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ČEBYŠEV'S TYPE INEQUALITIES FOR POSITIVE LINEAR FUNCTIONALS ON HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Some inequalities of Čebyšev type for positive linear functionals of synchronous functions of selfadjoint elements in Hermitian Unital Banach $*$-Algebras are given. Applications for power function and logarithm are provided as well.

1. Introduction

We say that the functions $f,\ g:[a,b]\longrightarrow\mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$(f(t)-f(s))(g(t)-g(s))\geq(\leq)\,0 \text{ for each } t,\ s\in[a,b]\,.$$

It is obvious that, if $f,\ g$ are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

In 1882-1883, Čebyšev [4] and [5] proved that if the n-tuples $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_n)$ are monotonic in the same (opposite) sense, then

$$\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\geq(\leq)\,0, \tag{1.1}$$

where $\mathbf{p}=(p_1,...,p_n)$ are positive weights.

In the special case $\mathbf{p}=\mathbf{a}\geq\mathbf{0}$, it appears that the inequality (1.1) has been obtained by Laplace long before Čebyšev (see for example [22, p. 240]).

The inequality (1.1) was mentioned by Hardy, Littlewood and Pólya in their book [21] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are synchronous (asynchronous), this means that

$$(a_i-a_j)(b_i-b_j)\geq(\leq)\,0 \text{ for any } i,\ j\in\{1,\ldots,n\}\,, \tag{1.2}$$

then (1.1) holds true as well.

For other recent results on the Čebyšev inequality in either discrete or integral form see [3], [7], [8], [9], [10], [11], [22], [24], [25], [28], [29], [30], and the references therein.

The following result provides an inequality of Čebyšev type for functions of selfadjoint operators [15] (see also [14, p. 73] or [16, p. 73]):

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Theorem 1. *Let A be a selfadjoint operator with* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some real numbers $m < M$. If f, $g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on* $[m, M]$, *then*

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle f(A) x, x \rangle \langle g(A) x, x \rangle \tag{1.3}$$

for any $x \in H$ with $\|x\| = 1$.

Assume that A is a positive operator on the Hilbert space H and p, $q > 0$. Then for each $x \in H$ with $\|x\| = 1$ we have by (1.3) the inequality

$$\left\langle A^{p+q} x, x \right\rangle \geq \langle A^p x, x \rangle \langle A^q x, x \rangle . \tag{1.4}$$

If A is positive definite then the inequality (1.4) also holds for p, $q < 0$. If A is positive definite and either $p > 0$, $q < 0$ or $p < 0$, $q > 0$, then the reverse inequality holds in (1.4).

Assume that A is positive definite and $p > 0$. Then by (1.3) we have

$$\langle A^p \log Ax, x \rangle \geq \langle A^p x, x \rangle \langle \log Ax, x \rangle \tag{1.5}$$

for each $x \in H$ with $\|x\| = 1$. If $p < 0$ then the reverse inequality holds in (1.5).

The following result that is related to the Čebyšev inequality also holds [15] (see also [14, p. 73] or [16, p. 73]):

Theorem 2. *Let A be a selfadjoint operator with* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some real numbers $m < M$.*

If f, $g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous on $[m, M]$, *then*

$$\begin{aligned} & \langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle \\ & \geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] [g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle] \end{aligned} \tag{1.6}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous, then

$$\begin{aligned} & \langle f(A) x, x \rangle \langle g(A) x, x \rangle - \langle f(A) g(A) x, x \rangle \\ & \geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] [\langle g(A) x, x \rangle - g(\langle Ax, x \rangle)] \end{aligned} \tag{1.7}$$

for any $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint operator with $\operatorname{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$. If f, $g : [m, M] \longrightarrow \mathbb{R}$ are continuous, *synchronous and one is convex while the other is concave on* $[m, M]$, then by Jensen's inequality for convex (concave) functions and by (1.6) we have

$$\begin{aligned} & \langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle \\ & \geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] [g(\langle Ax, x \rangle) - \langle g(A) x, x \rangle] \geq 0 \end{aligned} \tag{1.8}$$

for any $x \in H$ with $\|x\| = 1$.

If f, g are asynchronous and *either both of them are convex or both of them concave* on $[m, M]$, then

$$\begin{aligned} & \langle f(A) x, x \rangle \langle g(A) x, x \rangle - \langle f(A) g(A) x, x \rangle \\ & \geq [\langle f(A) x, x \rangle - f(\langle Ax, x \rangle)] [\langle g(A) x, x \rangle - g(\langle Ax, x \rangle)] \geq 0 \end{aligned} \tag{1.9}$$

for any $x \in H$ with $\|x\| = 1$.

Assume that A is a positive operator on the Hilbert space H. If $p \in (0,1)$ and $q \in (1,\infty)$, then for each $x \in H$ with $\|x\| = 1$ we have the inequality

$$\begin{aligned} &\left\langle A^{p+q}x, x\right\rangle - \langle A^p x, x\rangle \langle A^q x, x\rangle \\ &\geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\,[\langle Ax, x\rangle^p - \langle A^p x, x\rangle] \geq 0. \end{aligned} \tag{1.10}$$

If A is positive definite and $p > 1$, $q < 0$, then

$$\begin{aligned} &\langle A^p x, x\rangle \langle A^q x, x\rangle - \left\langle A^{p+q}x, x\right\rangle \\ &\geq [\langle A^q x, x\rangle - \langle Ax, x\rangle^q]\,[\langle A^p x, x\rangle - \langle Ax, x\rangle^p] \geq 0 \end{aligned} \tag{1.11}$$

for each $x \in H$ with $\|x\| = 1$.

Assume that A is positive definite and $p > 1$. Then also

$$\begin{aligned} &\langle A^p \log Ax, x\rangle - \langle A^p x, x\rangle \langle \log Ax, x\rangle \\ &\geq [\langle A^p x, x\rangle - \langle Ax, x\rangle^p]\,[\log \langle Ax, x\rangle - \langle \log Ax, x\rangle] \geq 0 \end{aligned} \tag{1.12}$$

for each $x \in H$ with $\|x\| = 1$.

In order to extend these results in the more general case of Hermitian unital Banach $*$-algebras and for positive linear functionals we need the following preparation.

2. Some Facts on Hermitian Unital Banach $*$-Algebra

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0,\infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [32] (see also [2, Theorem 41.5])

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [27], Tanahashi and Uchiyama [33] proved the following fundamental properties (see also [20]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5].

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\, z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz.$$

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [20], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [33, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [27] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

Now, assume that $f(\cdot)$ is analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$ assume that $f(z) \geq 0$ for any $z \in I$. If $u \in A$ such that $\sigma(u) \subset I$, then by (SMT) we have

$$\sigma(f(u)) = f(\sigma(u)) \subset f(I) \subset [0, \infty)$$

meaning that $f(u) \geq 0$ in the order of A.

Therefore, we can state the following fact that will be used to establish various inequalities in A, see also [18].

Lemma 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and for the real interval $I \subset G$, assume that $f(z) \geq g(z)$ for any $z \in I$. Then for any $u \in A$ with $\sigma(u) \subset I$ we have $f(u) \geq g(u)$ in the order of A.*

Definition 1. *Assume that A is a Hermitian unital Banach $*$-algebra. A linear functional $\psi : A \to \mathbb{C}$ is positive if for $a \geq 0$ we have $\psi(a) \geq 0$. We say that it is normalized if $\psi(1) = 1$. The functional ψ is called faithful if $a \geq 0$ and $\psi(a) = 0$ implies that $a = 0$. The functional ψ is called tracial if $\psi(ab) = \psi(ba)$ for any $a, b \in A$*

We observe that the positive linear functional ψ preserves the order relation, namely if $a \geq b$ then $\psi(a) \geq \psi(b)$ and if $\beta \geq a \geq \alpha$ with α, β real numbers, then $\beta \geq \psi(a) \geq \alpha$, provided ψ is normalized. If the positive linear functional ψ is faithful and $a > 0$ then $\psi(a) > 0$.

3. Čebyšev Type Inequalities for Positive Functionals

We have the following result:

Theorem 3. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f and g are synchronous (asynchronous) on the interval I and φ, $\psi : A \to \mathbb{C}$ are positive normalized linear functionals on A, then for any selfadjoint elements a, $b \in A$ with $\sigma(a)$, $\sigma(b) \subset I$,*

$$\varphi(f(a)g(a)) + \psi(f(b)g(b)) \geq (\leq)\, \varphi(f(a))\psi(g(b)) + \varphi(g(a))\psi(f(b)). \tag{3.1}$$

In particular, we have

$$\varphi(f(a)g(a)) + \psi(f(a)g(a)) \geq (\leq)\, \varphi(f(a))\psi(g(a)) + \varphi(g(a))\psi(f(a)). \tag{3.2}$$

Proof. We consider only the case of synchronous functions. In this case we have that

$$f(t)g(t) + f(s)g(s) \geq f(t)g(s) + f(s)g(t) \tag{3.3}$$

for each t, $s \in I$.

Fix $s \in I$. By utilizing Lemma 1 and the inequality (3.3) we obtain in the order of A that

$$f(a)g(a) + f(s)g(s) \geq g(s)f(a) + f(s)g(a)$$

for any $a \in A$ with $\sigma(a) \subset I$ and any $s \in I$.

If we apply to this inequality the functional $\varphi : A \to \mathbb{C}$ that is positive, normalized and linear, we get

$$\varphi(f(a)g(a)) + f(s)g(s) \geq g(s)\varphi(f(a)) + f(s)\varphi(g(a)) \tag{3.4}$$

for any $s \in I$.

If we apply Lemma 1 again and the inequality (3.4), we get in the order of A that

$$\varphi(f(a)g(a)) + f(b)g(b) \geq \varphi(f(a))g(b) + \varphi(g(a))f(b) \tag{3.5}$$

for any $b \in A$ with $\sigma(b) \subset I$.

If we apply to this inequality the functional $\psi : A \to \mathbb{C}$ that is positive, normalized and linear, we get the desired result (3.1). □

Remark 1. *If we take in (3.1) $\psi = \varphi$, then we get*

$$\varphi(f(a)g(a)) + \varphi(f(b)g(b)) \geq (\leq)\, \varphi(f(a))\varphi(g(b)) + \varphi(g(a))\varphi(f(b)) \tag{3.6}$$

and in particular, for $b = a$ we get the Čebyšev type inequality

$$\varphi(f(a)g(a)) \geq (\leq)\, \varphi(f(a))\varphi(g(a)) \tag{Ce}$$

for $\varphi : A \to \mathbb{C}$ a positive normalized linear functional on A and for any selfadjoint element $a \in A$ with $\sigma(a) \subset I$.

If ψ is a positive linear functional (non-necessary normalized) and $\psi(1) > 0$ then

$$\psi(1)\psi(f(a)g(a)) \geq (\leq)\, \psi(f(a))\psi(g(a)), \tag{Ce1}$$

provided f and g are synchronous (asynchronous) on the interval I and the selfadjoint element $a \in A$ with $\sigma(a) \subset I$.

If $0 < p \in A$ and $\varphi : A \to \mathbb{C}$ is a positive linear functional on A with $\varphi(p) > 0$ (it suffices for φ to be faithful), then the functional $\psi_p(x) := \varphi\left(p^{1/2}xp^{1/2}\right)$ is a positive linear functional with $\psi_p(1) = \varphi(p) > 0$ and by (Ce) we have

$$\varphi(p)\varphi\left(p^{1/2}f(a)g(a)p^{1/2}\right) \geq (\leq)\varphi\left(p^{1/2}f(a)p^{1/2}\right)\varphi\left(p^{1/2}g(a)p^{1/2}\right). \tag{Cep}$$

Moreover, if φ is tracial then by (Cep) we have

$$\varphi(p)\varphi(pf(a)g(a)) \geq (\leq)\varphi(pf(a))\varphi(pg(a)). \tag{Cept}$$

We will use only the inequality (Cep) to exemplify the Čebyšev type inequalities obtained above.

Proposition 1. *Let $0 < p \in A$ and $\varphi : A \to \mathbb{C}$ a faithful positive linear functional on A.*

(i) If $0 \leq a \in A$ and $\alpha, \beta > 0$, then

$$\varphi(p)\varphi\left(p^{1/2}a^{\alpha+\beta}p^{1/2}\right) \geq \varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right). \tag{3.7}$$

(ii) If $0 < a \in A$ and $\alpha, \beta < 0$, then the inequality (3.7) also holds.

(iii) If $0 < a \in A$ and either $\alpha > 0$, $\beta < 0$ or $\alpha < 0$, $\beta > 0$, then the reverse inequality holds in (3.7).

(iv) If $0 < a \in A$ and $\alpha > 0$, then

$$\varphi(p)\varphi\left(p^{1/2}(a^{\alpha}\ln a)p^{1/2}\right) \geq \varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)\varphi\left(p^{1/2}(\ln a)p^{1/2}\right). \tag{3.8}$$

(v) If $0 < a \in A$ and $\alpha < 0$,then the reverse inequality holds in (3.8).

These results generalize the corresponding inequalities from (1.3)-(1.5).

Let $w_j \in A$, $j = 1, ..., k$ satisfy the property

$$\sum_{j=1}^{k} w_j^* w_j = 1_H. \tag{3.9}$$

The map $\varphi_w : A \to \mathbb{C}$ defined by

$$\varphi_w(x) := \sum_{j=1}^{k} \varphi\left(w_j^* x w_j\right),$$

where $\varphi : A \to \mathbb{C}$ is a positive normalized linear functionals on A, is linear, positive and normalized.

Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f and g are synchronous (asynchronous) on the interval I, then by (Ce) we have for $\varphi : A \to \mathbb{C}$ a positive normalized linear functional on A

$$\sum_{j=1}^{k} \varphi\left(w_j^* f(a) g(a) w_j\right) \geq (\leq) \sum_{j=1}^{k} \varphi\left(w_j^* f(a) w_j\right) \sum_{j=1}^{k} \varphi\left(w_j^* g(a) w_j\right) \tag{3.10}$$

for any selfadjoint element $a \in A$ with $\sigma(a) \subset I$.

If, moreover $0 \leq a \in A$, and $\alpha, \beta > 0$, then by (3.10) we get

$$\varphi\left(\sum_{j=1}^{k} w_j^* a^{\alpha+\beta} w_j\right) \geq \varphi\left(\sum_{j=1}^{k} w_j^* a^{\alpha} w_j\right)\varphi\left(\sum_{j=1}^{k} w_j^* a^{\beta} w_j\right). \tag{3.11}$$

Moreover, if $0 < a \in A$, φ is faithful and either $\alpha > 0$, $\beta < 0$ or or $\alpha < 0$, $\beta > 0$, then the reverse inequality holds in (3.11).

In the later case, if we take the square root in (3.11) and use the arithmetic mean-geometric mean inequality, we get

$$\varphi^{1/2}\left(\sum_{j=1}^{k} w_j^* a^{\alpha+\beta} w_j\right) \le \varphi\left(\sum_{j=1}^{k} w_j^* \left(\frac{a^\alpha + a^\beta}{2}\right) w_j\right). \tag{3.12}$$

4. Some Related Results

We have:

Theorem 4. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. If f and g are synchronous (asynchronous) on the interval I and φ, $\psi : A \to \mathbb{C}$ are positive normalized linear functionals on A, then for any selfadjoint elements a, $b \in A$ with $\sigma(a), \sigma(b) \subset I$,*

$$\psi(f(b)g(b)) + f(\varphi(a))g(\varphi(a)) \ge f(\varphi(a))\psi(g(b)) + g(\varphi(a))\psi(f(b)). \tag{4.1}$$

In particular,

$$\varphi(f(b)g(b)) + f(\varphi(a))g(\varphi(a)) \ge f(\varphi(a))\varphi(g(b)) + g(\varphi(a))\varphi(f(b)) \tag{4.2}$$

and

$$\psi(f(a)g(a)) + f(\varphi(a))g(\varphi(a)) \ge f(\varphi(a))\psi(g(a)) + g(\varphi(a))\psi(f(a)). \tag{4.3}$$

Proof. Since $\sigma(a)$ is compact and $\sigma(a) \subset I$, there exist the real numbers $m < M$ such that $\sigma(a) \subseteq [m, M] \subset I$. This implies that $m \le a \le M$ in the order of A, hence by taking the functional φ we have $\varphi(a) \in [m, M] \subset I$.

We consider only the case of synchronous functions. In this case we have that

$$[f(z) - f(\varphi(a))][g(z) - g(\varphi(a))] \ge 0$$

for any $z \in I$.

This can be written as

$$f(z)g(z) + f(\varphi(a))g(\varphi(a)) \ge f(\varphi(a))g(z) + g(\varphi(a))f(z) \tag{4.4}$$

for any $z \in I$.

By utilizing Lemma 1 and the inequality (4.4) we obtain in the order of A that

$$f(b)g(b) + f(\varphi(a))g(\varphi(a)) \ge f(\varphi(a))g(b) + g(\varphi(a))f(b) \tag{4.5}$$

for any $\sigma(b) \subset I$.

If we take the functional ψ in (4.5), then we get the desired result (4.1). □

Corollary 1. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that $\varphi : A \to \mathbb{C}$ is a positive normalized linear functional on A and the selfadjoint element $a \in A$ with $\sigma(a) \subset I$.*

(i) If f and g are synchronous on the interval I, then

$$\begin{aligned} &\varphi(f(a)g(a)) - \varphi(f(a))\varphi(g(a)) \\ &\ge (f(\varphi(a)) - \varphi(f(a)))(\varphi(g(a)) - g(\varphi(a))). \end{aligned} \tag{4.6}$$

(ii) If f and g are asynchronous on the interval I, then

$$\begin{aligned} &\varphi(f(a))\varphi(g(a)) - \varphi(f(a)g(a)) \\ &\ge (\varphi(f(a)) - f(\varphi(a)))(\varphi(g(a)) - g(\varphi(a))). \end{aligned} \tag{4.7}$$

Proof. If we take in (4.2) $b=a$, we get

$$\varphi(f(a)g(a))+f(\varphi(a))g(\varphi(a))\geq f(\varphi(a))\varphi(g(a))+g(\varphi(a))\varphi(f(a)). \tag{4.8}$$

By subtracting in (4.8) $\varphi(f(a))\varphi(g(a))$ and rearranging, we get

$$\begin{aligned}&\varphi(f(a)g(a))-\varphi(f(a))\varphi(g(a))\\ &\geq f(\varphi(a))\varphi(g(a))+g(\varphi(a))\varphi(f(a))\\ &-f(\varphi(a))g(\varphi(a))-\varphi(f(a))\varphi(g(a))\\ &=(f(\varphi(a))-\varphi(f(a)))(\varphi(g(a))-g(\varphi(a))),\end{aligned}$$

which proves (4.6).

The inequality can be proved in a similar way and we omit the details. □

We need the following Jensen's type inequality for convex functions on a Hermitian unital Banach $*$-algebra :

Lemma 2. *Let $f(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I\subset G$. If f is convex (in the usual sense) on the interval I and $\psi: A\to\mathbb{C}$ is a positive normalized linear functional on A, then for any selfadjoint element $c\in A$ with $\sigma(c)\subset I$,*

$$\psi(f(c))\geq f(s)+f'(s)(\psi(c)-s) \tag{4.9}$$

for any $s\in I$.

In particular, we have the Jensen inequality

$$\psi(f(c))\geq f(\psi(c)). \tag{4.10}$$

Proof. Since f is differentiable and convex on I we have by the gradient inequality that

$$f(t)\geq f(s)+(t-s)f'(s)$$

for any t, $s\in I$.

Fix $s\in I$ and apply Lemma 1 for the analytic functions $f(z)$ and $g_s(z):=f(s)+f'(s)(z-s)$ to get for $c\in A$ with $\sigma(c)\subset I$ that the following inequality holds

$$f(c)\geq f(s)+f'(s)(c-s) \tag{4.11}$$

in the order of A and for any $s\in I$.

If we take the functional ψ on (3.5) we get

$$\begin{aligned}\psi(f(c))&\geq\psi[f(s)+f'(s)(c-s)]\\ &=f(s)\psi(1)+f'(s)(\psi(c)-s\psi(1))\\ &=f(s)\psi+f'(s)(\psi(c)-s)\end{aligned}$$

and the inequality (4.9) is proved.

Since $\sigma(c)$ is compact and $\sigma(c)\subset I$, then there exists the real numbers m, M with $\sigma(c)\subseteq[m,M]\subset I$. This means that we have $m\leq c\leq M$ in the order of A and by taking the functional ψ, we have $m\leq\psi(c)\leq M$, meaning that $\psi(c)\in[m,M]\subset I$. Therefore, by taking $s=\psi(c)\in[m,M]$ in (4.9) we get (4.10). □

We can establish now some refinements of the Čebyšev type inequality (Ce) when some convexity properties are assumed.

Corollary 2. *Let $f(z)$ and $g(z)$ be analytic in G, an open subset of $\mathbb{C}$ and the real interval $I \subset G$. Assume that $\varphi : A \to \mathbb{C}$ is a positive normalized linear functional on A and the selfadjoint element $a \in A$ with $\sigma(a) \subset I$.*

(i) If f and g are synchronous on the interval I and one is convex while the other is concave on I, then

$$\begin{aligned}\varphi(f(a)g(a)) - \varphi(f(a))\varphi(g(a)) \\ \geq (f(\varphi(a)) - \varphi(f(a)))(\varphi(g(a)) - g(\varphi(a))) \geq 0.\end{aligned} \tag{4.12}$$

(ii) If f and g are asynchronous and either both of them are convex or both of them concave on the interval I, then

$$\begin{aligned}\varphi(f(a))\varphi(g(a)) - \varphi(f(a)g(a)) \\ \geq (\varphi(f(a)) - f(\varphi(a)))(\varphi(g(a)) - g(\varphi(a))) \geq 0.\end{aligned} \tag{4.13}$$

The proof follows by Corollary 1 and Lemma 2.

Assume that $a \in A$ is a selfadjoint element with $\sigma(a) \subset I$.

If ψ is a positive linear functional (non-necessary normalized) and $\psi(1) > 0$ then

$$\begin{aligned}&\frac{\psi(f(a)g(a))}{\psi(1)} - \frac{\psi(f(a))}{\psi(1)}\frac{\psi(g(a))}{\psi(1)} \\ &\geq \left(f\left(\frac{\psi(a)}{\psi(1)}\right) - \frac{\psi(f(a))}{\psi(1)}\right)\left(\frac{\psi(g(a))}{\psi(1)} - g\left(\frac{\psi(a)}{\psi(1)}\right)\right) \geq 0\end{aligned} \tag{4.14}$$

provided f and g are synchronous on the interval I and one is convex while the other is concave on I.

If f and g are asynchronous and either both of them are convex or both of them concave on the interval I, then

$$\begin{aligned}&\frac{\psi(f(a))}{\psi(1)}\frac{\psi(g(a))}{\psi(1)} - \frac{\psi(f(a)g(a))}{\psi(1)} \\ &\geq \left(\frac{\psi(f(a))}{\psi(1)} - f\left(\frac{\psi(a)}{\psi(1)}\right)\right)\left(\frac{\psi(g(a))}{\psi(1)} - g\left(\frac{\psi(a)}{\psi(1)}\right)\right) \geq 0.\end{aligned} \tag{4.15}$$

If $0 < p \in A$ and $\varphi : A \to \mathbb{C}$ is a positive linear functional on A with $\varphi(p) > 0$ (it suffices for φ to be faithful), then

$$\begin{aligned}&\frac{\varphi\left(p^{1/2}f(a)g(a)p^{1/2}\right)}{\varphi(p)} - \frac{\varphi\left(p^{1/2}f(a)p^{1/2}\right)}{\varphi(p)}\frac{\varphi\left(p^{1/2}g(a)p^{1/2}\right)}{\varphi(p)} \\ &\geq \left(f\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi(p)}\right) - \frac{\varphi\left(p^{1/2}f(a)p^{1/2}\right)}{\varphi(p)}\right) \\ &\times \left(\frac{\varphi\left(p^{1/2}g(a)p^{1/2}\right)}{\varphi(p)} - g\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi(p)}\right)\right) \\ &\geq 0\end{aligned} \tag{4.16}$$

provided f and g are synchronous on the interval I and one is convex while the other is concave on I.

If f and g are asynchronous and either both of them are convex or both of them concave on the interval I, then

$$\begin{aligned}
(4.17)\qquad & \frac{\varphi\left(p^{1/2}f\left(a\right)p^{1/2}\right)}{\varphi\left(p\right)}\frac{\varphi\left(p^{1/2}g\left(a\right)p^{1/2}\right)}{\varphi\left(p\right)}-\frac{\varphi\left(p^{1/2}f\left(a\right)g\left(a\right)p^{1/2}\right)}{\varphi\left(p\right)}\\
&\geq\left(\frac{\varphi\left(p^{1/2}f\left(a\right)p^{1/2}\right)}{\varphi\left(p\right)}-f\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)\right)\\
&\times\left(\frac{\varphi\left(p^{1/2}g\left(a\right)p^{1/2}\right)}{\varphi\left(p\right)}-g\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)\right)\\
&\geq 0.
\end{aligned}$$

We can give the following simple examples:

If $0\leq a\in A$, $\alpha\in(0,1)$ and $\beta\geq 1$, then by (4.16) we have

$$\begin{aligned}
(4.18)\qquad & \frac{\varphi\left(p^{1/2}a^{\alpha+\beta}p^{1/2}\right)}{\varphi\left(p\right)}-\frac{\varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)}{\varphi\left(p\right)}\frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)}\\
&\geq\left(\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)^{\alpha}-\frac{\varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)}{\varphi\left(p\right)}\right)\\
&\times\left(\frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)}-\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)^{\beta}\right)\\
&\geq 0
\end{aligned}$$

provided $0<p\in A$ and $\varphi:A\to\mathbb{C}$ is a positive linear functional on A with $\varphi\left(p\right)>0$.

If $0<a\in A$, $\alpha<0$ and $\beta\geq 1$, then by (4.17)

$$\begin{aligned}
(4.19)\qquad & \frac{\varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)}{\varphi\left(p\right)}\frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)}-\frac{\varphi\left(p^{1/2}a^{\alpha+\beta}p^{1/2}\right)}{\varphi\left(p\right)}\\
&\geq\left(\frac{\varphi\left(p^{1/2}a^{\alpha}p^{1/2}\right)}{\varphi\left(p\right)}-\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)^{\alpha}\right)\\
&\times\left(\frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)}-\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)^{\beta}\right)\\
&\geq 0,
\end{aligned}$$

provided $0<p\in A$ and $\varphi:A\to\mathbb{C}$ is a positive linear functional on A with $\varphi\left(p\right)>0$.

If $0 < a \in A$ and $\beta \geq 1$, then by (4.16)

$$\begin{aligned}
&\frac{\varphi\left(p^{1/2}\left(a^{\beta}\ln a\right)p^{1/2}\right)}{\varphi\left(p\right)} - \frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)}\frac{\varphi\left(p^{1/2}\left(\ln a\right)p^{1/2}\right)}{\varphi\left(p\right)} \\
&\geq \left(\ln\left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right) - \frac{\varphi\left(p^{1/2}\left(\ln a\right)p^{1/2}\right)}{\varphi\left(p\right)}\right) \\
&\times \left(\frac{\varphi\left(p^{1/2}a^{\beta}p^{1/2}\right)}{\varphi\left(p\right)} - \left(\frac{\varphi\left(p^{1/2}ap^{1/2}\right)}{\varphi\left(p\right)}\right)^{\beta}\right) \\
&\geq 0
\end{aligned} \tag{4.20}$$

provided $0 < p \in A$ and $\varphi : A \to \mathbb{C}$ is a positive linear functional on A with $\varphi\left(p\right) > 0$.

These results generalize the corresponding inequalities from (1.10)-(1.12).

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