}{2}$, and

$$\int_a^x |t-a-m_h(x)|^p dt = \frac{[m_h(x)]^{p+1} + (x-a-m_h(x))^{p+1}}{p+1} \tag{2.22}$$

for $\frac{a+b}{2} \leq x \leq b$.

Using the similar way we also have,

$$\int_x^b |t-b-m_h(x)|^p dt = \frac{(m_h(x)+b-x)^{p+1} + (-1)^{p+1} [m_h(x)]^{p+1}}{p+1} \tag{2.23}$$

for $a \leq x \leq \frac{a+b}{2}$, and

$$\int_x^b |t-b-m_h(x)|^p dt = \frac{(m_h(x)+b-x)^{p+1} - [m_h(x)]^{p+1}}{p+1} \tag{2.24}$$

for $\frac{a+b}{2} \leq x \leq b$.

Using the identities (2.19)-(2.21) and (2.23) for the case $a \leq x \leq \frac{a+b}{2}$ and using the identities (2.19), (2.20) (2.22) and (2.24) for the case $\frac{a+b}{2} \leq x \leq b$, we obtain required results (2.15) and (2.16). □

Corollary 2. *If we choose $x=\frac{a+b}{2}$ in Theorem 2, then we have the inequality*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{(b-a)^2}{2^{4+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left\{\left[\frac{(q+3)\left|f''(a)\right|^q+(q+1)\left|f''(b)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{(q+1)\left|f''(a)\right|^q+(q+3)\left|f''(b)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 3. *If we choose $h=0$ in Theorem 2 , then we have the following inequality for $a\leq x\leq b$*

$$\begin{aligned}&\left|f(x)-\left(x-\frac{a+b}{2}\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{1}{2(b-a)^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left\{(x-a)^3\left[\left(\frac{b-a}{q+1}-\frac{x-a}{q+2}\right)\left|f'(a)\right|^q+\frac{x-a}{q+2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\ &\left.+(b-x)^3\left[\frac{b-x}{q+2}\left|f'(a)\right|^q+\left(\frac{b-a}{q+1}-\frac{b-x}{q+2}\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4. *Let us $x=a$ and $x=b$ in Theorem 2. Subsequently, if we add the obtained rersult and use the triangle inequality for the modulus, we get the inequality for $h\in[0,2]$*

$$\begin{aligned}&\left|\frac{h-2}{2}\frac{b-a}{4}\left(f'(b)-f'(a)\right)+\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ \leq\ &\frac{(b-a)^2}{2^{3+\frac{1}{p}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left((2-h)^{p+1}+h^{p+1}\right)^{\frac{1}{p}}\\ &\times\left\{\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 5. *If we take $h=0$ in Corollary 4 , then we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{(b-a)}{4}\left[f'(a)+f'(b)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{(b-a)^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}+\left[\frac{\left|f''(b)\right|^q+(q+1)\left|f''(a)\right|^q}{(q+1)(q+2)}\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 6. *If we take* $h=2$ *in Corollary 4 , then we have following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{(b-a)^2}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left[\frac{|f''(b)|^q+(q+1)\,|f''(a)|^q}{(q+1)\,(q+2)}\right]^{\frac{1}{q}}+\left[\frac{|f''(b)|^q+(q+1)\,|f''(a)|^q}{(q+1)\,(q+2)}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

3. Applications to Some Special Means

Let us recall the following means:
(a) The *arithmatic mean:*

$$A=A(a,b):=\tfrac{a+b}{2},\qquad a,b\geq 0$$

(b) The *Geometric mean*:

$$G=G(a,b):=\sqrt{ab},\qquad a,b\geq 0$$

(c) The *Harmonic mean*:

$$H=H(a,b):=\tfrac{2}{\frac{1}{a}+\frac{1}{b}},\qquad a,b>0$$

(d) The *Logarithmic mean*:

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{b-a}{\ln b-\ln a} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

(e) The *Identric mean*:

$$I=L(a,b):=\begin{cases} a & \text{if } a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

(f) The $p-$*logarithmic mean*:

$$L_p=L_p(a,b):=\begin{cases} a & \text{if } a=b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b\end{cases},\qquad a,b>0$$

where $p\in\mathbb{R}\backslash\{-1,0\}$.

The following simple relationships are known in literature

$$H\leq G\leq L\leq I\leq A.$$

It is also known that L_p is monotonically increasing in $p\in\mathbb{R}$ with $L_0=I$ and $L_{-1}=L$.

Proposition 1. *Let $a, b \in \mathbb{R}$, $0 < a < b$, $n \in Z$ and $|n(n-1)| \geq 3$. Then, we have*

$$\begin{aligned}
&\left|\frac{n(h-2)}{2}(x-A)x^{n-1} + x^n - \frac{n.h}{2}L_{n-1}^{n-1}(x-A) - L_n^n\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{|n(n-1)|\,a^{n-2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3} + \right.\right. \\
& \left. (b-a)\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6}\right] \\
& + |n(n-1)|\,b^{n-2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
& \left.\left. + (b-a)\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a)\frac{[h(x-A)]^3}{3}\right]\right\}
\end{aligned}$$

for all $a \leq x \leq A$ with $h \in [0,2]$ and

$$\begin{aligned}
&\left|\frac{n(h-2)}{2}(x-A)x^{n-1} + x^n - \frac{n.h}{2}L_{n-1}^{n-1}(x-A) - L_n^n\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{|n(n-1)|\,a^{n-2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right.\right. \\
& \left. + (b-a)\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a)\frac{[h(x-A)]^3}{3}\right] \\
& + |n(n-1)|\,b^{n-2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
& \left.\left. (b-a)\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6}\right]\right\}
\end{aligned}$$

for all $A \leq x \leq b$ with $h \in [0,2]$.

Proof. The proof is immediate from Theorem 1 applied for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in Z$, $|n(n-1)| \geq 3$. $\square$

Remark 7. *If we choose $h = 0$ in Proposition 1, then we have the inequality*

$$\begin{aligned}
&\left|x^n - n(x-A)x^{n-1} - L_n^n\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{|n(n-1)|\,a^{n-2}\left[\frac{(b-x)^4-(x-a)^4}{4} + + (b-a)\frac{(x-a)^3}{3}\right]\right. \\
& \left. + |n(n-1)|\,b^{n-2}\left[\frac{(x-a)^4-(b-x)^4}{4} + (b-a)\frac{(b-x)^3}{3}\right]\right\}
\end{aligned}$$

for $x \in [a,b]$.

Proposition 2. *Let $a, b \in (0, \infty)$ and $a < b$. Then,,we have*

$$\begin{aligned}
&\left|\ln I + \frac{h(x-A)}{2L} - \frac{(h-2)(x-A)}{2x} - \ln x\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right.\right. \\
&\left. + (b-a)\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} + \frac{[h(x-A)]^4}{6}\right] \\
&+\frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
&\left.\left. + (b-a)\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} - \frac{[h(x-A)]^4}{6} - (b-a)\frac{[h(x-A)]^3}{3}\right]\right\}
\end{aligned}$$

for all $a \leq x \leq A$ with $h \in [0, 2]$ and

$$\begin{aligned}
&\left|\ln I + \frac{h(x-A)}{2L} - \frac{(h-2)(x-A)}{2x} - \ln x\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right.\right. \\
&\left. + (b-a)\frac{(x-a)^3}{3} - (b-a)\,h(x-A)\frac{(x-a)^2}{2} - \frac{[h(x-A)]^4}{6} + (b-a)\frac{[h(x-A)]^3}{3}\right] \\
&+\frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} - h(x-A)\frac{(x-a)^3+(b-x)^3}{3}\right. \\
&\left.\left. (b-a)\frac{(b-x)^3}{3} + (b-a)\,h(x-A)\frac{(b-x)^2}{2} + \frac{[h(x-A)]^4}{6}\right]\right\}
\end{aligned}$$

for all $A \leq x \leq b$ with $h \in [0, 2]$.

Proof. The assertion follows from Theorem 1 applied to the mapping $f : (0, \infty) \to (-\infty, 0)$, $f(x) = -lnx$ and the details are omitted. □

Remark 8. *If we choose $h = 0$ in Proposition 2, then we have the inequality,*

$$\begin{aligned}
&\left|\ln I + \frac{(x-A)}{x} - \ln x\right| \\
\leq\ & \frac{1}{2(b-a)^2}\left\{\frac{1}{a^2}\left[\frac{(b-x)^4-(x-a)^4}{4} + (b-a)\frac{(x-a)^3}{3}\right]\right. \\
&\left.+\frac{1}{b^2}\left[\frac{(x-a)^4-(b-x)^4}{4} + (b-a)\frac{(b-x)^3}{3}\right]\right\}
\end{aligned}$$

for $x \in [a, b]$.

4. **Concluding Remarks**

In this study, first of all, using practical identity for twice differentiable functions proved by Erden et al., we present somenew upper bounds for generalized Ostrowski type inequalities by taking advantageous of mappings whose second derivatives absolute values are convex. Moreover, we provide these inequalities for special means.

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