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TWO POINTS TAYLOR'S TYPE REPRESENTATIONS WITH INTEGRAL REMAINDERS

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Abstract. In this paper we establish some two points Taylor's type representations with integral remainders and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

1. Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 1. *Let $I\subset\mathbb{R}$ be a closed interval, $c\in I$ and let n be a positive integer. If $f:I\longrightarrow\mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $y\in I$*

$$f(y)=T_n(f;c,y)+R_n(f;c,y), \tag{1.1}$$

where $T_n(f;c,y)$ is Taylor's polynomial, i.e.,

$$T_n(f;c,y):=\sum_{k=0}^{n}\frac{(y-c)^k}{k!}f^{(k)}(c). \tag{1.2}$$

Note that $f^{(0)}:=f$ and $0!:=1$ and the remainder is given by

$$R_n(f;c,y):=\frac{1}{n!}\int_c^y(y-t)^n f^{(n+1)}(t)\,dt. \tag{1.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For related results, see [1]-[5], [11]-[14], [18]-[19] and [22].

Let $a,\ b>0$, then we have the equality:

$$\ln b-\ln a=\sum_{k=1}^{n}\frac{(-1)^{k-1}(b-a)^k}{ka^k}+(-1)^n\int_a^b\frac{(b-t)^n}{t^{n+1}}dt,\ n\geq 1. \tag{1.4}$$

Indeed, if we consider the function $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=\ln x$, then,

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},\ n\geq 1,\ x>0,$$

$$T_n(\ln;a,x)=\ln a+\sum_{k=1}^{n}\frac{(-1)^{k-1}(x-a)^k}{ka^k},\ a>0$$

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and

$$R_n(\ln; a, x) = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}} dt.$$

Now, using (1.1), we have the equality,

$$\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k} + (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}} dt.$$

Choosing in the last equality $x = b$, we get (1.4).

Consider the function $f : \mathbb{R} \longrightarrow (0, \infty)$, $f(y) = \exp y$. Then for any $c \in \mathbb{R}$ we have

$$T_n(\exp; c, y) = \sum_{k=0}^{n} \frac{(y-c)^k}{k!} \exp c$$

and

$$R_n(\exp; c, y) := \frac{1}{n!} \int_c^y (y-t)^n \exp t dt.$$

On applying Taylor's formula (1.1) we have

$$\exp y - \exp c - \sum_{k=1}^{n} \frac{(y-c)^k}{k!} \exp c = \frac{1}{n!} \int_c^y (y-t)^n \exp t dt \tag{1.5}$$

for any c, $y \in \mathbb{R}$.

If we take $y = \ln x$, $c = \ln a$ where x, $a > 0$ then we get

$$x - a - a \sum_{k=1}^{n} \frac{(\ln x - \ln a)^k}{k!} = \frac{1}{n!} \int_{\ln a}^{\ln x} (\ln x - t)^n \exp t dt.$$

By using the change of variable, $s = \exp t$, we have

$$\int_{\ln a}^{\ln x} (\ln x - t)^n \exp t dt = \int_a^x (\ln x - \ln s)^n ds$$

giving that

$$b - a - a \sum_{k=1}^{n} \frac{(\ln b - \ln a)^k}{k!} = \frac{1}{n!} \int_a^b (\ln b - \ln s)^n ds, \tag{1.6}$$

for any b, $a > 0$.

Now, if $n \geq 2$ then by (1.6) we have

$$\frac{b-a}{a} - \sum_{k=1}^{n} \frac{(\ln b - \ln a)^k}{k!} = \frac{1}{n!a} \int_a^b (\ln b - \ln s)^n ds,$$

namely

$$\ln b - \ln a = \frac{b-a}{a} - \sum_{k=2}^{n} \frac{(\ln b - \ln a)^k}{k!} - \frac{1}{n!a} \int_a^b (\ln b - \ln s)^n ds, \tag{1.7}$$

for any b, $a > 0$.

By taking in (1.4) and (1.7) $n = 2m+1$, we get the following equalities of interest

$$\ln b - \ln a = \sum_{k=1}^{2m+1} \frac{(-1)^{k-1}(b-a)^k}{ka^k} - \int_a^b \frac{(b-t)^{2m+1}}{t^{2m+2}} dt, \; m \geq 0 \tag{1.8}$$

and

$$\text{(1.9)}\quad \ln b-\ln a = \frac{b-a}{a}-\sum_{k=2}^{2m+1}\frac{(\ln b-\ln a)^k}{k!}-\frac{1}{(2m+1)!a}\int_a^b(\ln b-\ln s)^{2m+1}ds,\ m\geq 1.$$

Since for any a, $b>0$

$$\int_a^b\frac{(b-t)^{2m+1}}{t^{2m+2}}dt\geq 0 \text{ and } \int_a^b(\ln b-\ln s)^{2m+1}ds\geq 0,$$

then we have from (1.8) that

$$\ln b-\ln a\leq\frac{b-a}{a}+\sum_{k=2}^{2m+1}\frac{(-1)^{k-1}(b-a)^k}{ka^k},\ m\geq 1 \tag{1.10}$$

and from (1.9) that

$$\ln b-\ln a\leq\frac{b-a}{a}-\sum_{k=2}^{2m+1}\frac{(\ln b-\ln a)^k}{k!},\ m\geq 1. \tag{1.11}$$

The case $m=1$ provides the following inequalities

$$\ln b-\ln a\leq\frac{b-a}{a}-\frac{(b-a)^2}{2a^2}+\frac{(b-a)^3}{3a^3} \tag{1.12}$$

and

$$\ln b-\ln a\leq\frac{b-a}{a}-\frac{(\ln b-\ln a)^2}{2}-\frac{(\ln b-\ln a)^3}{6} \tag{1.13}$$

for any a, $b>0$.

Now, if $0<a<b$ then by (1.7) we have

$$\ln b-\ln a\leq\frac{b-a}{a}-\sum_{k=2}^{n}\frac{(\ln b-\ln a)^k}{k!} \tag{1.14}$$

for any $n\geq 2$.

If $0<a<b$ and $n=2m$, then by (1.4) we have

$$\ln b-\ln a\geq\sum_{k=1}^{2m}\frac{(-1)^{k-1}(b-a)^k}{ka^k},\ m\geq 1 \tag{1.15}$$

while in the case $n=2m+1$ we have

$$\ln b-\ln a\leq\sum_{k=1}^{2m+1}\frac{(-1)^{k-1}(b-a)^k}{ka^k},\ m\geq 0. \tag{1.16}$$

In this paper we establish some two points Taylor's type representations with integral remainders and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

2. Some Two Points Identities

The following identity can be stated:

Theorem 2. *Let $f : I \to \mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct x, a, $b \in \mathring{I}$ and for any $\lambda \in \mathbb{R}\setminus\{0,1\}$ we have the representation*

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda) f^{(k)}(a)(x-a)^k + (-1)^k \lambda f^{(k)}(b)(b-x)^k\right] + S_{n,\lambda}(x,a,b), \tag{2.1}$$

where the remainder $S_{n,\lambda}(x,a,b)$ is given by

$$S_{n,\lambda}(x,a,b) := \frac{1}{n!}\left[(1-\lambda)(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n ds + (-1)^{n+1}\lambda(b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right]. \tag{2.2}$$

Proof. Using Taylor's representation with the integral remainder (1.1) we can write the following two identities

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt \tag{2.3}$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + \frac{(-1)^{n+1}}{n!}\int_x^b f^{(n+1)}(t)(t-x)^n dt \tag{2.4}$$

for any x, a, $b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t = (1-s)c + sd$, $s \in [0,1]$ that

$$\int_c^d h(t)\,dt = (d-c)\int_0^1 h((1-s)c+sd)\,ds.$$

Therefore,

$$\int_a^x f^{(n+1)}(t)(x-t)^n dt = (x-a)\int_0^1 f^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^n ds = (x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n ds$$

and

$$\begin{aligned}&\int_x^b f^{(n+1)}(t)(t-x)^n\,dt\\&=(b-x)\int_0^1 f^{(n+1)}((1-s)x+sb)((1-s)x+sb-x)^n\,ds\\&=(b-x)^{n+1}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds.\end{aligned}$$

The identities (2.3) and (2.4) can then be written as

$$\begin{aligned}f(x)&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^k\\&\quad+\frac{1}{n!}(x-a)^{n+1}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n\,ds\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}f(x)&=\sum_{k=0}^{n}\frac{(-1)^k}{k!}f^{(k)}(b)(b-x)^k\\&\quad+(-1)^{n+1}\frac{(b-x)^{n+1}}{n!}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds.\end{aligned}\tag{2.6}$$

Now, if we multiply (2.5) with $1-\lambda$ and (2.6) with λ and add the resulting equalities, a simple calculation yields the desired identity (2.1). □

Remark 1. *If we take in (2.1)* $x=\frac{a+b}{2}$, *with* $a,b\in\mathring{I}$, *then we have for any* $\lambda\in\mathbb{R}\setminus\{0,1\}$ *that*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=(1-\lambda)f(a)+\lambda f(b)\\&\quad+\sum_{k=1}^{n}\frac{1}{2^k k!}\left[(1-\lambda)f^{(k)}(a)+(-1)^k\lambda f^{(k)}(b)\right](b-a)^k\\&\quad+\tilde{S}_{n,\lambda}(a,b),\end{aligned}\tag{2.7}$$

where the remainder $\tilde{S}_{n,\lambda}(a,b)$ *is given by*

$$\begin{aligned}&\tilde{S}_{n,\lambda}(a,b)\\&:=\frac{1}{2^{n+1}n!}(b-a)^{n+1}\left[(1-\lambda)\int_0^1 f^{(n+1)}\left((1-s)a+s\frac{a+b}{2}\right)(1-s)^n\,ds\right.\\&\left.+(-1)^{n+1}\lambda\int_0^1 f^{(n+1)}\left((1-s)\frac{a+b}{2}+sb\right)s^n ds\right].\end{aligned}\tag{2.8}$$

In particular, for $\lambda=\frac{1}{2}$ *we have*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=\frac{f(a)+f(b)}{2}\\&\quad+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right](b-a)^k\\&\quad+\tilde{S}_n(a,b),\end{aligned}\tag{2.9}$$

where the remainder $\tilde{S}_n(a,b)$ *is given by*

$$\tilde{S}_n(a,b) := \frac{1}{2^{n+2}n!}(b-a)^{n+1}\left[\int_0^1 f^{(n+1)}\left((1-s)a+s\frac{a+b}{2}\right)(1-s)^n ds + (-1)^{n+1}\int_0^1 f^{(n+1)}\left((1-s)\frac{a+b}{2}+sb\right)s^n ds\right]. \tag{2.10}$$

Corollary 1. *With the assumptions in Theorem 2 we have for each distinct* x, a, $b \in \mathring{I}$

$$f(x) = \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right] + \frac{(b-x)(x-a)}{b-a} \times \sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\} + L_n(x,a,b), \tag{2.11}$$

where

$$L_n(x,a,b) := \frac{(b-x)(x-a)}{n!(b-a)}\left[(x-a)^n\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n ds + (-1)^{n+1}(b-x)^n\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right]$$

and

$$f(x) = \frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right] + \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right\} + P_n(x,a,b), \tag{2.12}$$

where

$$P_n(x,a,b) := \frac{1}{n!(b-a)}\left[(x-a)^{n+2}\int_0^1 f^{(n+1)}((1-s)a+sx)(1-s)^n ds + (-1)^{n+1}(b-x)^{n+2}\int_0^1 f^{(n+1)}((1-s)x+sb)s^n ds\right],$$

respectively.

The proof is obvious. Choose $\lambda = (x-a)/(b-a)$ and $\lambda = (b-x)/(b-a)$, respectively, in Theorem 2. The details are omitted.

Corollary 2. *With the assumption in Theorem 2 we have for each* $\lambda \in [0,1]$ *and any distinct* a, $b \in \mathring{I}$ *that*

$$f((1-\lambda)a+\lambda b) = (1-\lambda)f(a)+\lambda f(b)+\lambda(1-\lambda) \times \sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}f^{(k)}(a)+(-1)^k(1-\lambda)^{k-1}f^{(k)}(b)\right](b-a)^k + S_{n,\lambda}(a,b), \tag{2.13}$$

where the remainder $S_{n,\lambda}(a,b)$ is given by

$$S_{n,\lambda}(a,b) := \frac{1}{n!}(1-\lambda)\lambda(b-a)^{n+1}\left[\lambda^{n}\int_{0}^{1} f^{(n+1)}((1-s\lambda)a+s\lambda b)(1-s)^{n}ds + (-1)^{n+1}(1-\lambda)^{n}\int_{0}^{1} f^{(n+1)}((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)s^{n}ds\right]. \tag{2.14}$$

We also have

$$f((1-\lambda)b+\lambda a) = (1-\lambda)f(a)+\lambda f(b) + \sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k+1}f^{(k)}(a)+(-1)^{k}\lambda^{k+1}f^{(k)}(b)\right](b-a)^{k} + P_{n,\lambda}(a,b), \tag{2.15}$$

where the remainder $P_{n,\lambda}(a,b)$ is given by

$$P_{n,\lambda}(a,b) := \frac{1}{n!}(b-a)^{n+1}\left[(1-\lambda)^{n+2}\int_{0}^{1} f^{(n+1)}((1-s+\lambda s)a+(1-\lambda)sb)(1-s)^{n}ds + (-1)^{n+1}\lambda^{n+2}\int_{0}^{1} f^{(n+1)}((1-s)\lambda a+(1-\lambda+\lambda s)b)s^{n}ds\right]. \tag{2.16}$$

The case $n=0$, namely when the function f is locally absolutely continuous on $\mathring{I}$ with the derivative f' existing almost everywhere in $\mathring{I}$ is important and produces the following simple identities for each distinct x, a, $b \in \mathring{I}$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$

$$f(x) = (1-\lambda)f(a)+\lambda f(b)+S_{\lambda}(x,a,b), \tag{2.17}$$

where the remainder $S_{\lambda}(x,a,b)$ is given by

$$S_{\lambda}(x,a,b) := (1-\lambda)(x-a)\int_{0}^{1} f'((1-s)a+sx)ds - \lambda(b-x)\int_{0}^{1} f'((1-s)x+sb)ds. \tag{2.18}$$

We then have for each distinct x, a, $b \in \mathring{I}$

$$f(x) = \frac{1}{b-a}[(b-x)f(a)+(x-a)f(b)]+L(x,a,b), \tag{2.19}$$

where

$$L(x,a,b) := \frac{(b-x)(x-a)}{b-a}\left[\int_{0}^{1} f'((1-s)a+sx)ds - \int_{0}^{1} f'((1-s)x+sb)ds\right] \tag{2.20}$$

and

$$f(x) = \frac{1}{b-a}[(x-a)f(a)+(b-x)f(b)]+P(x,a,b), \tag{2.21}$$

where

$$P(x,a,b) := \frac{1}{b-a}\left[(x-a)^2\int_0^1 f'((1-s)a+sx)\,ds-(b-x)^2\int_0^1 f'((1-s)x+sb)\,ds\right]. \tag{2.22}$$

We also have

$$f((1-\lambda)a+\lambda b)=(1-\lambda)f(a)+\lambda f(b)+S_\lambda(a,b), \tag{2.23}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b) := (1-\lambda)\lambda(b-a)\left[\int_0^1 f'((1-s\lambda)a+s\lambda b)\,ds - \int_0^1 f'((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)\,ds\right] \tag{2.24}$$

and

$$f((1-\lambda)b+\lambda a)=(1-\lambda)f(a)+\lambda f(b)+P_\lambda(a,b), \tag{2.25}$$

where the remainder $P_\lambda(a,b)$ is given by

$$P_\lambda(a,b) := (b-a)\left[(1-\lambda)^2\int_0^1 f'((1-s+\lambda s)a+(1-\lambda)sb)\,ds - \lambda^2\int_0^1 f'((1-s)\lambda a+(1-\lambda+\lambda s)b)\,ds\right]. \tag{2.26}$$

Moreover, if we take in (2.17) $x=\frac{a+b}{2}$ for each distinct $a, b\in \mathring{I}$ and $\lambda\in \mathbb{R}\setminus\{0,1\}$, then we have

$$f\left(\frac{a+b}{2}\right)=(1-\lambda)f(a)+\lambda f(b)+S_\lambda(a,b), \tag{2.27}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b) := \frac{1}{2}(b-a)\times\left[(1-\lambda)\int_0^1 f'\left((1-s)a+s\frac{a+b}{2}\right)ds-\lambda\int_0^1 f'\left((1-s)\frac{a+b}{2}+sb\right)ds\right]. \tag{2.28}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right)=\frac{f(a)+f(b)}{2}+S(a,b), \tag{2.29}$$

where

$$S(a,b) := \frac{1}{4}(b-a)\times\left[\int_0^1 f'\left((1-s)a+s\frac{a+b}{2}\right)ds-\int_0^1 f'\left((1-s)\frac{a+b}{2}+sb\right)ds\right]. \tag{2.30}$$

3. Examples for Logarithm and Exponential

Consider the function $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=\ln x$, then,

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n},\ n\geq 1,\ x>0. \tag{3.1}$$

Using the identity (2.1) for this function we get for any x, a, $b>0$ and $\lambda\in\mathbb{R}\backslash\{0,1\}$ that

$$\begin{aligned}\ln x &= (1-\lambda)\ln a+\lambda\ln b\\ &+\sum_{k=1}^{n}\frac{1}{k}\left[(-1)^{k-1}(1-\lambda)\left(\frac{x}{a}-1\right)^k-\lambda\left(1-\frac{x}{b}\right)^k\right]\\ &+U_{n,\lambda}(x,a,b),\end{aligned} \tag{3.2}$$

where the remainder $U_{n,\lambda}(x,a,b)$ is given by

$$\begin{aligned}&U_{n,\lambda}(x,a,b)\\ &:=\left[(-1)^n(1-\lambda)(x-a)^{n+1}\int_0^1\frac{(1-s)^n}{((1-s)a+sx)^n}ds\right.\\ &\left.-\lambda(b-x)^{n+1}\int_0^1\frac{s^n}{((1-s)x+sb)^n}ds\right].\end{aligned} \tag{3.3}$$

Using the identity (2.7) for the function $f:(0,\infty)\longrightarrow\mathbb{R}$, $f(x)=\ln x$, then

$$\begin{aligned}\ln\left(\frac{a+b}{2}\right) &= (1-\lambda)\ln a+\lambda\ln b\\ &+\sum_{k=1}^{n}\frac{1}{2^k k}\left[(-1)^{k-1}\frac{1-\lambda}{a^k}-\frac{\lambda}{b^k}\right](b-a)^k\\ &+U_{n,\lambda}(a,b),\end{aligned} \tag{3.4}$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$\begin{aligned}&U_{n,\lambda}(a,b)\\ &:=\frac{1}{2^{n+1}}(b-a)^{n+1}\left[(1-\lambda)\int_0^1\frac{(-1)^n(1-s)^n}{\left((1-s)a+s\frac{a+b}{2}\right)^{n+1}}ds\right.\\ &\left.-\lambda\int_0^1\frac{s^n}{\left((1-s)\frac{a+b}{2}+sb\right)^{n+1}}ds\right].\end{aligned} \tag{3.5}$$

In particular, for $\lambda=\frac{1}{2}$, we have for all a, $b>0$ that

$$\begin{aligned}\ln\left(\frac{a+b}{2}\right) &= \frac{\ln a+\ln b}{2}\\ &+\sum_{k=1}^{n}\frac{1}{2^{k+1}k}\left[\frac{(-1)^{k-1}}{a^k}-\frac{1}{b^k}\right](b-a)^k+U_n(a,b),\end{aligned} \tag{3.6}$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$U_n(a,b) := \frac{1}{2^{n+2}}(b-a)^{n+1} \times \left[\int_0^1 \frac{(-1)^n (1-s)^n}{\left((1-s)a+s\frac{a+b}{2}\right)^{n+1}}ds - \int_0^1 \frac{s^n}{\left((1-s)\frac{a+b}{2}+sb\right)^{n+1}}ds\right]. \tag{3.7}$$

From (2.13) we have for any a, $b>0$ and $\lambda\in[0,1]$ that

$$0 \le \ln\left(\frac{(1-\lambda)a+\lambda b}{a^{1-\lambda}b^{\lambda}}\right) = \lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k} - \frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k + U_{n,\lambda}(a,b), \tag{3.8}$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$U_{n,\lambda}(a,b) := (1-\lambda)\lambda(b-a)^{n+1}\left[\lambda^n\int_0^1 \frac{(-1)^n(1-s)^n}{((1-s\lambda)a+s\lambda b)^{n+1}}ds - (1-\lambda)^n\int_0^1 \frac{s^n}{((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)^{n+1}}ds\right]. \tag{3.9}$$

Consider the function $f:\mathbb{R}\longrightarrow(0,\infty)$, $f(y)=\exp y$, then,

$$f^{(n)}(y) = \exp y,\ n\ge 1,\ x>0. \tag{3.10}$$

If we write the equality (2.1) for this function we get for any y, c, $d\in\mathbb{R}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$ that

$$\exp y = (1-\lambda)\exp c + \lambda\exp d + \sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)(y-c)^k\exp c + (-1)^k\lambda(d-y)^k\exp d\right] + R_{n,\lambda}(y,c,d), \tag{3.11}$$

where the remainder $R_{n,\lambda}(y,c,d)$ is given by

$$R_{n,\lambda}(y,c,d) := \frac{1}{n!}\left[(1-\lambda)(y-c)^{n+1}\int_0^1(1-s)^n\exp((1-s)c+sy)\,ds + (-1)^{n+1}\lambda(d-y)^{n+1}\int_0^1 s^n\exp((1-s)y+sd)\,ds\right]. \tag{3.12}$$

Let $x,\ a,\ b>0$. If we take in (3.11) and (3.12) $y=\ln x,\ c=\ln a$ and $d=\ln b$, then we get for any $\lambda\in\mathbb{R}\backslash\{0,1\}$ that

$$x=(1-\lambda)a+\lambda b+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)a(\ln x-\ln a)^{k}+(-1)^{k}\lambda b(\ln b-\ln x)^{k}\right]+R_{n,\lambda}(x,a,b), \tag{3.13}$$

where the remainder $R_{n,\lambda}(x,a,b)$ is given by

$$R_{n,\lambda}(x,a,b):=\frac{1}{n!}\left[(1-\lambda)(\ln x-\ln a)^{n+1}\int_0^1(1-s)^n a^{1-s}x^s ds+(-1)^{n+1}\lambda(\ln b-\ln x)^{n+1}\int_0^1 s^n x^{1-s}b^s ds\right]. \tag{3.14}$$

If we write the equality (2.13) for the function $f:\mathbb{R}\longrightarrow(0,\infty)$, $f(y)=\exp y$, we get for any $c,\ d\in\mathbb{R}$ and $\lambda\in\mathbb{R}\backslash\{0,1\}$ that

$$\exp((1-\lambda)c+\lambda d)=(1-\lambda)\exp c+\lambda\exp d+\lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}\exp c+(-1)^{k}(1-\lambda)^{k-1}\exp d\right](d-c)^{k}+T_{n,\lambda}(c,d), \tag{3.15}$$

where

$$T_{n,\lambda}(c,d):=\frac{1}{n!}(d-c)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1\exp((1-s+\lambda s)c+(1-\lambda)sd)(1-s)^n ds+(-1)^{n+1}\lambda^{n+2}\int_0^1\exp((1-s)\lambda c+(1-\lambda+\lambda s)d)s^n ds\right]. \tag{3.16}$$

Let $x,\ a,\ b>0$. If we take in (3.15) and (3.16) $y=\ln x,\ c=\ln a$ and $d=\ln b$, then we get for any $\lambda\in\mathbb{R}\backslash\{0,1\}$ that

$$a^{1-\lambda}b^{\lambda}=(1-\lambda)a+\lambda b+\lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}a+(-1)^{k}(1-\lambda)^{k-1}b\right](\ln b-\ln a)^{k}+T_{n,\lambda}(a,b), \tag{3.17}$$

where

$$T_{n,\lambda}(a,b):=\frac{1}{n!}(\ln b-\ln a)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1 a^{1-s+\lambda s}b^{(1-\lambda)s}(1-s)^n ds+(-1)^{n+1}\lambda^{n+2}\int_0^1 a^{(1-s)\lambda}b^{1-\lambda+\lambda s}s^n ds\right]. \tag{3.18}$$

If $\lambda \in [0,1]$ and a, $b > 0$ then we have from (3.17) that

$$0 \le (1-\lambda) a + \lambda b - a^{1-\lambda} b^{\lambda} = \lambda (1-\lambda) \sum_{k=1}^{n} \frac{1}{k!} \left[(-1)^{k-1} (1-\lambda)^{k-1} b - \lambda^{k-1} a \right] (\ln b - \ln a)^{k} - T_{n,\lambda}(a,b). \tag{3.19}$$

4. Some Inequalities

We have the following inequality:

Theorem 3. *Let $f : I \to \mathbb{R}$ be $(2m+1)$-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(2m+1)}$, with $m \ge 0$, be locally absolutely continuous on $\mathring{I}$. If $f^{(2m+2)}(t) \ge (\le) 0$ for almost every $t \in \mathring{I}$, then for each distinct x, a, $b \in \mathring{I}$ and for any $\lambda \in [0,1]$ we have*

$$f(x) \ge (\le) (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a) (x-a)^{k} + (-1)^{k} \lambda f^{(k)}(b) (b-x)^{k} \right]. \tag{4.1}$$

Proof. From Theorem 2 we have for each distinct x, a, $b \in \mathring{I}$ and for any $\lambda \in [0,1]$ that

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a) (x-a)^{k} + (-1)^{k} \lambda f^{(k)}(b) (b-x)^{k} \right] + S_{2m+1,\lambda}(x,a,b), \tag{4.2}$$

where the remainder $S_{2m+1,\lambda}(x,a,b)$ is given by

$$S_{2m+1,\lambda}(x,a,b) := \frac{1}{(2m+1)!} \left[(1-\lambda)(x-a)^{2m+2} \int_0^1 f^{(2m+2)}((1-s)a + sx)(1-s)^{2m+1} ds + \lambda (b-x)^{2m+2} \int_0^1 f^{(2m+2)}((1-s)x + sb) s^{2m+1} ds \right].$$

If $f^{(2m+2)}(t) \ge (\le) 0$ for almost every $t \in \mathring{I}$, then for each distinct x, a, $b \in \mathring{I}$ we have

$$\int_0^1 f^{(2m+2)}((1-s)a + sx)(1-s)^{2m+1} ds \ge (\le) 0$$

and

$$\int_0^1 f^{(2m+2)}((1-s)x + sb) s^{2m+1} ds \ge (\le) 0,$$

which implies that $S_{2m+1,\lambda}(x,a,b) \ge (\le) 0$ for each distinct x, a, $b \in \mathring{I}$.

Using the identity (4.2) we deduce the desired result (4.1). □

Corollary 3. *With the assumptions of Theorem 3 for the function $f : I \to \mathbb{R}$ then for each distinct $a, b \in \mathring{I}$ and for any $\lambda \in [0,1]$ we have*

$$f((1-\lambda)a+\lambda b) \geq (\leq)(1-\lambda)f(a)+\lambda f(b) + \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k!}\left[\lambda^{k-1}f^{(k)}(a)+(-1)^k(1-\lambda)^{k-1}f^{(k)}(b)\right](b-a)^k. \tag{4.3}$$

Remark 2. *If the function $f : I \to \mathbb{R}$ is twice differentiable convex (concave) on $\mathring{I}$ then for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in [0,1]$ we have from (4.1) that*

$$f(x) \geq (\leq)(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)f'(a)(x-a)-\lambda f'(b)(b-x). \tag{4.4}$$

From (4.3) we have that

$$f((1-\lambda)a+\lambda b) \geq (\leq)(1-\lambda)f(a)+\lambda f(b)+\lambda(1-\lambda)\left[f'(a)-f'(b)\right](b-a)$$

that is equivalent to

$$\lambda(1-\lambda)\left[f'(b)-f'(a)\right](b-a) \geq (\leq)(1-\lambda)f(a)+\lambda f(b)-f((1-\lambda)a+\lambda b) \tag{4.5}$$

for any $a, b \in \mathring{I}$ and for any $\lambda \in [0,1]$.

We get from (3.2) and (3.3) for any $x, a, b > 0$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$ that

$$\begin{aligned}\ln x &= (1-\lambda)\ln a+\lambda\ln b \\ &+ \sum_{k=1}^{2m+1}\frac{1}{k}\left[(-1)^{k-1}(1-\lambda)\left(\frac{x}{a}-1\right)^k-\lambda\left(1-\frac{x}{b}\right)^k\right] \\ &+ U_{2m+1,\lambda}(x,a,b),\end{aligned} \tag{4.6}$$

where the remainder $U_{2m+1,\lambda}(x,a,b)$ is given by

$$\begin{aligned}&U_{2m+1,\lambda}(x,a,b) \\ &:= -\left[(1-\lambda)(x-a)^{2m+2}\int_0^1\frac{(1-s)^{2m+1}}{((1-s)a+sx)^{2m+1}}ds\right. \\ &\left.+\lambda(b-x)^{2m+2}\int_0^1\frac{s^{2m+1}}{((1-s)x+sb)^{2m+1}}ds\right].\end{aligned} \tag{4.7}$$

If $x, a, b > 0$ and $\lambda \in [0,1]$, then $U_{2m+1,\lambda}(x,a,b) \leq 0$ and by (4.6) we get

$$\begin{aligned}\ln x &\leq (1-\lambda)\ln a+\lambda\ln b \\ &+ \sum_{k=1}^{2m+1}\frac{1}{k}\left[(-1)^{k-1}(1-\lambda)\left(\frac{x}{a}-1\right)^k-\lambda\left(1-\frac{x}{b}\right)^k\right].\end{aligned} \tag{4.8}$$

From (3.8) we have for any $a, b > 0$, $m \geq 0$ and $\lambda \in [0,1]$ that

$$\begin{aligned}0 &\leq \ln\left(\frac{A_\lambda(a,b)}{G_\lambda(a,b)}\right) \\ &\leq \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k}-\frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k,\end{aligned} \tag{4.9}$$

where $A_\lambda(a,b) := (1-\lambda)a + \lambda b$ is the weighted arithmetic mean and $G_\lambda(a,b) := a^{1-\lambda}b^\lambda$ is the weighted geometric mean. For $\lambda = \frac{1}{2}$ we recapture the arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$, respectively.

By taking the exponential in (4.9) we have

$$1 \le \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \le \exp\left[\lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k} - \frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k\right], \tag{4.10}$$

for any $a,\ b>0$, $m \ge 0$ and $\lambda \in [0,1]$.

In particular, we have

$$1 \le \frac{A(a,b)}{G(a,b)} \le \exp\left[\frac{1}{4}\sum_{k=1}^{2m+1}\frac{1}{2^{k-1}k}\left[\frac{(-1)^{k-1}b^k - a^k}{a^k b^k}\right](b-a)^k\right], \tag{4.11}$$

for any $a,\ b>0$ and $m \ge 0$.

If we take in (4.10) $m=0$, then we get

$$1 \le \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \le \exp\left[\lambda(1-\lambda)\frac{(b-a)^2}{ab}\right] \tag{4.12}$$

for any $a,\ b>0$ and $\lambda \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{4.13}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

Using Kantorovich's constant we can write the inequality (4.12) as

$$1 \le \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right] \tag{4.14}$$

for any $a,\ b>0$ and $\lambda \in [0,1]$. That has been obtained in [6].

In particular, we have [6]

$$1 \le \frac{A(a,b)}{G(a,b)} \le \exp\left(K\left(\frac{a}{b}\right)-1\right) \tag{4.15}$$

for any $a,\ b>0$.

Let $x,\ a,\ b>0$ and $m \ge 0$. Then we get from (3.13) and (3.14) for any $\lambda \in \mathbb{R}\setminus\{0,1\}$ that

$$\begin{aligned} x &= (1-\lambda)a + \lambda b \\ &+ \sum_{k=1}^{2m+1}\frac{1}{k!}\left[(1-\lambda)a(\ln x - \ln a)^k + (-1)^k\lambda b(\ln b - \ln x)^k\right] \\ &+ R_{2m+1,\lambda}(x,a,b), \end{aligned} \tag{4.16}$$

where the remainder $R_{2m+1,\lambda}(x,a,b)$ is given by

$$R_{2m+1,\lambda}(x,a,b) := \frac{1}{(2m+1)!}\left[(1-\lambda)(\ln x-\ln a)^{2m+2}\int_0^1 (1-s)^{2m+1}a^{1-s}x^s ds + \lambda(\ln b-\ln x)^{2m+2}\int_0^1 s^{2m+1}x^{1-s}b^s ds\right]. \tag{4.17}$$

If $x,\ a,\ b>0,\ m\geq 0$ and $\lambda\in[0,1]$, then $R_{2m+1,\lambda}(x,a,b)\geq 0$ and by (4.16) we have

$$x\geq(1-\lambda)a+\lambda b + \sum_{k=1}^{2m+1}\frac{1}{k!}\left[(1-\lambda)a(\ln x-\ln a)^k+(-1)^k\lambda b(\ln b-\ln x)^k\right]. \tag{4.18}$$

If $\lambda\in[0,1]$ and $a,\ b>0,\ m\geq 0$ then we have from (3.19) that

$$0\leq(1-\lambda)a+\lambda b-a^{1-\lambda}b^{\lambda} = \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k!}\left[(-1)^{k-1}(1-\lambda)^{k-1}b-\lambda^{k-1}a\right](\ln b-\ln a)^k - T_{2m+1,\lambda}(a,b), \tag{4.19}$$

where

$$T_{2m+1,\lambda}(a,b) := \frac{1}{n!}(\ln b-\ln a)^{2m+2}\left[(1-\lambda)^{2m+3}\int_0^1 a^{1-s+\lambda s}b^{(1-\lambda)s}(1-s)^{2m+1}ds + \lambda^{2m+3}\int_0^1 a^{(1-s)\lambda}b^{1-\lambda+\lambda s}s^{2m+1}ds\right]. \tag{4.20}$$

Since $T_{2m+1,\lambda}(a,b)\geq 0$ if $\lambda\in[0,1]$ and $a,\ b>0,\ m\geq 0$, then from (4.19) we get

$$0\leq A_{\lambda}(a,b)-G_{\lambda}(a,b) \leq \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k!}\left[(-1)^{k-1}(1-\lambda)^{k-1}b-\lambda^{k-1}a\right](\ln b-\ln a)^k. \tag{4.21}$$

In particular, we have for any $a,\ b>0$ and $m\geq 0$ that

$$0\leq A(a,b)-G(a,b)\leq\frac{1}{4}\sum_{k=1}^{2m+1}\frac{1}{2^{k-1}k!}\left[(-1)^{k-1}b-a\right](\ln b-\ln a)^k. \tag{4.22}$$

If we take $m=0$ in (4.21), then we get

$$0\leq A_{\lambda}(a,b)-G_{\lambda}(a,b)\leq\lambda(1-\lambda)(b-a)(\ln b-\ln a), \tag{4.23}$$

for any $a,\ b>0$ and $\lambda\in[0,1]$, that has been obtained in [6].

In particular, we have [6]

$$0\leq A(a,b)-G(a,b)\leq\frac{1}{4}(b-a)(\ln b-\ln a), \tag{4.24}$$

for any $a,\ b>0$.

For other recent inequalities between the weighted arithmetic mean and geometric mean see [6]-[10], [15]-[17], [20]-[21] and [23].

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