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FUNCTIONS GENERATING $(m, M, \Psi)-$SCHUR-CONVEX SUMS

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Abstract. The notion of $(m, M, \Psi)-$Schur-convexity is introduced and functions generating $(m, M, \Psi)-$Schur-convex sums are investigated. An extension of the Hardy-Littlewood-Pólya majorization theorem is obtained. A counterpart of the result of Ng stating that a function generates $(m, M, \Psi)-$Schur-convex sums if and only if it is $(m, M, \psi)-$Wright-convex is proved and a characterization of $(m, M, \psi)-$Wright-convex functions is given.

1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. Assume that D is a convex subset of X and c is a positive constant. A function $f : D \to \mathbb{R}$ is called:

– *strongly convex with modulus* c if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0,1]$;

– *strongly Wright-convex with modulus* c if

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) - 2ct(1-t)\|x-y\|^2 \tag{2}$$

for all $x, y \in D$ and $t \in [0,1]$;

– *strongly Jensen-convex with modulus* c if (1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \quad x, y \in D. \tag{3}$$

The usual concepts of convexity, Wright-convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions have been introduced by Polyak [21] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [10], [15], [19], [21], [22], [23], [26]). Let us mention also the paper [18] by the second author which is a survey article devoted to strongly convex functions and related classes of functions.

In [1] the first author introduced the following concepts of $(m, \psi)-$lower convex, $(M, \psi)-$upper convex and $(m, M, \psi)-$convex functions (see also [2], [3], [4]): Assume that D is a convex subset of a real linear space X, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called $(m, \psi)-$*lower convex* ($(M, \psi)-$*upper convex*) if the function $f - m\psi$ (the function $M\psi - f$) is convex. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)-$*convex* if it

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is $(m,\psi)-$lower convex and $(M,\psi)-$upper convex. Denote the above classes of functions by:

$$\begin{aligned}\mathcal{L}(D,m,\psi) &= \{f: D\to\mathbb{R} \mid f-m\psi \text{ is convex}\},\\ \mathcal{U}(D,M,\psi) &= \{f: D\to\mathbb{R} \mid M\psi-f \text{ is convex}\}\\ \mathcal{B}(D,m,M,\psi) &= \mathcal{L}(D,m,\psi)\cap\mathcal{U}(D,M,\psi).\end{aligned}$$

Let us observe that if $f\in\mathcal{B}(D,m,M,\psi)$ then $f-m\psi$ and $M\psi-f$ are convex and then $(M-m)\psi$ is also convex, implying that $M\geq m$ whenever ψ is not trivial (i.e. is not the zero function).

If $m>0$ and $(X,\|\cdot\|)$ is an inner product space (that is, the norm $\|\cdot\|$ in X is induced by an inner product: $\|x\|=\sqrt{\langle x,x\rangle}$) the notions of $m-\|\cdot\|^2-$lower convexity and strong convexity with modulus m coincide. Namely, in this case the following characterization was proved in [19]: A function f is strongly convex with modulus c if and only if $f-c\|\cdot\|^2$ is convex (for $X=\mathbb{R}^n$ this result can be also found in [8, Prop. 1.1.2]). However, if $(X,\|\cdot\|)$ is not an inner product space, then the two notions are different. There are functions $f\in\mathcal{L}(D,m,\|\cdot\|^2)$ which are not strongly convex with modulus m, as well as there are functions strongly convex with modulus m which do not belong to $\mathcal{L}(D,m,\|\cdot\|^2)$ (see the examples given in [6]).

If $M>0$ and $f\in\mathcal{U}(D,M,\psi)$, then f is a difference of two convex functions. Such functions are called *d.c. convex* or *δ-convex* and play an important role in convex analysis (cf. e.g. [25] and the reference therein). Functions from the class $\mathcal{U}(D,M,\|\cdot\|^2)$ with $M>0$ were also investigated in [13] under the name *approximately concave functions*.

In [5] Dragomir and Ionescu introduced the concept of g-convex dominated functions, where g is a given convex function. Namely, a function f is called *g-convex dominated*, if the functions $g+f$ and $g-f$ are convex. Note that this concept can be obtained as a particular case of $(m,M,\Psi)-$convexity by choosing $m=-1$, $M=1$ and $\psi=g$. Observe also (cf. [1]), that in the case where $I\subset\mathbb{R}$ is an open interval and $f,\psi: I\to\mathbb{R}$ are twice differentiable, then $f\in\mathcal{B}(I,m,M,\psi)$ if and only if

$$m\psi''(t)\leq f''(t)\leq M\psi''(t),\quad \text{for all } \ t\in I.$$

In particular, if $I\subset(0,\infty)$, $f: I\to\mathbb{R}$ is twice differentiable and $\psi(t)=-\ln t$, then $f\in\mathcal{B}(I,m,M,-\ln)$ if and only if

$$m\leq t^2f''(t)\leq M,\quad \text{for all } \ t\in I, \tag{4}$$

which is a convenient condition to verify in applications.

Let $I\subset\mathbb{R}$ be an interval and $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in I^n$, where $n\geq 2$. Following I. Schur (cf. e.g. [24], [12]) we say that x *is majorized by* y, and write $x\preceq y$, if there exists a doubly stochastic $n\times n$ matrix P (i.e. matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x=y\cdot P$. A function $F: I^n\to\mathbb{R}$ is said to be *Schur-convex* if $F(x)\leq F(y)$ whenever $x\preceq y$, $x,y\in I^n$.

It is known, by the classical works of Schur [24], Hardy–Littlewood–Pólya [7] and Karamata [9] that if a function $f: I\to\mathbb{R}$ is convex then it *generates Schur-convex sums*, that is the function $F: I^n\to\mathbb{R}$ defined by

$$F(x)=F(x_1,\ldots,x_n)=f(x_1)+\cdots+f(x_n)$$

is Schur-convex. It is also known that the convexity of f is a sufficient but not necessary condition under which F is Schur-convex. A full characterization of functions generating Schur-convex sums was given by C. T. Ng [16]. Namely, he proved that a function $f : I \to \mathbb{R}$ generates Schur-convex sums if and only if it is Wright-convex. Recently Nikodem, Rajba and Wąsowicz [20] obtained similar results connectet with strong convexity in inner product spaces.

The aim of this paper is to present some generalizations and counterparts of the mentioned above results related to $(m, \psi)-$lower convexity, $(M, \psi)-$upper convexity and $(m, M, \psi)-$convexity. We introduce the notion of $(m, M, \Psi)-$Schur-convex functions and give a sufficient and necessary condition for a function f to generate $(m, M, \Psi)-$Schur-convex sums. As a corollary we obtain a counterpart of the classical Hardy-Littlewood-Pólya majorization theorem. Finally we introduce the concept of $(m, M, \psi)-$Wright-convex functions, prove a representation theorem for them and present an Ng-type characterization of functions generating $(m, M, \Psi)-$Schur-convex sums. It is worth to underline, that our results concern a few different classes of functions related to convexity and are formulated in vector spaces, that is in much more general setting than the original ones.

2. Main results

Let X be a real vector space. Similarly as in the classical case we define the majorization in the product space X^n. Namely, given two n–tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X^n$ we say that x *is majorized by* y, written $x \preceq y$, if

$$(x_1, \dots, x_n) = (y_1, \dots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix P.

In what follows we will assume that D is a convex subset of a real vector space X, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. For any $n \geq 2$ define $\Psi_n : D^n \to \mathbb{R}$ by $\Psi_n(x_1, \dots, x_n) = \psi(x_1) + \cdots + \psi(x_n)$, $x_1, \dots, x_n \in D$. We say that a function $F : D^n \to \mathbb{R}$ is $(m, M, \Psi_n)-$*Schur-convex* if for all $x, y \in D$

$$x \preceq y \implies F(x) \leq F(y) - m\big(\Psi_n(y) - \Psi_n(x)\big) \tag{5}$$

and

$$x \preceq y \implies F(x) \geq F(y) - M\big(\Psi_n(y) - \Psi_n(x)\big). \tag{6}$$

In only condition (5) (condition (6)) is satisfied, we say that F is $(m, \Psi_n)-$lower ($(M, \Psi_n)-$upper) Schur-convex.

Note that if $x \preceq y$ then $\Psi_n(x) \leq \Psi_n(y)$. It follows from the fact that the function ψ is convex and so it generates Schur-convex sums Ψ_n.

Given a function $f : D \to \mathbb{R}$ and an integer $n \geq 2$ we define the function $F_n : D^n \to \mathbb{R}$ by

$$F_n(x_1, \dots, x_n) = f(x_1) + \cdots + f(x_n), \quad x_1, \dots, x_n \in D. \tag{7}$$

Now, we will prove that $(m, M, \psi)-$convex functions generate $(m, M, \Psi_n)-$Schur-convex sums.

Theorem 1. *(i) If $f \in \mathcal{L}(D, m, \psi)$, then the function F_n defined by* (7) *is $(m, \Psi_n)-$lower Schur-convex;*

(ii) If $f \in \mathcal{U}(D, M, \psi)$, then the function F_n defined by (7) is $(M, \Psi_n)-$upper Schur-convex;

(iii) If $f \in \mathcal{B}(D, m, M, \psi)$, then the function F_n defined by (7) is $(m, M, \Psi_n)-$Schur-convex.

Proof. To prove (i) fix $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in D^n with $x \preceq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \ldots, n.$$

Since $f \in \mathcal{L}(D, m, \psi)$, the function $g = f - m\psi$ is convex and hence

$$\begin{aligned} g(x_1) + \cdots + g(x_n) &= \sum_{j=1}^{n} g\Big(\sum_{i=1}^{n} t_{ij} y_i\Big) \le \sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} g(y_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} g(y_i) = \sum_{i=1}^{n} g(y_i) \sum_{j=1}^{n} t_{ij} = g(y_1) + \cdots + g(y_n). \end{aligned}$$

Consequently,

$$\begin{aligned} F_n(x) &= f(x_1) + \cdots + f(x_n) \\ &= g(x_1) + \cdots + g(x_n) + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\ &\le g(y_1) + \cdots + g(y_n) + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\ &= f(y_1) + \cdots + f(y_n) - m\big(\psi(y_1) + \cdots + \psi(y_n)\big) + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\ &= F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big). \end{aligned}$$

This shows that F_n satisfies (5), i.e. it is $(m, \Psi_n)-$lower Schur-convex.

The proof of part (ii) is similar. Since $f \in \mathcal{U}(D, M, \psi)$, the function $h = M\psi - f$ is convex. Hence, for x and y as previously, we have

$$\begin{aligned} F_n(x) &= f(x_1) + \cdots + f(x_n) \\ &= +M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - h(x_1) - \cdots - h(x_n) \\ &\ge M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - h(y_1) - \cdots - h(y_n) \\ &= M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - M\big(\psi(y_1) + \cdots + \psi(y_n)\big) + f(y_1) + \cdots + f(y_n) \\ &= F_n(y) - M\big(\Psi_n(y) - \Psi_n(x)\big). \end{aligned}$$

Part (iii) follows from (i) and (ii). □

As an immediate consequence of the abowe theorem, we obtain the following counterpart of the classical Hardy-Littlewood-Pólya majorization theorem [7].

Corollary 2. *Let $I \subset \mathbb{R}$ be an interval and $n \ge 2$. Assume that $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in I^n$ satisfy:*

(a) $x_1 \le \cdots \le x_n$, $y_1 \le \cdots \le y_n$;

(b) $y_1 + \cdots + y_k \le x_1 + \cdots + x_k$, $k = 1, ..., n-1$;

(c) $y_1 + \cdots + y_n = x_1 + \cdots + x_n$.

Assume also that $f, \psi : I \to \mathbb{R}$ and ψ is convex.

(i) If $f \in \mathcal{L}(D, m, \psi)$, *then*

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n) - m\big(\Psi_n(y) - \Psi_n(x)\big);$$

(ii) If $f \in \mathcal{U}(D, M, \psi)$, *then*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n) - M\big(\Psi_n(y) - \Psi_n(x)\big);$$

(iii) If $f \in \mathcal{B}(D, m, M, \psi)$, *then*

$$\begin{aligned} f(y_1) + \cdots + f(y_n) - M\big(\Psi_n(y) - \Psi_n(x)\big) &\leq f(x_1) + \cdots + f(x_n) \\ &\leq f(y_1) + \cdots + f(y_n) - m\big(\Psi_n(y) - \Psi_n(x)\big). \end{aligned}$$

Proof. Note that assumptions (a)-(c) imply $x \preceq y$ (see e.g. [12]) and apply Theorem 1. □

Remark 3. Specifying the functions ψ and f in the above Corollary 2, one can get various analytic inequalities. For example, if $I \subset (0, \infty)$ and $f \in \mathcal{B}(I, m, M, -\ln)$, then for all $(x_1, ..., x_n), (y_1, ..., y_n) \in I^n$ satisfying conditions (a)-(c), we get

$$m \ln \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right) \leq \sum_{i=1}^{n} f(y_i) - \sum_{i=1}^{n} f(x_i) \leq M \ln \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right),$$

or, equivalently,

$$\prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^m \leq \frac{\exp\left[\sum_{i=1}^{n} f(y_i)\right]}{\exp\left[\sum_{i=1}^{n} f(x_i)\right]} \leq \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^M. \tag{8}$$

If we take, for instance, $I = [k, K] \subset (0, \infty)$ and $f(t) = \frac{1}{p(p-1)} t^p$, with $p > 0$, $p \neq 1$, then $t^2 f''(t) = t^p \in [k^p, K^p]$, which means (cf. (4)) that $f \in \mathcal{B}(I, k^p, K^p, -\ln)$. Therefore, by (8), we then have

$$\prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{p(p-1)k^p} \leq \frac{\exp\left(\sum_{i=1}^{n} y_i^p\right)}{\exp\left(\sum_{i=1}^{n} x_i^p\right)} \leq \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{p(p-1)K^p}.$$

One can give other examples by choosing $f(t) = t^q$ with $q < 0$, $f(t) = t \ln t$, etc.

We say that a function $f : D \to \mathbb{R}$ is $(m, \psi)-$*lower Jensen-convex* ($(M, \psi)-$*upper Jensen-convex*) if the function $f - m\psi$ (the function $M\psi - f$) is Jensen-convex, i.e. satisfies (3) with $c = 0$. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)-$*Jensen-convex* if it is $(m, \psi)-$lower Jensen-convex and $(M, \psi)-$upper Jensen-convex.

In the next theorem we show that functions generating $(m, M, \Psi_n)-$Schur-convex sums must be $(m, M, \psi)-$Jensen–convex.

Theorem 4. *Let* $f : D \to \mathbb{R}$.

(i) If for some $n \geq 2$ *the function* F_n *defined by* (7) *is* $(m, \Psi_n)-$*lower Schur-convex, then* f *is* $(m, \psi)-$*lower Jensen-convex;*

(ii) If for some $n \geq 2$ *the function* F_n *defined by* (7) *is* $(M, \Psi_n)-$*upper Schur-convex, then* f *is* $(M, \psi)-$*upper Jensen-convex;*

(iii) If for some $n \geq 2$ *the function* F_n *defined by* (7) *is* $(m, M, \Psi_n)-$ *Schur-convex, then* f *is* $(m, M, \psi)-$ *Jensen-convex.*

Proof. To prove (i) take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \dots, y_2), \quad x = (x_1, x_2, y_2, \dots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2)$, $x = (x_1, x_2)$). One can check easily that $x \preceq y$. Therefore, by (5),

$$F_n(x) \le F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is

$$2f\Big(\frac{y_1+y_2}{2}\Big) \le f(y_1) + f(y_2) - m\Big(\psi(y_1) + \psi(y_2) - 2\psi\big(\frac{y_1+y_2}{2}\big)\Big).$$

Hence, for $g = f - m\psi$ we have

$$\begin{aligned} 2g\Big(\frac{y_1+y_2}{2}\Big) &= 2f\Big(\frac{y_1+y_2}{2}\Big) - 2m\psi\Big(\frac{y_1+y_2}{2}\Big) \\ &\le f(y_1) + f(y_2) - m\big((\psi(y_1) + \psi(y_2)\big) = g(y_1) + g(y_2) \end{aligned}$$

which means that f is $(m, \psi)-$lower Jensen-convex.
The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). □

We say that a function $f : D \to \mathbb{R}$ is $(m, \psi)-$*lower Wright-convex* ($(M, \psi)-$*upper Wright-convex*) if the function $f - m\psi$ (the function $M\psi - f$) is Wright-convex, i.e. satisfies (2) with $c = 0$. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)-$*Wright-convex* if it is $(m, \psi)-$lower Wright-convex and $(M, \psi)-$upper Wright-convex.

As was shown above in Theorem 1 and Theorem 2, if a function $f : D \to \mathbb{R}$ is $(m, M, \psi)-$convex, then for every $n \ge 2$ the corresponding function F_n defined by (7) is $(m, M, \Psi_n)-$Schur-convex and if for some $n \ge 2$ the function F_n is $(m, M, \Psi_n)-$Schur-convex, then f is $(m, M, \psi)-$Jensen-convex. The next theorem characterizes all the functions f for which F_n are $(m, M, \Psi_n)-$ Schur–convex. It is a counterpart of the result of Ng [16] on functions generating Schur–convex sums.

Recall also that a subset D of a vector space X is said to be algebraically open if for every $x \in D$ and for every $y \in X$ there exists $\varepsilon > 0$ such that

$$\{ty + (1-t)x \mid t \in (-\varepsilon, \varepsilon)\} \subset D.$$

Theorem 5. *Let $f : D \to \mathbb{R}$, where D is an algebraically open convex subset of a vector space X. Then:*

(i) If f is $(m, \psi)-$lower Wright-convex, then for every $n \ge 2$ the function F_n defined by (7) is $(m, \Psi_n)-$lower Schur-convex. Conversely, if for some $n \ge 2$ the function F_n is $(m, \Psi_n)-$lower Schur-convex, then f is $(m, \psi)-$lower Wright-convex;

(ii) If f is $(M, \psi)-$upper Wright-convex, then for every $n \ge 2$ the function F_n defined by (7) is $(M, \Psi_n)-$upper Schur-convex. Conversely, if for some $n \ge 2$ the function F_n is $(M, \Psi_n)-$upper Schur-convex, then f is $(M, \psi)-$upper Wright-convex;

(iii) If f is $(m, M, \psi)-$ Wright-convex, then for every $n \ge 2$ the function F_n defined by (7) is $(m, M, \Psi_n)-$ Schur-convex. Conversely, if for some $n \ge 2$ the function F_n is $(m, M, \Psi_n)-$ Schur-convex, then f is $(m, M, \psi)-$ Wright-convex.

Proof. To prove (i) assume that f is $(m, \psi)-$lower Wright-convex and fix an $n \ge 2$. Since the function $g = f - m\psi$ is Wright-convex, it is of the form $g = g_1 + a$, where g_1 is convex and a is additive (cf. [11]; here the assumption that D is algebraically open is needed).

Therefore it generates Schur-convex sums. Thus, for $x = (x_1, \dots, x_n) \preceq y = (y_1, \dots, y_n)$, we have

$$g(x_1) + \cdots + g(x_n) \le g(y_1) + \cdots + g(y_n).$$

Hence

$$f(x_1)+\cdots+f(x_n)-m\big(\psi(x_1)+\cdots+\psi(x_n)\big) \le g(y_1)+\cdots+g(y_n)-m\big(\psi(y_1)+\cdots+\psi(y_n)\big),$$

which means that

$$F_n(x) \le F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is F_n is $(m, \Psi_n)-$lower Schur-convex. Now, assume that for some $n \ge 2$ the function F_n is $(m, \Psi_n)-$lower Schur-convex. Take $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \dots, n$. Then $x = (x_1, \dots, x_n) \preceq y = (y_1, \dots, y_n)$. Therefore, by (5),

$$F_n(x) \le F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is

$$f(ty_1+(1-t)y_2)+f((1-t)y_1+ty_2) \le f(y_1)+f(y_2)-m\big(\psi(y_1)+\psi(y_2)-\psi(x_1)-\psi(x_2)\big).$$

Hence, for $g = f - m\psi$ we get

$$\begin{aligned} & g(ty_1 + (1-t)y_2) + g((1-t)y_1 + ty_2) \\ = \;& f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) - m\psi(ty_1 + (1-t)y_2) - m\psi((1-t)y_1 + ty_2) \\ \le \;& f(y_1) + f(y_2) - m\psi(y_1) - m\psi(y_2) = g(y_1) + g(y_2). \end{aligned}$$

Thus g is Wright-convex, which means that f is $(m, \psi)-$lower Wright-convex.
The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). □

Remark 6. In the special case where $(X, \|\cdot\|)$ is an inner product space, $\psi = \|\cdot\|^2$ and $m = c > 0$, the parts (i) of the above Theorems 1, 4, 5 reduces to the results obtained in [20] for strong Schur-convexity. For $m = 0$ and $X = \mathbb{R}^n$ they coincide with the Ng theorem [16].

Finally, we give a representation theorem for $(m, M, \psi)-$Wright-convex functions. It is known (and easy to check) that every convex function is Wright-convex, and every Wright-convex function is Jensen-convex, but not the converse (some examples can be found in [18]). In [16] Ng proved that a function f defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = f_1 + a$, where f_1 is a convex function and a is an additive function (see also [18]). Kominek [11] extended that result to functions defined on algebraically open subset of a vector space. Analogous result for strongly Wright-convex functions was obtained in [14]. In the next theorem we give a similar representation for $(m, M, \psi)-$Wright-convex functions. In the proof we will use the following fact:

Lemma 7. *Assume that $f, g : D \to \mathbb{R}$ are convex functions, $a : X \to \mathbb{R}$ is additive and $a(x) = f(x) - g(x)$ for all $x \in D$. Then a is an affine function on D.*

Proof. Fix $x, y \in D$ and consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(s) = a(sx + (1-s)y) = f(sx + (1-s)y) - g(sx + (1-s)y), \; s \in [0,1].$$

As a difference of convex functions on $[0,1]$, φ is continuous on $(0,1)$. Fix any $t \in (0,1)$ and take a sequence (q_n) on rational numbers in $(0,1)$ tending to t. By the additivity of a we have

$$a(q_n x + (1-q_n)y) = q_n a(x) + (1-q_n)a(y),$$

whence

$$\varphi(q_n) = q_n a(x) + (1-q_n)a(y).$$

Going to the limit we get

$$\varphi(t) = ta(x) + (1-t)a(y).$$

Hence

$$a(tx + (1-t)y) = ta(x) + (1-t)a(y),$$

which proves that a is affine on D.

□

Theorem 8. *Let $f : D \to \mathbb{R}$, where D is an algebraically open convex subset of a vector space X. Then:*

(i) f is $(m,\psi)-$lower Wright-convex if and only if $f = g_1 + a_1$, where $g_1 \in \mathcal{L}(D,m,\psi)$ and $a_1 : X \to \mathbb{R}$ is additive;

(ii) f is $(M,\psi)-$upper Wright-convex if and only if $f = g_2 + a_2$, where $g_2 \in \mathcal{U}(D,M,\psi)$ and $a_2 : X \to \mathbb{R}$ is additive;

(iii) f is $(m,M,\psi)-$ Wright-convex if and only if $f = g + a$, where $g \in \mathcal{B}(D,m,M,\psi)$ and $a : X \to \mathbb{R}$ is additive.

Proof. To prove (i) assume first that f is $(m,\psi)-$lower Wright-convex, that is $h = f - m\psi$ is Wright-convex. By the Ng representation theorem [16] (extended by Kominek [11] to functions defined on algebraically open domains), there exist a convex function $h_1 : D \to \mathbb{R}$ and an additive function $a_1 : X \to \mathbb{R}$ such that $h = h_1 + a_1$ on D. Then $g_1 = h_1 + m\psi$ belongs to $\mathcal{L}(D,m,\psi)$ and

$$f = h + m\psi = h_1 + a_1 + m\psi = g_1 + a_1,$$

which was to be proved. Conversely, if $f = g_1 + a_1$ with some $g_1 \in \mathcal{L}(D,m,\psi)$ and a_1 additive, then $f - m\psi = g_1 - m\psi + a_1$ is Wright-convex as a sum of a convex function and an additive function. This shows that f is $(m,\psi)-$lower Wright-convex.
The proof of part (ii) is analogous.

Part (iii). If $f = g + a$, where $g \in \mathcal{B}(D,m,M,\psi)$ and $a : X \to \mathbb{R}$ is additive, then, by (i) and (ii) f is $(m,\psi)-$lower Wright-convex and $(M,\psi)-$upper Wright-convex. Consequently, it is $(m,M,\psi)-$Wright-convex.

The proof in the opposite direction is more delicate. If f is $(m,M,\psi)-$Wright-convex, then $f - m\psi$ and $M\psi - f$ are Wright-convex. Then

$$f - m\psi = h_1 + a_1 \quad \text{and} \quad M\psi - f = h_2 + a_2$$

with some convex functions h_1, h_2 and additive functions a_1, a_2. Hence

$$a_1 + a_2 = (M - m)\psi - (h_1 + h_2)$$

which, by Lemma 5, implies that $A = a_1 + a_2$ is affine. Denote $a = a_1$ and $g = f - a$. Then

$$g - m\psi = f - a - m\psi = h_1,$$

which implies that $g \in \mathcal{L}(D, m, \psi)$ because h_1 is convex. Also

$$M\psi - g = M\psi - f + a = h_2 + a_2 + a = h_2 + A,$$

which implies that $g \in \mathcal{U}(D, m, \psi)$ because $h_2 + A$ is convex. Thus $g \in \mathcal{B}(D, m, \psi)$ and $f = g + a$, which finishes the proof.

□

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