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JENSEN'S AND HERMITE-HADAMARD'S TYPE INEQUALITIES FOR LOWER AND STRONGLY CONVEX FUNCTIONS ON NORMED SPACES

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Abstract. In this paper we obtain some Jensen's and Hermite-Hadamard's type inequalities for lower, upper and strongly convex functions defined on convex subsets in normed linear spaces. The case of inner product space is of interest since in these case the concepts of lower convexity and strong convexity coincides. Applications for univariate functions of real variable and the connections with earlier Hermite-Hadamard's type inequalities are also provided.

1. Introduction

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy the properties:

(i) $g'_{\pm}(a,b)(s) = (\triangledown_{\pm} f[(1-s)a + sb])(b-a)$, $s \in (0,1)$;
(ii) $g'_{+}(a,b)(0) = (\triangledown_{+} f(a))(b-a)$;
(iii) $g'_{-}(a,b)(1) = (\triangledown_{-} f(b))(b-a)$;

where $(\triangledown_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives.* Recall that

$$(\triangledown_{+} f(x))(y) := \lim_{h\to 0+}\left[\frac{f(x+hy) - f(x)}{h}\right],$$
$$(\triangledown_{-} f(x))(y) := \lim_{k\to 0-}\left[\frac{f(x+ky) - f(x)}{k}\right],\quad x,\ y \in X.$$

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := (\triangledown_{+} f_0(y))(x) = \lim_{t\to 0+}\left[\frac{\|y+tx\|^2 - \|y\|^2}{2t}\right]$;
(v) $\langle x, y\rangle_i := (\triangledown_{-} f_0(y))(x) = \lim_{s\to 0-}\left[\frac{\|y+sx\|^2 - \|y\|^2}{2s}\right]$;

for any $x,\ y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [3]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

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(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x,y\rangle_p$ if $\alpha, \beta \geq 0$ and $x,\ y \in X$;
(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\| \|y\|$ for all $x,\ y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x,x\rangle_p + \langle y,x\rangle_p$ if $x,\ y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all $x,\ y \in X$;
(va) $\langle x+y,z\rangle_p \leq \|x\| \|z\| + \langle y,z\rangle_p$ for all $x,\ y,\ z \in X$;
(vaa) The mapping $\langle \cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all $x, y \in X$.

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left[(1-t)a + tb\right] dt \leq \frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$. For other related results see the monograph on line [11].

Applying inequality (HH) for the convex function $f_0(x) = \|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x + ty\|^2 dt \leq \frac{\|x\|^2 + \|y\|^2}{2} \tag{1.1}$$

for any $x,\ y \in X$. The same (HH) inequality applied for $f_1(x) = \|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}, \quad x,\ y \in X. \tag{1.2}$$

The distance between the first and second term in (1.1) has the lower and upper bounds [6]

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i\right] \\ &\leq \int_0^1 \|(1-t)x + ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{4}\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right] \end{aligned} \tag{1.3}$$

while the distance between the second and third term in (1.1) has the same upper and lower bounds, namely [7]

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\langle y-x, y+x\rangle_s - \langle y-x, y+x\rangle_i\right] \\ &\leq \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \leq \frac{1}{4}\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right] \end{aligned} \tag{1.4}$$

for any $x,\ y \in X$. The multiplicative constants $\frac{1}{8}$ and $\frac{1}{4}$ are best possible in (1.3) and (1.4).

2. Some Jensen's Type Inequalities

Let $(X, \|\cdot\|)$ be a real or complex normed linear space, $C \subseteq X$ a convex subset of X and $f : C \to \mathbb{R}$. Let $m,\ M \in \mathbb{R}$. The mapping f will be called *m-lower convex* on C if $f - \frac{m}{2}\|\cdot\|^2$ is a convex mapping on C. The mapping f will be called *M-upper convex* on C if $\frac{M}{2}\|\cdot\|^2 - f$ is a convex mapping on C. The mapping f will be called *(m, M)- convex* on C if it is both m-lower convex and M-upper convex on C. Note that if f is (m, M)-convex on C, then $m \le M$.

Further, assume that c is a positive constant. A function $f : C \to \mathbb{R}$ is called: *strongly convex with modulus c* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in C$ and $t \in [0, 1]$. Also, it is called: *strongly Jensen-convex with modulus c* if (2.1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \text{ for all } x,\ y \in C. \tag{2.2}$$

The usual concepts of convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions have been introduced by Polyak [18] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [2], [15], [17], [18], [19], [20], [21]). Let us mention also the paper [16] by the second author which is a survey article devoted to strongly convex functions and related classes of functions.

Denote by $\mathcal{SC}(C, c)$ the class of all functions $f : C \to \mathbb{R}$ strongly convex with modulus c and by $\mathcal{LC}(C, m)$ the class of all functions $f : C \to \mathbb{R}$ m-lower convex. It is known that [17], if X is an inner product space then

$$\mathcal{SC}(C, \frac{m}{2}) = \mathcal{LC}(C, m).$$

However, in arbitrary normed spaces the above classes differ in general.

The following examples shows that neither $\mathcal{LC}(C, m)$ is included in $\mathcal{SC}(C, \frac{m}{2})$, nor conversely.

Example 1 ([17])**.** *Let $X = \mathbb{R}^2$ and $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$. Take $f = \|\cdot\|^2$. Then $g = f - \|\cdot\|^2$ is convex being the zero function. However, f is not strongly convex with modulus 1. Indeed, for $x = (1, 0)$ and $y = (0, 1)$ we have*

$$f\left(\frac{x+y}{2}\right) = 1 > 0 = \frac{f(x)+f(y)}{2} - \frac{1}{4}\|x-y\|^2,$$

which contradicts (2.1).

Example 2. *Let $X = \mathbb{R}^2$ and $\|x\| = |x_1| + |x_2|$, for $x = (x_1, x_2)$. Take $f(x) = x_1^2 + x_2^2$. Then f is strongly convex with modulus $c = \frac{1}{2}$. Indeed, for arbitrary $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ we have*

$$f\left(\frac{x+y}{2}\right) = \frac{1}{4}\left(x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2\right)$$

and

$$\begin{aligned}&\frac{f(x)+f(y)}{2}-\frac{1}{2}\cdot\frac{1}{4}\|x-y\|^2\\&=\frac{3}{8}\left(x_1^2+y_1^2+x_2^2+y_2^2\right)+\frac{1}{4}\left(x_1y_1+x_2y_2-|x_1-y_1||x_2-y_2|\right).\end{aligned}$$

Hence

$$\frac{f(x)+f(y)}{2}-\frac{1}{2}\cdot\frac{1}{4}\|x-y\|^2-f\left(\frac{x+y}{2}\right)=\frac{1}{8}\left(|x_1-y_1|-|x_2-y_2|\right)^2\geq 0.$$

This shows that f is strongly midconvex with modulus $c=\frac{1}{2}$. Since f is continuous, it is also strongly convex with modulus $c=\frac{1}{2}$. On the other hand, the function $g=f-\frac{1}{2}\|\cdot\|^2$ is not convex. Indeed, for $x=(-1,1)$ and $y=(1,1)$ we have

$$g\left(\frac{x+y}{2}\right)=\frac{1}{2}>0=\frac{g(x)+g(y)}{2}.$$

The following Jensen's type inequality holds [4].

Proposition 1. *Let $f:C\subseteq X\to\mathbb{R}$, C be convex on X, $x_i\in C$, $p_i\geq 0$ $(i=1,...,n)$ with $\sum_{i=1}^{n}p_i=1$.*

(i) *If f is m-lower convex on C, then we have the following inequality (for $m\geq 0$ - refinement of Jensen's inequality)*

$$\frac{m}{2}\left[\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right]\leq\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right).\tag{2.3}$$

(ii) *If f is M-upper convex on C, then we have the following inequality (which is a counterpart of Jensen's inequality if f is convex)*

$$\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\leq\frac{M}{2}\left[\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right].\tag{2.4}$$

(iii) *If f is (m,M)-convex on C, then we have the following sandwich inequality*

$$\begin{aligned}\frac{m}{2}\left[\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right]&\leq\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right)\\&\leq\frac{M}{2}\left[\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right].\end{aligned}\tag{2.5}$$

The following corollary for inner product spaces holds.

Corollary 1. *Let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space, $\|\cdot\|:=\sqrt{\langle\cdot,\cdot\rangle}$, $C\subseteq X$ a convex subset on X, $f:C\to\mathbb{R}$ and $x_i\in C$, $p_i\geq 0$ $(i=1,...,n)$ with $\sum_{i=1}^{n}p_i=1$.*

(i) *If f is m-lower convex on C, then*

$$\frac{m}{2}\sum_{1\leq i<j\leq n}p_ip_j\|x_i-x_j\|^2\leq\sum_{i=1}^{n}p_if(x_i)-f\left(\sum_{i=1}^{n}p_ix_i\right).\tag{2.6}$$

(*ii*) *If f is M-upper convex on C, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \frac{M}{2} \sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\|^2 . \tag{2.7}$$

(*iii*) *If f is (m, M)-convex on C, then*

$$\begin{aligned} \frac{m}{2} \sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\|^2 &\le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\le \frac{M}{2} \sum_{1\le i<j\le n} p_i p_j \|x_i - x_j\|^2 . \end{aligned} \tag{2.8}$$

The case of the mappings defined on real intervals have been obtained by Andrica and Raşa in [1].

Furthermore, let us assume that $\Delta(x) := \max_{1\le i<j\le n} \|x_i - x_j\|$ and $\delta(x) := \min_{1\le i<j\le n} \|x_i - x_j\|$. The following corollary also holds.

Corollary 2. *Let X, C, f, x_i, p_i $(i = 1, ..., n)$ be as in Corollary 1.*

(*i*) *If f is m-lower convex on C with $m > 0$, then we have the following refinement of Jensen's inequality:*

$$0 < \frac{m}{4}\left(1 - \sum_{i=1}^{n} p_i^2\right)(\delta(x))^2 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right). \tag{2.9}$$

(*ii*) *If f is convex and M-upper convex on C, then we have the following converse inequality:*

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \frac{M}{4}\left(1 - \sum_{i=1}^{n} p_i^2\right)(\Delta(x))^2 . \tag{2.10}$$

(*iii*) *If f is (m, M)-convex on C with $m > 0$, then we have the following estimate*

$$\begin{aligned} \frac{m}{4}\left(1 - \sum_{i=1}^{n} p_i^2\right)(\delta(x))^2 &\le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\le \frac{M}{4}\left(1 - \sum_{i=1}^{n} p_i^2\right)(\Delta(x))^2 . \end{aligned} \tag{2.11}$$

We can state the following Jensen's type inequality for strongly convex function with modulus c.

Theorem 1. *Let $f : C \subseteq X \to \mathbb{R}$ be a function strongly convex with modulus c on C that is open and convex in X, the normed linear space $(X, \|\cdot\|)$, $x_i \in C$, $p_i \ge 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$ and $\overline{x}_p := \sum_{i=1}^{n} p_i x_i \in C$. Then we have the following refinement of Jensen's inequality*

$$\sum_{i=1}^{n} p_i f(x_i) - f(\overline{x}_p) \ge c \sum_{i=1}^{n} p_i \|x_i - \overline{x}_p\|^2 . \tag{2.12}$$

If $y \in C$ is such that

$$\sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(y) \ge \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(x_i), \tag{2.13}$$

where $(\triangledown_+ f(\cdot))(\cdot)$ is Gâteaux lateral derivative of f, then the following refinement of Slater's inequality holds

$$f(y) - c\sum_{i=1}^n p_i \|x_i - y\|^2 \geq \sum_{i=1}^n p_i f(x_i). \tag{2.14}$$

We have the following reverse of Jensen's inequality as well

$$\begin{aligned}\sum_{i=1}^n p_i (\triangledown_+ f(x_i))(x_i) - \sum_{i=1}^n p_i (\triangledown_+ f(x_i))(\overline{x}_p) - \sum_{i=1}^n p_i \|x_i - \overline{x}_p\|^2 \\ \geq \sum_{i=1}^n p_i f(x_i) - f(\overline{x}_p).\end{aligned} \tag{2.15}$$

Proof. By the definition of c-strongly convex function on C, we have

$$t(f(y) - f(x)) \geq f((1-t)x + ty) - f(x) + ct(1-t)\|x-y\|^2$$

for any x, $y \in C$ and $t \in [0,1]$. This implies that

$$f(y) - f(x) \geq \frac{f(x + t(y-x)) - f(x)}{t} + c(1-t)\|x-y\|^2$$

for $t \in (0,1)$.

Since f is convex on open convex subset C, then the lateral derivative $(\triangledown_+ f(x))(y-x)$ exists for any x, $y \in C$ and by taking the limit over $t \to 0+$ we get the gradient inequality

$$f(y) - f(x) \geq (\triangledown_+ f(x))(y-x) + c\|y-x\|^2 \tag{2.16}$$

for any x, $y \in C$.

If we take in (2.16) $y = x_i$, $i \in \{1,...,n\}$ and $x = \overline{x}_p$, then we get

$$f(x_i) - f(\overline{x}_p) \geq (\triangledown_+ f(\overline{x}_p))(x_i - \overline{x}_p) + c\|x_i - \overline{x}_p\|^2 \tag{2.17}$$

for any $i \in \{1,...,n\}$.

Multiply (2.17) by $p_i \geq 0$, $i \in \{1,...,n\}$ and sum over i from 1 to n to get

$$\sum_{i=1}^n p_i f(x_i) - f(\overline{x}_p) \geq \sum_{i=1}^n p_i (\triangledown_+ f(\overline{x}_p))(x_i - \overline{x}_p) + c\sum_{i=1}^n p_i \|x_i - \overline{x}_p\|^2. \tag{2.18}$$

This is an inequality of interest in itself.

Since $(\triangledown_+ f(\overline{x}_p))(\cdot)$ is a subadditive and positive homogeneous functional on X we have

$$\begin{aligned}\sum_{i=1}^n p_i (\triangledown_+ f(\overline{x}_p))(x_i - \overline{x}_p) &\geq (\triangledown_+ f(\overline{x}_p))\left(\sum_{i=1}^n p_i x_i - \overline{x}_p\right) \\ &= (\triangledown_+ f(\overline{x}_p))(0) = 0\end{aligned}$$

and by the inequality (2.18) we get the desired result (2.12).

From (2.16) we have for any x_i, $y \in C$, $i \in \{1,...,n\}$, that

$$f(y) \geq f(x_i) + (\triangledown_+ f(x_i))(y - x_i) + c\|x_i - y\|^2$$

for any $i \in \{1,...,n\}$. If we multiply this inequality by $p_i \geq 0$, $i \in \{1,...,n\}$ and sum over i from 1 to n, then we get

$$f(y) \geq \sum_{i=1}^n p_i f(x_i) + \sum_{i=1}^n p_i (\triangledown_+ f(x_i))(y - x_i) + c\sum_{i=1}^n p_i \|x_i - y\|^2. \tag{2.19}$$

By the subadditivity of $(\nabla_+ f(x_i))(\cdot)$ we have

$$(\nabla_+ f(x_i))(y - x_i) \geq (\nabla_+ f(x_i))(y) - (\nabla_+ f(x_i))(x_i),$$

which implies that

$$\sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(y - x_i) \geq \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(y) - \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(x_i),$$

and by (2.19) we get

$$f(y) \geq \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(y) - \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(x_i) + c \sum_{i=1}^{n} p_i \|x_i - y\|^2, \tag{2.20}$$

for any x_i, $y \in C$, $i \in \{1, ..., n\}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. This is an inequality of interest in itself.

If the condition (2.13) is valid for some $y \in C$, then by (2.20) we get the desired result (2.14).

Now, if we take in (2.20) $y = \overline{x}_p \in C$, then we get

$$f(\overline{x}_p) \geq \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(\overline{x}_p) - \sum_{i=1}^{n} p_i (\nabla_+ f(x_i))(x_i) + c \sum_{i=1}^{n} p_i \|x_i - \overline{x}_p\|^2 \tag{2.21}$$

that is equivalent to the desired result (2.15). □

Remark 1. *For inequalities in terms of the Gâteaux derivatives for convex functions on linear spaces see* [8] *while for Slater's type inequalities for convex functions defined on linear spaces and applications, see* [9]*. The inequalities (2.12)-(2.15) are improvements of the corresponding inequalities for convex functions on normed spaces in which the term* $c \sum_{i=1}^{n} p_i \|x_i - y\|^2$ *vanishes. We observe that, if* X *is an inner product space, then by using the inner product properties we have that*

$$\sum_{i=1}^{n} p_i \|x_i - \overline{x}_p\|^2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 = \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\|^2$$

and the inequality (2.3) for $m > 0$ *and the inequality (2.12) for* $c = \frac{m}{2}$ *are the same. However, in the general case of normed spaces they are different. Inequality (2.12) in the case where* X *is an inner product space was obtained in* [13].

3. Some Hermite-Hadamard's Type Inequalities

We have:

Theorem 2. *Let* $f : C \subseteq X \to \mathbb{R}$*, where* C *is a convex subset in the normed linear space* $(X, \|\cdot\|)$ *and* x*,* $y \in C$ *with* $x \neq y$*. Assume also that* $0 < m < M$*.*

(*i*) *If f is m-lower convex on C, then*

$$0 \leq \frac{1}{16} m \left[\langle y-x, y+x \rangle_s - \langle y-x, y+x \rangle_i \right] \tag{3.1}$$
$$\leq \frac{1}{2} m \left[\int_0^1 \|(1-t)x + ty\|^2 dt - \left\| \frac{x+y}{2} \right\|^2 \right]$$
$$\leq \int_0^1 f[(1-t)x + ty] dt - f\left(\frac{x+y}{2} \right)$$

and

$$0 \leq \frac{1}{16} m \left[\langle y-x, y+x \rangle_s - \langle y-x, y+x \rangle_i \right] \tag{3.2}$$
$$\leq \frac{1}{2} m \left[\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \right]$$
$$\leq \frac{f(x) + f(y)}{2} - \int_0^1 f[(1-t)x + ty] dt.$$

(*ii*) *If f is M-upper convex on C, then*

$$\int_0^1 f[(1-t)x + ty] dt - f\left(\frac{x+y}{2} \right) \tag{3.3}$$
$$\leq \frac{1}{2} M \left[\int_0^1 \|(1-t)x + ty\|^2 dt - \left\| \frac{x+y}{2} \right\|^2 \right]$$
$$\leq \frac{1}{8} M \left[\langle y-x, y \rangle_i - \langle y-x, x \rangle_s \right]$$

and

$$\frac{f(x) + f(y)}{2} - \int_0^1 f[(1-t)x + ty] dt \tag{3.4}$$
$$\leq \frac{1}{2} M \left[\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \right]$$
$$\leq \frac{1}{8} M \left[\langle y-x, y \rangle_i - \langle y-x, x \rangle_s \right].$$

(*iii*) *If f is (m, M)-convex on C, then the pair of inequalities (3.1), (3.3) and (3.2), (3.4) hold simultaneously.*

Proof. (i) The first two inequalities in (3.1) and (3.2) follows by (1.3) and (1.4).

Since f is m-lower convex on C, hence $g(x) := f(x) - \frac{1}{2} m \|x\|^2$ is convex on C. By using the Hermite-Hadamard inequality (HH) we have

$$f\left(\frac{x+y}{2} \right) - \frac{1}{2} m \left\| \frac{x+y}{2} \right\|^2 \leq \int_0^1 \left(f[(1-t)x + ty] - \frac{1}{2} m \|(1-t)x + ty\|^2 \right) dt$$
$$\leq \frac{1}{2} \left[f(x) - \frac{1}{2} m \|x\|^2 + f(y) - \frac{1}{2} m \|y\|^2 \right],$$

which imply the third inequalities in (3.1) and (3.2).

(ii) Follows in a similar way by considering the convex function $h(x) := \frac{1}{2} M \|x\|^2 - f(x)$, $x \in C$. □

Remark 2. *If the positivity condition for m is dropped, then only the third inequalities in (3.1) and (3.2) remain true. If M is not positive, then only the first inequalities in (3.3) and (3.4) remain true.*

Corollary 3. *Let $f : C \subseteq X \to \mathbb{R}$, where C is a convex subset in the inner product space $(X, \langle\cdot,\cdot\rangle)$ and $x, y \in C$ with $x \neq y$. Assume also that $0 < m < M$.*

(i) If f is m-lower convex on C, then

$$\frac{1}{24} m \|x-y\|^2 \leq \int_0^1 f[(1-t)x+ty]\,dt - f\left(\frac{x+y}{2}\right) \tag{3.5}$$

and

$$\frac{1}{12} m \|x-y\|^2 \leq \frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]\,dt. \tag{3.6}$$

(ii) If f is M-upper convex on C, then

$$\int_0^1 f[(1-t)x+ty]\,dt - f\left(\frac{x+y}{2}\right) \leq \frac{1}{24} M \|x-y\|^2 \tag{3.7}$$

and

$$\frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{M}{12} \|x-y\|^2. \tag{3.8}$$

(iii) If f is (m, M)-convex on C, then the pair of inequalities (3.5), (3.7) and (3.6), (3.8) hold simultaneously.

Proof. Since $(X, \langle\cdot,\cdot\rangle)$ is an inner product, then for any x, y

$$\begin{aligned}
&\int_0^1 \|(1-t)x+ty\|^2\,dt \\
&= \int_0^1 \left[(1-t)^2\|x\|^2 + 2t(1-t)\operatorname{Re}\langle x,y\rangle + t^2\|y\|^2\right] dt \\
&= \|x\|^2 \int_0^1 (1-t)^2\,dt + 2\operatorname{Re}\langle x,y\rangle \int_0^1 t(1-t)\,dt + \|y\|^2 \int_0^1 t^2 dt \\
&= \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right).
\end{aligned}$$

Therefore, for any x, y

$$\begin{aligned}
&\int_0^1 \|(1-t)x+ty\|^2\,dt - \left\|\frac{x+y}{2}\right\|^2 \\
&= \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right) - \frac{1}{4}\left(\|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right) \\
&= \frac{1}{12}\left(\|x\|^2 - 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right) = \frac{1}{12}\|x-y\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x+ty\|^2\,dt \\
&= \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right) - \frac{1}{3}\left(\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right) \\
&= \frac{1}{6}\|x-y\|^2.
\end{aligned}$$

By using Theorem 2 we get the desired results (3.5)-(3.8). □

Remark 3. *If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is twice differentiable and $f''(t)\geq m$ for any $t\in[a,b]$, then by (3.5) and (3.6) we have*

$$\frac{1}{24}m(b-a)^2\leq\frac{1}{b-a}\int_a^b f(s)\,ds-f\left(\frac{a+b}{2}\right),\ \text{see [5, Eq. (5.1)]}\tag{3.9}$$

and

$$\frac{1}{12}m(b-a)^2\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds,\ \text{see [10], [11, p. 40]}.\tag{3.10}$$

If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is twice differentiable and $f''(t)\leq M$ for any $t\in[a,b]$, then by (3.7) and (3.8) we have

$$\frac{1}{b-a}\int_a^b f(s)\,ds-f\left(\frac{a+b}{2}\right)\leq\frac{1}{24}M(b-a)^2,\ \text{see [5, Eq. (5.1)]}\tag{3.11}$$

and

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds\leq\frac{1}{12}M(b-a)^2,\ \text{see [10], [11, p. 40]}.\tag{3.12}$$

We also have:

Theorem 3. *Let $f:C\subseteq X\to\mathbb{R}$ be a function strongly convex with modulus c on C that is open and convex in the normed linear space $(X,\|\cdot\|)$ and $x,y\in C$ with $x\neq y$. Then*

$$\frac{f(x)+f(y)}{2}-\int_0^1 f[(1-t)x+ty]\,dt\geq\frac{1}{6}c\|x-y\|^2\tag{3.13}$$

and

$$\int_0^1 f[(1-t)x+ty]\,dt-f\left(\frac{x+y}{2}\right)\geq\frac{1}{12}c\|x-y\|^2.\tag{3.14}$$

Proof. If we integrate condition (2.1) in the definition of strongly convex functions, we have

$$\begin{aligned}\frac{f(x)+f(y)}{2}-\int_0^1 f(tx+(1-t)y)dt&\geq c\|x-y\|^2\int_0^1 t(1-t)dt\\&=\frac{1}{6}c\|x-y\|^2\end{aligned}\tag{3.15}$$

for any $x,y\in C$, which proves (3.13).

By taking in the definition of strong convexity $t=\frac{1}{2}$, we have

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\geq\frac{c}{4}\|a-b\|^2,\tag{3.16}$$

for all $a,b\in C$.

If we take in (3.16) $a=(1-t)x+ty$ and $b=tx+(1-t)y$, then we get

$$\frac{f((1-t)x+ty)+f(tx+(1-t)y)}{2}-f\left(\frac{x+y}{2}\right)\geq c\left(t-\frac{1}{2}\right)^2\|x-y\|^2\tag{3.17}$$

for any $x,y\in C$ and $t\in[0,1]$.

Integrating over $t \in [0,1]$, we have

$$\frac{1}{2}\left[\int_0^1 f((1-t)x+ty)dt + \int_0^1 f(tx+(1-t)y)dt\right] - f\left(\frac{x+y}{2}\right) \geq c\|x-y\|^2 \int_0^1 \left(t-\frac{1}{2}\right)^2 dt \tag{3.18}$$

and since

$$\int_0^1 f((1-t)x+ty)dt = \int_0^1 f(tx+(1-t)y)dt \text{ and } \int_0^1 \left(t-\frac{1}{2}\right)^2 dt = \frac{1}{12},$$

then by (3.18) we get (3.14). □

Remark 4. *If $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is a function strongly convex with modulus c on the interval $[a,b]$, then by (3.13) and (3.14) we get*

$$\frac{f(a)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \geq \frac{1}{6}c(b-a)^2 \tag{3.19}$$

and

$$\frac{1}{b-a}\int_a^b f(s)\,ds - f\left(\frac{a+b}{2}\right) \geq \frac{1}{12}c(b-a)^2. \tag{3.20}$$

These inequalities were obtained in this form in [13].

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