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MULTIPLICATIVE REFINEMENTS AND REVERSES OF YOUNG AND HÖLDER'S INEQUALITIES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. In this paper we obtain some multiplicative refinements and reverses of the celebrated Young and Hölder's inequalities in the general setting of Hermitian unital Banach $*$-algebras and for positive linear functionals defined on such algebras.

1. Introduction

The famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu\in[0,1]$, then

$$G_{\nu}(a,b):=a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b=:A_{\nu}(a,b) \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

Assume that $a,\ b>0$, $\nu\in[0,1]$. The following multiplicative refinement and reverse of the arithmetic mean-geometric mean inequality holds

$$\mathcal{S}\left(\left(\frac{a}{b}\right)^{r}\right)\leq\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\leq\mathcal{S}\left(\frac{a}{b}\right), \tag{FT}$$

where, $\mathcal{S}$ is *Specht's ratio* and $r=\min\{1-\nu,\nu\}$. *Specht's ratio $\mathcal{S}$,* was introduced in 1960 in [22], and is defined by

$$\mathcal{S}(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}\text{ if }h\in(0,1)\cup(1,\infty),\\ \\ 1\text{ if }h=1.\end{cases} \tag{S}$$

Recall that $\mathcal{S}$ satisfies the properties

$$\lim_{h\to1}\mathcal{S}(h)=1,\ \mathcal{S}(h)=\mathcal{S}\left(\frac{1}{h}\right)>1 \tag{1.2}$$

for $h>0$, $h\neq1$, is *decreasing* on $(0,1)$ and *increasing* on $(1,\infty)$.

The second inequality in (FT) is due to Tominaga, 2002 [24], while the first is due to Furuichi, 2012, [14].

Zuo et al. 2011, [25] and Liao et al. 2015, [13] obtained the following multiplicative refinement and reverse inequalities

$$\mathcal{K}^{r}\left(\frac{a}{b}\right)\leq\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\leq\mathcal{K}^{R}\left(\frac{a}{b}\right) \tag{ZL}$$

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where $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$. *Kantorovich's constant* $\mathcal{K}$ is defined by

$$\mathcal{K}(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{K}$$

$\mathcal{K}$ is *decreasing* on $(0,1)$ and *increasing* on $[1,\infty)$, $\mathcal{K}(h) \geq 1$ for any $h>0$ and $\mathcal{K}(h) = \mathcal{K}\left(\frac{1}{h}\right)$ for any $h>0$.

We also have [25]

$$\mathcal{S}\left(\left(\frac{a}{b}\right)^r\right) \leq \mathcal{K}^r\left(\frac{a}{b}\right), \tag{Co}$$

which shows that the lower bound in (ZL) is better than the same bound in (FT).

The inequalities (ZL) can be directly obtained from the following more general result obtained by the author in 2006 [5] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^n \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right] \\ & \leq \frac{1}{P_n}\sum_{j=1}^n p_j\Phi(x_j) - \Phi\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \\ & \leq n \max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^n \Phi(x_j) - \Phi\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right], \end{aligned} \tag{1.3}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^n p_j > 0$.

For $n=2$, we deduce from (1.3) that

$$\begin{aligned} & 2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ & \leq \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y] \\ & \leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{1.4}$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$. Now, if we write the inequality (1.4) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get (ZL).

In 2015, Alzer & Fonseca & Kovačec, [1] and Dragomir, [10] obtained independently and by using different techniques the following logarithmic upper and lower bounds for the quotient of the arithmetic mean and geometric mean:

$$\begin{aligned} & \exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \\ & \leq \frac{A_\nu(a,b)}{G_\nu(a,b)} \\ & \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \end{aligned} \tag{1.5}$$

for any a, $b > 0$, $\nu \in [0,1]$.

A different reverse in terms of the exponential was also obtained recently in the paper [6]

$$\frac{A_\nu(a,b)}{G_\nu(a,b)} \le \exp\left[4\nu(1-\nu)\left(\mathcal{K}\left(\frac{a}{b}\right)-1\right)\right], \tag{1.6}$$

for any a, $b > 0$, $\nu \in [0,1]$.

If upper and lower bounds are assumed for the positive numbers a, b namely a, $b \in [\gamma,\Gamma] \subset (0,\infty)$ and $\nu \in [0,1]$, then [7]

$$\frac{A_\nu(a,b)}{G_\nu(a,b)} \le \max\{h_{\gamma,\Gamma}(\nu), h_{\gamma,\Gamma}(1-\nu)\}, \tag{1.7}$$

where

$$h_{\gamma,\Gamma}(\nu) := \frac{(1-\nu)\gamma + \nu\Gamma}{\gamma^{1-\nu}\Gamma^\nu}. \tag{1.8}$$

In order to extend these results in the abstract setting of Hermitian unital Banach $*$-algebras and for positive linear functionals we need the following preparation.

2. Some Facts on Hermitian Unital Banach $*$-Algebra

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \ge 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that a is *positive* and write $a > 0$ if $a \ge 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \ge b$ means that $a - b \ge 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [21] (see also [2, Theorem 41.5])

$$a^*a \ge 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [20], Tanahashi and Uchiyama [23] proved the following fundamental properties (see also [14]):

(i) If a, $b \in A$, then $a \ge 0$, $b \ge 0$ imply $a + b \ge 0$ and $\alpha \ge 0$ implies $\alpha a \ge 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \ge 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \ge b > 0$ or $a > b \ge 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \le a$ if and only if $cbc \le cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \le a$, then $0 < a^{-1} \le b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\, z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [14], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [23, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [20] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

We define the following means for $\nu \in [0,1]$, see also [14] for different notations:

$$a \nabla_\nu b := (1-\nu)a + \nu b,\ \ a,\ \ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a,b),

$$a !_\nu b := \left((1-\nu)a^{-1} + \nu b^{-1}\right)^{-1},\ \ a,\ \ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a,b) and

$$a \sharp_\nu b := a^{1/2}\left(a^{-1/2} b a^{-1/2}\right)^{\upsilon} a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a,b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [14], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a$, $b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;
(xiii) If $0 < a$, $b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)c = (c^*ac)!(c^*bc) \text{ and } c^*(a\sharp b)c = (c^*ac)\sharp(c^*bc);$$

(xiv) If $0 < a$, $b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = (a^{-1})\nabla_\nu(b^{-1}) \text{ and } (a^{-1})\sharp_\nu(b^{-1}) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [14] the following inequality between the weighted means introduced above:

$$a\nabla_\nu b \geq a\sharp_\nu b \geq a!_\nu b \tag{HGA}$$

for any $0 < a,\, b \in A$ and $\nu \in [0,1]$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0,\infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [11].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$. The functional ψ is called faithful if $a \geq 0$ and $\psi(a) = 0$ implies that $a = 0$.*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with $\alpha,\, \beta$ real numbers, then $\beta \geq \psi(a) \geq \alpha$, provided ψ is normalized. If the positive linear functional ψ is faithful and $a > 0$ then $\psi(a) > 0$.

If we use the first inequality in (HGA), then we can state the *Young type inequality*

$$x^p \sharp_{1/q} y^q \leq \frac{1}{p} x^p + \frac{1}{q} y^q \tag{Y}$$

for any $0 \leq x,\, y \in A$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We have the following *Hölder's type inequality* for positive functionals as well [12]:

Theorem 1. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a faithful normalized positive linear functional. If $0 \leq a,\, b \in A$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\psi\left(a^p \sharp_{1/q} b^q\right) \leq \psi^{1/p}(a^p)\, \psi^{1/q}(b^q). \tag{H}$$

In particular,

$$\psi^2\left(a^2 \sharp_{1/2} b^2\right) \leq \psi\left(a^2\right) \psi\left(b^2\right). \tag{Sc}$$

In the following we obtain some multiplicative refinements and reverses of the celebrated Young and Hölder's inequalities in the general setting of Hermitian unital Banach $*$-algebras and for positive linear functionals defined on such algebras.

3. Multiplicative Refinements and Reverses

We consider the function $g_\nu : (0,\infty) \to (0,\infty)$ defined for $\nu \in (0,1)$ by

$$g_\nu(t) = \frac{1-\nu+\nu t}{t^\nu} = \frac{A_\nu(1,t)}{G_\nu(1,t)} = (1-\nu)t^{-\nu} + \nu t^{1-\nu}. \tag{3.1}$$

For $[k,K] \subset (0,\infty)$, define the quantities

$$\Gamma_\nu(k,K) := \begin{cases} \frac{A_\nu(1,k)}{G_\nu(1,k)} \text{ if } K<1, \\ \\ \max\left\{\frac{A_\nu(1,k)}{G_\nu(1,k)}, \frac{A_\nu(1,K)}{G_\nu(1,K)}\right\} \text{ if } k \le 1 \le K, \\ \\ \frac{A_\nu(1,K)}{G_\nu(1,K)} \text{ if } 1<k \end{cases} \tag{3.2}$$

$$= \begin{cases} (1-\nu)k^{-\nu} + \nu k^{1-\nu} \text{ if } K<1, \\ \\ \max\left\{(1-\nu)k^{-\nu} + \nu k^{1-\nu}, (1-\nu)K^{-\nu} + \nu K^{1-\nu}\right\} \\ \text{if } k \le 1 \le K, \\ \\ (1-\nu)K^{-\nu} + \nu K^{1-\nu} \text{ if } 1<k \end{cases}$$

and

$$\gamma_\nu(k,K) := \begin{cases} \frac{A_\nu(1,K)}{G_\nu(1,K)} \text{ if } K<1, \\ \\ 1 \text{ if } k \le 1 \le K, \\ \\ \frac{A_\nu(1,k)}{G_\nu(1,k)} \text{ if } 1<k, \end{cases} \tag{3.3}$$

$$= \begin{cases} (1-\nu)K^{-\nu} + \nu K^{1-\nu} \text{ if } K<1, \\ \\ 1 \text{ if } k \le 1 \le K, \\ \\ (1-\nu)k^{-\nu} + \nu k^{1-\nu} \text{ if } 1<k. \end{cases}$$

The following lemma holds.

Lemma 2. *For $[k,K] \subset (0,\infty)$ we have*

$$\max_{t\in[k,K]} g_\nu(t) = \Gamma_\nu(k,K) \tag{3.4}$$

and

$$\min_{t\in[k,K]} g_\nu(t) = \gamma_\nu(k,K). \tag{3.5}$$

Proof. The function g_ν is differentiable and

$$g_\nu'(t) = (1-\nu)\nu t^{-\nu-1}(t-1),$$

which shows that the function g_ν is decreasing on $(0,1)$ and increasing on $[1,\infty)$. We have $g_\nu(1) = 1$, $\lim_{t\to 0+} g_\nu(t) = +\infty$, $\lim_{t\to\infty} g_\nu(t) = +\infty$ and $g_\nu\left(\frac{1}{t}\right) = g_{1-\nu}(t)$ for any $t>0$ and $\nu \in (0,1)$.

Therefore, by considering the 3 possible situations for the location of the interval $[k, K]$ and the number 1 we get the desired bounds (3.4) and (3.5). □

We have the following multiplicative refinement and reverse of Young's inequality:

Theorem 2. *Let $0 < x$, $y \in A$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that there exists the constant m, $M > 0$ such that*

$$m^q x^p \leq y^q \leq M^q x^p, \tag{3.6}$$

then

$$\gamma_{1/q}\left(m^q, M^q\right) x^p \sharp_{1/q} y^q \leq \frac{1}{p} x^p + \frac{1}{q} y^q \leq \Gamma_{1/q}\left(m^q, M^q\right) x^p \sharp_{1/q} y^q, \tag{3.7}$$

where the functions γ and Γ are defined by (3.2) and (3.3).

Proof. From the above Lemma 2 we have

$$\gamma_\nu\left(k, K\right) z^\nu \leq 1 - \nu + \nu z \leq \Gamma_\nu\left(k, K\right) z^\nu \tag{3.8}$$

for any real $z \in [k, K] \subset (0, \infty)$ and for any $\nu \in [0, 1]$.

Let $u \in A$ with spectrum $\sigma(u) \subset [k, K] \subset (0, \infty)$. Then by applying Lemma 1 for the corresponding analytic functions in the right half open plane $\{\operatorname{Re} z > 0\}$ involved in the inequality (3.8) we conclude that we have in the order of A that

$$\gamma_\nu\left(k, K\right) u^\nu \leq 1 - \nu + \nu u \leq \Gamma_\nu\left(k, K\right) u^\nu \tag{3.9}$$

for any $\nu \in [0, 1]$.

Since x is invertible, then by multiplying both sides of (3.6) with $x^{-p/2} > 0$, we get $m^q \leq x^{-p/2} y^q x^{-p/2} \leq M^q$ and by taking $\nu = 1/q$, $u = x^{-p/2} y^q x^{-p/2}$, $k = m^q$ and $K = M^q$ we get in the order of A that

$$\begin{aligned} \gamma_{1/q}\left(m^q, M^q\right) \left(x^{-p/2} y^q x^{-p/2}\right)^{1/q} &\leq \frac{1}{p} + \frac{1}{q} x^{-p/2} y^q x^{-p/2} \\ &\leq \Gamma_{1/q}\left(m^q, M^q\right) \left(x^{-p/2} y^q x^{-p/2}\right)^{1/q}. \end{aligned} \tag{3.10}$$

If we multiply both sides of (3.10) by $x^{p/2} > 0$, then we get

$$\begin{aligned} &\gamma_{1/q}\left(m^q, M^q\right) x^{p/2} \left(x^{-p/2} y^q x^{-p/2}\right)^{1/q} x^{p/2} \\ &\leq \frac{1}{p} x^p + \frac{1}{q} y^q \\ &\leq \Gamma_{1/q}\left(m^q, M^q\right) x^{p/2} \left(x^{-p/2} y^q x^{-p/2}\right)^{1/q} x^{p/2} \end{aligned}$$

and the inequality (3.7) is proved. □

Corollary 1. *Let $0 < x$, $y \in A$. Assume that there exists the constant m, $M > 0$ such that*

$$m^2 x^2 \leq y^2 \leq M^2 x^2, \tag{3.11}$$

then

$$
\begin{cases} \frac{1+M^2}{2M} x^2 \sharp y^2 \ \text{if } M<1, \\ \\ x^2 \sharp y^2 \ \text{if } m \le 1 \le M, \\ \\ \frac{1+m^2}{2m} x^2 \sharp y^2 \ \text{if } 1<m, \end{cases} \tag{3.12}
$$

$$
\le \frac{x^2+y^2}{2}
$$

$$
\le \begin{cases} \frac{1+m^2}{2m} x^2 \sharp y^2 \ \text{if } M<1, \\ \\ \max\left\{\frac{1+m^2}{2m}, \frac{1+M^2}{2M}\right\} x^2 \sharp y^2 \ \text{if } m \le 1 \le M, \\ \\ \frac{1+M^2}{2M} x^2 \sharp y^2 \ \text{if } 1<m. \end{cases}
$$

Proof. If follows by taking $p=q=2$ in Theorem 2 and observing that

$$
\Gamma_{1/2}\left(m^2, M^2\right) = \begin{cases} \frac{A\left(1,m^2\right)}{G\left(1,m^2\right)} \text{ if } M<1, \\ \\ \max\left\{\frac{A\left(1,m^2\right)}{G\left(1,m^2\right)}, \frac{A\left(1,M^2\right)}{G\left(1,M^2\right)}\right\} \text{ if } m \le 1 \le M, \\ \\ \frac{A\left(1,M^2\right)}{G\left(1,M^2\right)} \text{ if } 1<m, \end{cases}
$$

$$
= \begin{cases} \frac{1+m^2}{2m} \text{ if } M<1, \\ \\ \max\left\{\frac{1+m^2}{2m}, \frac{1+M^2}{2M}\right\} \text{ if } m \le 1 \le M, \\ \\ \frac{1+M^2}{2M} \text{ if } 1<m \end{cases}
$$

and

$$
\gamma_{1/2}\left(m^2, M^2\right) = \begin{cases} \frac{A\left(1,M^2\right)}{G\left(1,M^2\right)} \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ \frac{A\left(1,m^2\right)}{G\left(1,m^2\right)} \text{ if } 1<m, \end{cases}
$$

$$
= \begin{cases} \frac{1+M^2}{2M} \text{ if } M<1, \\ \\ 1 \text{ if } m \le 1 \le M, \\ \\ \frac{1+m^2}{2m} \text{ if } 1<m. \end{cases}
$$

□

Remark 1. *Let $0<x$, $y \in A$. Assume that there exists the constant m_1, M_1, m_2, $M_2>0$ such that $m_1 \le x \le M_1$ and $m_2 \le y \le M_2$. Then $m_1^p \le x^p \le M_1^p$ and*

$m_2^q \le y^q \le M_2^2$ which implies that

$$\frac{m_2^q}{M_1^p} x^p \le y^q \le \frac{M_2^q}{m_1^p} x^p$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Also,

$$\left(\frac{m_2}{M_1}\right)^2 x^2 \le y^2 \le \left(\frac{M_2}{m_1}\right)^2 x^2.$$

Therefore, by (3.7) we have

$$\gamma_{1/q}\left(\frac{m_2^q}{M_1^p}, \frac{M_2^q}{m_1^p}\right) x^p \sharp_{1/q} y^q \le \frac{1}{p} x^p + \frac{1}{q} y^q \le \Gamma_{1/q}\left(\frac{m_2^q}{M_1^p}, \frac{M_2^q}{m_1^p}\right) x^p \sharp_{1/q} y^q \tag{3.13}$$

where the functions γ and Γ are defined by (3.2) and (3.3).

By (3.12) we also have

$$\begin{cases} \frac{m_1^2+M_2^2}{2m_1M_2} x^2 \sharp y^2 \text{ if } \frac{M_2}{m_1} < 1, \\ \\ x^2 \sharp y^2 \text{ if } \frac{m_2}{M_1} \le 1 \le \frac{M_2}{m_1}, \\ \\ \frac{M_1^2+m_2^2}{2m_2M_1} x^2 \sharp y^2 \text{ if } 1 < \frac{m_2}{M_1}, \end{cases} \tag{3.14}$$
$$\le \frac{x^2+y^2}{2}$$
$$\le \begin{cases} \frac{M_1^2+m_2^2}{2m_2M_1} x^2 \sharp y^2 \text{ if } \frac{M_2}{m_1} < 1, \\ \\ \max\left\{\frac{M_1^2+m_2^2}{2m_2M_1}, \frac{m_1^2+M_2^2}{2M_2m_1}\right\} x^2 \sharp y^2 \text{ if } \frac{m_2}{M_1} \le 1 \le \frac{M_2}{m_1}, \\ \\ \frac{m_1^2+M_2^2}{2M_2m_1} x^2 \sharp y^2 \text{ if } 1 < \frac{m_2}{M_1}. \end{cases}$$

The following multiplicative reverse of Hölder's inequality holds:

Theorem 3. *Assume that A is a Hermitian unital Banach $*$-algebra and $\psi : A \to \mathbb{C}$ a faithful normalized positive linear functional. If $0 \le a, b \in A$ such that there exists the constant $k_1,\ K_1,\ k_2,\ K_2 > 0$ with $k_1 \le a \le K_1$ and $k_2 \le b \le K_2$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$1 \le \frac{\psi^{1/p}(a^p)\,\psi^{1/q}(b^q)}{\psi\left(a^p \sharp_{1/q} b^q\right)} \le \max\left\{\frac{A_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}, \frac{A_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}\right\}. \tag{3.15}$$

In particular,

$$1 \le \frac{\psi\left(a^2\right)\psi\left(b^2\right)}{\psi^2\left(a^2 \sharp b^2\right)} \le \frac{1}{4}\left(\frac{k_1^2k_2^2 + K_1^2K_2^2}{k_1k_2K_1K_2}\right)^2. \tag{3.16}$$

Proof. We have $0 < k_1 \leq \psi^{1/p}(a^p) \leq K_1$ and $0 < k_2 \leq \psi^{1/q}(b^q) \leq K_2$. These imply that $\frac{k_1}{K_1} \leq \frac{a}{\psi^{1/p}(a^p)} \leq \frac{K_1}{k_1}$ and $\frac{k_2}{K_2} \leq \frac{b}{\psi^{1/q}(b^q)} \leq \frac{K_2}{k_2}$ and $\frac{k_1}{K_1}, \frac{k_2}{K_2} \leq 1 \leq \frac{K_1}{k_1}, \frac{K_2}{k_2}$. Consider $x = \frac{a}{\psi(a)}$, $y = \frac{b}{\psi(b)}$, $m_1 = \frac{k_1}{K_1}$, $M_1 = \frac{K_1}{k_1}$, $m_2 = \frac{k_2}{K_2}$ and $M_2 = \frac{K_2}{k_2}$. Also, observe that

$$\frac{m_2^q}{M_1^p} = \frac{\left(\frac{k_2}{K_2}\right)^q}{\left(\frac{K_1}{k_1}\right)^p} = \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p \leq 1$$

and

$$\frac{M_2^q}{m_1^p} = \frac{\left(\frac{K_2}{k_2}\right)^q}{\left(\frac{k_1}{K_1}\right)^p} = \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p \geq 1.$$

Using the inequality (3.13) we have

$$\begin{aligned}
&\frac{1}{p}\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p + \frac{1}{q}\left(\frac{b}{\psi^{1/q}(b^q)}\right)^q \\
&\leq \Gamma_{1/q}\left(\left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right) \\
&\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p \sharp_{1/q} \left(\frac{a}{\psi^{1/p}(a^p)}\right)^q \\
&= \max\left\{\frac{A_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}, \frac{A_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}\right\} \\
&\left(\frac{a}{\psi^{1/p}(a^p)}\right)^p \sharp_{1/q} \left(\frac{a}{\psi^{1/p}(a^p)}\right)^q,
\end{aligned}$$

namely

$$\begin{aligned}
(3.17)\qquad &\frac{1}{p}\frac{a^p}{\psi(a^p)} + \frac{1}{q}\frac{b^q}{\psi(b^q)} \\
&\leq \max\left\{\frac{A_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{k_2}{K_2}\right)^q \left(\frac{k_1}{K_1}\right)^p\right)}, \frac{A_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}{G_{1/q}\left(1, \left(\frac{K_2}{k_2}\right)^q \left(\frac{K_1}{k_1}\right)^p\right)}\right\} \\
&\times \frac{a^p \sharp_{1/q} b^q}{\psi^{1/q}(a^p)\,\psi^{1/q}(b^q)}.
\end{aligned}$$

If we take in (3.17) the functional ψ, then we get the desired result (3.15).

For $p=q=2$ we get

$$\begin{aligned}
1 &\leq \frac{\psi^{1/2}\left(a^2\right)\psi^{1/2}\left(b^2\right)}{\psi\left(a^2\sharp b^2\right)} \\
&\leq \max\left\{\frac{A\left(1,\left(\frac{k_2}{K_2}\right)^2\left(\frac{k_1}{K_1}\right)^2\right)}{G\left(1,\left(\frac{k_2}{K_2}\right)^2\left(\frac{k_1}{K_1}\right)^2\right)},\frac{A\left(1,\left(\frac{K_2}{k_2}\right)^2\left(\frac{K_1}{k_1}\right)^2\right)}{G\left(1,\left(\frac{K_2}{k_2}\right)^2\left(\frac{K_1}{k_1}\right)^2\right)}\right\} \\
&= \frac{k_1^2k_2^2+K_1^2K_2^2}{2k_1k_2K_1K_2},
\end{aligned}$$

which proves the desired result (3.16). □

4. Further Bounds

By the use of the multiplicative inequalities from the introduction we have further upper and lower bounds for the quotient

$$\frac{A_\nu\left(1,t\right)}{G_\nu\left(1,t\right)}$$

with $t>0$ and $\nu\in[0,1]$.

Indeed, by (FT), (ZL), (1.5) and (1.6) we have the following upper bounds

$$\frac{A_\nu\left(1,t\right)}{G_\nu\left(1,t\right)}\leq\begin{cases}\mathcal{S}\left(t\right),\\ \mathcal{K}^R\left(t\right),\\ \exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{\max\{1,t\}}{\min\{1,t\}}-1\right)^2\right],\\ \exp\left[4\nu\left(1-\nu\right)\left(\mathcal{K}\left(t\right)-1\right)\right]\end{cases}\tag{4.1}$$

for any $t>0$ and $\nu\in[0,1]$, where $R=\max\{1-\nu,\nu\}$.

By using the inequalities (ZL) and (1.5) we have the lower bounds

$$\left.\begin{array}{r}\mathcal{K}^r\left(t\right),\\ \exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(1-\frac{\min\{1,t\}}{\max\{1,t\}}\right)^2\right]\end{array}\right\}\leq\frac{A_\nu\left(1,t\right)}{G_\nu\left(1,t\right)}\tag{4.2}$$

for any $t>0$ and $\nu\in[0,1]$, where $r=\min\{1-\nu,\nu\}$.

Observe that for $0<m<M$ and $q>1$, by making use of the definition (3.2) we have

$$\begin{aligned}
&\Gamma_{1/q}\left(m^q,M^q\right)\\
&=\begin{cases}\frac{A_{1/q}(1,m^q)}{G_{1/q}(1,m^q)}\text{ if }M<1,\\ \\ \max\left\{\frac{A_{1/q}(1,m^q)}{G_{1/q}(1,m^q)},\frac{A_{1/q}(1,M^q)}{G_{1/q}(1,M^q)}\right\}\text{ if }m\leq1\leq M,\\ \\ \frac{A_{1/q}(1,M^q)}{G_{1/q}(1,M^q)}\text{ if }1<m.\end{cases}
\end{aligned}\tag{4.3}$$

Using the inequalities (4.1) we have the following upper bounds for $\Gamma_{1/q}\left(m^q,M^q\right)$.

If $0<m<M<1$, then

$$\Gamma_{1/q}\left(m^{q},M^{q}\right)\leq\left\{\begin{array}{l}\mathcal{S}\left(m^{q}\right),\\ \mathcal{K}^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(m^{q}\right),\\ \exp\left[\frac{1}{2pq}\nu\left(1-\nu\right)\left(\frac{1-m^{q}}{m^{q}}\right)^{2}\right],\\ \exp\left[\frac{4}{pq}\left(\mathcal{K}\left(m^{q}\right)-1\right)\right].\end{array}\right.\tag{4.4}$$

If $0<m\leq 1\leq M$, then

$$\begin{aligned}&\Gamma_{1/q}\left(m^{q},M^{q}\right)\\ &\leq\begin{array}{l}\max\left\{\mathcal{S}\left(m^{q}\right),\mathcal{S}\left(M^{q}\right)\right\},\\ \max\left\{\mathcal{K}^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(m^{q}\right),\mathcal{K}^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(M^{q}\right)\right\},\\ \max\left\{\exp\left[\frac{1}{2pq}\left(\frac{1-m^{q}}{m^{q}}\right)^{2}\right],\exp\left[\frac{1}{2pq}\left(M^{q}-1\right)^{2}\right]\right\},\\ \max\left\{\exp\left[\frac{4}{pq}\left(\mathcal{K}\left(m^{q}\right)-1\right)\right],\exp\left[\frac{4}{pq}\left(\mathcal{K}\left(M^{q}\right)-1\right)\right]\right\}.\end{array}\end{aligned}\tag{4.5}$$

If $1<m<M$, then

$$\Gamma_{1/q}\left(m^{q},M^{q}\right)\leq\left\{\begin{array}{l}\mathcal{S}\left(M^{q}\right),\\ \mathcal{K}^{\max\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(M^{q}\right),\\ \exp\left[\frac{1}{2pq}\left(M^{q}-1\right)^{2}\right],\\ \exp\left[\frac{4}{pq}\left(\mathcal{K}\left(M^{q}\right)-1\right)\right].\end{array}\right.\tag{4.6}$$

Also, we observe that for $0<m<M$ and $q>1$, by making use of the definition (3.3) we have that

$$\gamma_{1/q}\left(m^{q},M^{q}\right)=\left\{\begin{array}{l}\frac{A_{\nu}\left(1,M^{q}\right)}{G_{\nu}\left(1,M^{q}\right)}\text{ if }M<1,\\ \\ 1\text{ if }k\leq 1\leq K,\\ \\ \frac{A_{\nu}\left(1,m^{q}\right)}{G_{\nu}\left(1,m^{q}\right)}\text{ if }1<m.\end{array}\right.\tag{4.7}$$

Using the inequalities (4.2) we have the following lower bounds for $\gamma_{1/q}\left(m^{q},M^{q}\right)$.

If $0<m<M<1$, then

$$\left.\begin{array}{r}\mathcal{K}^{\min\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(M^{q}\right),\\ \exp\left[\frac{1}{2pq}\left(1-M^{q}\right)^{2}\right]\end{array}\right\}\leq\gamma_{1/q}\left(m^{q},M^{q}\right).\tag{4.8}$$

Finally, if $1<m<M$, then

$$\left.\begin{array}{r}\mathcal{K}^{\min\left\{\frac{1}{p},\frac{1}{q}\right\}}\left(m^{q}\right),\\ \exp\left[\frac{1}{2pq}\left(\frac{m^{q}-1}{m^{q}}\right)^{2}\right]\end{array}\right\}\leq\gamma_{1/q}\left(m^{q},M^{q}\right).\tag{4.9}$$

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