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TWO PARAMETERS AND TWO POINTS REPRESENTATIONS OF ABSOLUTELY CONTINUOUS FUNCTIONS WITH INTEGRAL REMAINDER

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Abstract. In this paper we establish some two parameters two points representations with integral remainders for absolutely continuous functions and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

1. Introduction

Throughout this paper the integrals are taken in the Lebesgue sense.

Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Then for any λ_1 and λ_2 complex numbers, we have [24]

$$f(x)=\frac{1}{2(b-a)}\left[(x-a)^2\lambda_1-(b-x)^2\lambda_2\right]+\frac{1}{b-a}\int_a^b f(t)\,dt+R(x,a,b;\lambda_1,\lambda_2), \tag{1.1}$$

where the reminder $R(x,a,b;\lambda_1,\lambda_2)$ is given by

$$R(x,a,b;\lambda_1,\lambda_2)=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2\right]dt.$$

With the above assumption for f, we have for any $\lambda\in\mathbb{C}$ that

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\left(x-\frac{a+b}{2}\right)\lambda+R(x,a,b;\lambda) \tag{1.2}$$

where

$$R(x,a,b;\lambda)=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda\right]dt. \tag{1.3}$$

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If we take $\lambda = 0$ in (1.3), then we get *Montgomery's identity for absolutely continuous functions*, namely

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^x (t-a) f'(t)\,dt + \frac{1}{b-a}\int_x^b (t-b) f'(t)\,dt, \tag{1.4}$$

for $x \in [a,b]$.

We have the following midpoint representation as well:

$$f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{8}(b-a)(\lambda_1 - \lambda_2) + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda_1\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda_2\right] dt \tag{1.5}$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \lambda\right] dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \lambda\right] dt. \tag{1.6}$$

Using the representation (1.1) we can prove the following Ostrowski type inequality:

Theorem 1 (Dragomir, 2003 [20])**.** *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x \in [a,b]$. Suppose that there exist the functions m_i, $M_i : [a,b] \to \mathbb{R}$ $\left(i = \overline{1,2}\right)$ with the properties:*

$$m_1(x) \leq f'(t) \leq M_1(x) \quad \text{for a.e. } t \in [a,x] \tag{1.7}$$

and

$$m_2(x) \leq f'(t) \leq M_2(x) \quad \text{for a.e. } t \in (x,b]. \tag{1.8}$$

Then we have the inequalities:

$$\begin{aligned} &\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2 - M_2(x)(b-x)^2\right] \\ &\leq f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{1}{2(b-a)}\left[M_1(x)(x-a)^2 - m_2(x)(b-x)^2\right]. \end{aligned} \tag{1.9}$$

The constant $\frac{1}{2}$ is sharp on both sides.

In the case that the derivative is globally bounded on $[a,b]$ by two constants, then we have:

Corollary 1. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is, there exists the constants $M>m$ such that*

$$-\infty<m\leq f'(t)\leq M<\infty \text{ for a.e. } t\in[a,b], \tag{1.10}$$

then we have the inequality

$$\begin{aligned}&\frac{1}{2(b-a)}\left[m(x-a)^2-M(b-x)^2\right]\\&\leq f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\&\leq\frac{1}{2(b-a)}\left[M(x-a)^2-m(b-x)^2\right]\end{aligned} \tag{1.11}$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

If we assume that $\|f'\|_\infty:=ess\sup_{t\in[a,b]}|f'(t)|<\infty$, then obviously we may choose in (1.11) $m=-\|f'\|_\infty$ and $M=\|f'\|_\infty$, obtaining Ostrowski's inequality for absolutely continuous functions whose derivatives are essentially bounded:

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|&\leq\frac{\|f'\|_\infty}{2(b-a)}\left[(x-a)^2+(b-x)^2\right]\\&=\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty,\end{aligned} \tag{1.12}$$

for all $x\in[a,b]$.

For other Ostrowski type inequalities see [1]-[19] and [21]-[44].

Let $f:I\to\mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$, the interior of the interval I. In this paper we consider the alternative problem of approximating an absolutely continuous function by using an affine combination of the values in two points $f(a)$, $f(b)$ where a, $b\in\mathring{I}$ and two free parameters δ, $\gamma\in\mathbb{C}$ as follows

$$f(x)\approx(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)(x-a)\delta-\lambda(b-x)\gamma$$

for $\lambda\in\mathbb{C}\setminus\{0,1\}$ and $x\in\mathring{I}$. Some inequalities for bounded derivatives and applications for weighted means are also given.

2. Some Identities

We start with the following representation result:

Theorem 2. *Let $f:I\to\mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$, the interior of the interval I. Then for any x, a, $b\in\mathring{I}$ and $\lambda\in\mathbb{C}\setminus\{0,1\}$, δ, $\gamma\in\mathbb{C}$ we have*

$$f(x)=(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)(x-a)\delta-\lambda(b-x)\gamma+S_\lambda(x,a,b;\delta,\gamma), \tag{2.1}$$

where the remainder $S_\lambda(x,a,b;\delta,\gamma)$ is given by

$$\begin{aligned}S_\lambda(x,a,b;\delta,\gamma)&:=(1-\lambda)(x-a)\int_0^1\left[f'((1-s)a+sx)-\delta\right]ds\\&+\lambda(b-x)\int_0^1\left[\gamma-f'((1-s)x+sb)\right]ds.\end{aligned} \tag{2.2}$$

Proof. For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t=(1-s)c+sd$, $s\in[0,1]$ that

$$\int_c^d h(t)\,dt=(d-c)\int_0^1 h((1-s)c+sd)\,ds.$$

Using this property we have

$$\begin{aligned}(1-\lambda)(x-a)&\int_0^1[f'((1-s)a+sx)-\delta]\,ds\\&=(1-\lambda)(x-a)\int_0^1 f'((1-s)a+sx)\,ds-(1-\lambda)(x-a)\delta\\&=(1-\lambda)\int_a^x f'(t)\,dt-(1-\lambda)(x-a)\delta\\&=(1-\lambda)[f(x)-f(a)]-(1-\lambda)(x-a)\delta\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\lambda(b-x)&\int_0^1[\gamma-f'((1-s)x+sb)]\,ds\\&=\lambda(b-x)\gamma-\lambda(b-x)\int_0^1 f'((1-s)x+sb)\,ds\\&=\lambda(b-x)\gamma-\lambda\int_x^b f'(t)\,dt=\lambda(b-x)\gamma-\lambda[f(b)-f(x)]\end{aligned}\tag{2.4}$$

for any $x,\ a,\ b\in\mathring{I}$ and $\lambda\in\mathbb{C}\setminus\{0,1\}$, $\delta,\ \gamma\in\mathbb{C}$.

If we add the equalities (2.3) and (2.4) we get

$$\begin{aligned}&(1-\lambda)(x-a)\int_0^1[f'((1-s)a+sx)-\delta]\,ds\\&+\lambda(b-x)\int_0^1[\gamma-f'((1-s)x+sb)]\,ds\\&=(1-\lambda)[f(x)-f(a)]-(1-\lambda)(x-a)\delta\\&+\lambda(b-x)\gamma-\lambda[f(b)-f(x)]\\&=f(x)-(1-\lambda)f(a)-\lambda f(b)-(1-\lambda)(x-a)\delta+\lambda(b-x)\gamma,\end{aligned}$$

which is equivalent to the desired result (2.1). □

Corollary 2. *Let $f:I\to\mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$. Then for any $x,\ a,\ b\in\mathring{I}$ and $\delta,\ \gamma\in\mathbb{C}$ we have*

$$\begin{aligned}f(x)=&\frac{1}{b-a}[(b-x)f(a)+(x-a)f(b)]+\frac{(b-x)(x-a)}{b-a}(\delta-\gamma)\\&+S_1(x,a,b;\delta,\gamma),\end{aligned}\tag{2.5}$$

where the remainder $S_1(x,a,b;\delta,\gamma)$ is given by

$$\begin{aligned}&S_1(x,a,b;\delta,\gamma)\\&:=\frac{(b-x)(x-a)}{b-a}\\&\times\left[\int_0^1[f'((1-s)a+sx)-\delta]\,ds+\int_0^1[\gamma-f'((1-s)x+sb)]\,ds\right].\end{aligned}\tag{2.6}$$

Alternatively, we have

$$f(x)=\frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right]+\frac{1}{b-a}\left[(x-a)^{2}\delta-(b-x)^{2}\gamma\right]+S_{2}(x,a,b;\delta,\gamma), \tag{2.7}$$

where the remainder $S_2(x,a,b;\delta,\gamma)$ *is given by*

$$S_{2}(x,a,b;\delta,\gamma):=\frac{1}{b-a}\left[(x-a)^{2}\int_{0}^{1}\left[f'((1-s)a+sx)-\delta\right]ds+(b-x)^{2}\int_{0}^{1}\left[\gamma-f'((1-s)x+sb)\right]ds\right]. \tag{2.8}$$

Proof. Follows by Theorem 2 on taking $\lambda=\frac{x-a}{b-a}$ and $\lambda=\frac{b-x}{b-a}$, respectively. □

The following particular case is of interest as well:

Corollary 3. *Let* $f:I\to\mathbb{C}$ *be a locally absolutely continuous function on* $\mathring{I}$. *Then for any* $a,\ b\in\mathring{I}$, $\lambda\in[0,1]$ *and* $\delta,\ \gamma\in\mathbb{C}$ *we have*

$$f((1-\lambda)a+\lambda b)=(1-\lambda)f(a)+\lambda f(b)+(1-\lambda)\lambda(b-a)(\delta-\gamma)+S_{1,\lambda}(a,b;\delta,\gamma), \tag{2.9}$$

where the remainder $S_{1,\lambda}(a,b;\delta,\gamma)$ *is given by*

$$S_{1,\lambda}(a,b;\delta,\gamma):=(1-\lambda)\lambda(b-a)\times\left[\int_{0}^{1}\left[f'((1-s\lambda)a+s\lambda b)-\delta\right]ds+\int_{0}^{1}\left[\gamma-f'((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)\right]ds\right]. \tag{2.10}$$

Alternatively, we have

$$f(\lambda a+(1-\lambda)b)=(1-\lambda)f(a)+\lambda f(b)+(b-a)\left[(1-\lambda)^{2}\delta-\lambda^{2}\gamma\right]+S_{2,\lambda}(a,b;\delta,\gamma), \tag{2.11}$$

where the remainder $S_{2,\lambda}(a,b;\delta,\gamma)$ *is given by*

$$S_{2,\lambda}(x,a,b;\delta,\gamma):=(b-a)\times\left[(1-\lambda)^{2}\int_{0}^{1}\left[f'((1-s+\lambda s)a+(1-\lambda)sb)-\delta\right]ds+\lambda^{2}\int_{0}^{1}\left[\gamma-f'((1-s)\lambda a+(1-\lambda+\lambda s)b)\right]ds\right]. \tag{2.12}$$

Remark 1. *Let* f *be as in Theorem 2, then for any* $a,\ b\in\mathring{I}$ *and* $\lambda\in\mathbb{C}\setminus\{0,1\}$, δ, $\gamma\in\mathbb{C}$ *we have*

$$f\left(\frac{a+b}{2}\right)=(1-\lambda)f(a)+\lambda f(b)+\frac{1}{2}(b-a)\left[(1-\lambda)\delta-\lambda\gamma\right]+S_{\lambda}(a,b;\delta,\gamma), \tag{2.13}$$

where the remainder $S_{\lambda}(a,b;\delta,\gamma)$ *is given by*

$$S_{\lambda}(a,b;\delta,\gamma):=\frac{1}{2}(b-a)\left[(1-\lambda)\int_0^1\left[f'\left((1-s)a+s\frac{a+b}{2}\right)-\delta\right]ds\right.$$
$$\left.+\lambda\int_0^1\left[\gamma-f'\left((1-s)\frac{a+b}{2}+sb\right)\right]ds\right]. \tag{2.14}$$

The case $\delta=\gamma=0$ in (2.1) produces the following simple identities for each distinct $x,\ a,\ b\in\mathring{I}$ and $\lambda\in\mathbb{C}\setminus\{0,1\}$

$$f(x)=(1-\lambda)f(a)+\lambda f(b)+S_{\lambda}(x,a,b), \tag{2.15}$$

where the remainder $S_{\lambda}(x,a,b)$ is given by

$$S_{\lambda}(x,a,b):=(1-\lambda)(x-a)\int_0^1 f'((1-s)a+sx)\,ds$$
$$-\lambda(b-x)\int_0^1 f'((1-s)x+sb)\,ds. \tag{2.16}$$

We then have for each distinct $x,\ a,\ b\in\mathring{I}$

$$f(x)=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+L(x,a,b), \tag{2.17}$$

where

$$L(x,a,b)$$
$$:=\frac{(b-x)(x-a)}{b-a}\left[\int_0^1 f'((1-s)a+sx)\,ds-\int_0^1 f'((1-s)x+sb)\,ds\right] \tag{2.18}$$

and

$$f(x)=\frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right]+P(x,a,b), \tag{2.19}$$

where

$$P(x,a,b)$$
$$:=\frac{1}{b-a}\left[(x-a)^2\int_0^1 f'((1-s)a+sx)\,ds-(b-x)^2\int_0^1 f'((1-s)x+sb)\,ds\right]. \tag{2.20}$$

We also have

$$f((1-\lambda)a+\lambda b)=(1-\lambda)f(a)+\lambda f(b)+S_{\lambda}(a,b), \tag{2.21}$$

where the remainder $S_{\lambda}(a,b)$ is given by

$$S_{\lambda}(a,b):=(1-\lambda)\lambda(b-a)\left[\int_0^1 f'((1-s\lambda)a+s\lambda b)\,ds\right.$$
$$\left.-\int_0^1 f'((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)\,ds\right] \tag{2.22}$$

and

$$f((1-\lambda)b+\lambda a)=(1-\lambda)f(a)+\lambda f(b)+P_{\lambda}(a,b), \tag{2.23}$$

where the remainder $P_\lambda(a,b)$ is given by

$$P_\lambda(a,b) := (b-a)\left[(1-\lambda)^2\int_0^1 f'((1-s+\lambda s)a+(1-\lambda)sb)\,ds - \lambda^2\int_0^1 f'((1-s)\lambda a+(1-\lambda+\lambda s)b)\,ds\right]. \tag{2.24}$$

Moreover, if we take in (2.15) $x=\frac{a+b}{2}$ for each distinct $a,\ b\in\mathring{I}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$, then we have

$$f\left(\frac{a+b}{2}\right) = (1-\lambda)f(a)+\lambda f(b)+S_\lambda(a,b), \tag{2.25}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b) := \frac{1}{2}(b-a) \times\left[(1-\lambda)\int_0^1 f'\left((1-s)a+s\frac{a+b}{2}\right)ds - \lambda\int_0^1 f'\left((1-s)\frac{a+b}{2}+sb\right)ds\right]. \tag{2.26}$$

In particular, for $\lambda=\frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2}+S(a,b), \tag{2.27}$$

where

$$S(a,b) := \frac{1}{4}(b-a) \times\left[\int_0^1 f'\left((1-s)a+s\frac{a+b}{2}\right)ds - \int_0^1 f'\left((1-s)\frac{a+b}{2}+sb\right)ds\right]. \tag{2.28}$$

3. Inequalities for Bounded Derivatives

Now, for $\phi,\ \Phi\in\mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [25])

$$\bar{U}_I(\phi,\Phi) := \left\{g: I\to\mathbb{C}\,|\,\operatorname{Re}\left[(\Phi-g(t))\left(\overline{g(t)}-\overline{\phi}\right)\right]\geq 0 \text{ for almost every } t\in I\right\}$$

and

$$\bar{\Delta}_I(\phi,\Phi) := \left\{g: I\to\mathbb{C}\,|\,\left|g(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for a.e. } t\in I\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_I(\phi,\Phi)$ and $\bar{\Delta}_I(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_I(\phi,\Phi) = \bar{\Delta}_I(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi-z)(\bar{z}-\phi)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^{2}-\left|z-\frac{\phi+\Phi}{2}\right|^{2}=\operatorname{Re}\left[\left(\Phi-z\right)\left(\bar{z}-\phi\right)\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4. *For any $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{I}\left(\phi,\Phi\right)=\{g:I\to\mathbb{C}\mid\ &\left(\operatorname{Re}\Phi-\operatorname{Re}g\left(t\right)\right)\left(\operatorname{Re}g\left(t\right)-\operatorname{Re}\phi\right)\\&+\left(\operatorname{Im}\Phi-\operatorname{Im}g\left(t\right)\right)\left(\operatorname{Im}g\left(t\right)-\operatorname{Im}\phi\right)\geq0\text{ for a.e. }t\in I\}.\end{aligned}\tag{3.2}$$

Now, if we assume that $\operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}\left(\phi\right)$ and $\operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}\left(\phi\right)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{I}\left(\phi,\Phi\right):=\{g:I\to\mathbb{C}\mid\ &\operatorname{Re}\left(\Phi\right)\geq\operatorname{Re}g\left(t\right)\geq\operatorname{Re}\left(\phi\right)\\&\text{and }\operatorname{Im}\left(\Phi\right)\geq\operatorname{Im}g\left(t\right)\geq\operatorname{Im}\left(\phi\right)\ \text{ for a.e. }t\in I\}.\end{aligned}\tag{3.3}$$

One can easily observe that $\bar{S}_{I}\left(\phi,\Phi\right)$ is closed, convex and

$$\emptyset\neq\bar{S}_{I}\left(\phi,\Phi\right)\subseteq\bar{U}_{I}\left(\phi,\Phi\right).\tag{3.4}$$

The following result holds:

Theorem 3. *Let $f:I\to\mathbb{C}$ be a locally absolutely continuous function on $\mathring{I}$ and with the property that there exists complex numbers $\phi,\Phi\in\mathbb{C}$ such that the derivative $f'\in\bar{U}_{I}\left(\phi,\Phi\right)$. Then for any $x,\ a,\ b\in\mathring{I}$ and $\lambda\in\mathbb{C}\backslash\left\{0,1\right\}$ we have*

$$\begin{aligned}&\left|f\left(x\right)-\left(1-\lambda\right)f\left(a\right)-\lambda f\left(b\right)-\frac{\phi+\Phi}{2}\left[x-\left(1-\lambda\right)a-\lambda b\right]\right|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\left[\left|1-\lambda\right|\left|x-a\right|+\left|\lambda\right|\left|b-x\right|\right]\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\begin{cases}\max\left\{\left|1-\lambda\right|,\left|\lambda\right|\right\}\left(\left|x-a\right|+\left|b-x\right|\right),\\ \\ \left(\left|1-\lambda\right|^{p}+\left|\lambda\right|^{p}\right)^{1/p}\left(\left|x-a\right|^{q}+\left|b-x\right|^{q}\right)^{1/q},\\ \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left(\left|1-\lambda\right|+\left|\lambda\right|\right)\max\left\{\left|x-a\right|,\left|b-x\right|\right\}.\end{cases}\end{aligned}\tag{3.5}$$

Proof. From the representation (2.1) we have

$$\begin{aligned}f\left(x\right)&=\left(1-\lambda\right)f\left(a\right)+\lambda f\left(b\right)+\left(1-\lambda\right)\left(x-a\right)\frac{\phi+\Phi}{2}-\lambda\left(b-x\right)\frac{\phi+\Phi}{2}\\&\quad+S_{\lambda}\left(x,a,b;\delta,\gamma\right)\\&=\left(1-\lambda\right)f\left(a\right)+\lambda f\left(b\right)+\frac{\phi+\Phi}{2}\left[x-\left(1-\lambda\right)a-\lambda b\right]\\&\quad+S_{\lambda}\left(x,a,b;\phi,\Phi\right),\end{aligned}\tag{3.6}$$

where the remainder $S_\lambda(x,a,b;\phi,\Phi)$ is given by

$$S_\lambda(x,a,b;\delta,\gamma) := (1-\lambda)(x-a)\int_0^1 \left[f'((1-s)a+sx) - \frac{\phi+\Phi}{2}\right]ds + \lambda(b-x)\int_0^1 \left[\frac{\phi+\Phi}{2} - f'((1-s)x+sb)\right]ds \tag{3.7}$$

and $x,\ a,\ b \in \mathring{I}$ while $\lambda \in \mathbb{C}\setminus\{0,1\}$, $\phi,\ \Phi \in \mathbb{C}$.

Therefore, by taking the modulus and utilizing the fact that $f' \in \bar{U}_I(\phi,\Phi)$, we have

$$\begin{aligned}
&\left|f(x) - (1-\lambda)f(a) - \lambda f(b) - \frac{\phi+\Phi}{2}[x-(1-\lambda)a-\lambda b]\right| \\
&= |S_\lambda(x,a,b;\phi,\Phi)| \\
&\leq \left|(1-\lambda)(x-a)\int_0^1\left[f'((1-s)a+sx) - \frac{\phi+\Phi}{2}\right]ds\right| \\
&+ \left|\lambda(b-x)\int_0^1\left[\frac{\phi+\Phi}{2} - f'((1-s)x+sb)\right]ds.\right| \\
&\leq |1-\lambda||x-a|\int_0^1\left|f'((1-s)a+sx) - \frac{\phi+\Phi}{2}\right|ds \\
&+ |\lambda||b-x|\int_0^1\left|\frac{\phi+\Phi}{2} - f'((1-s)x+sb)\right|ds \\
&\leq \frac{1}{2}|\Phi-\phi|\left[|1-\lambda||x-a| + |\lambda||b-x|\right]
\end{aligned}$$

for any $x,\ a,\ b \in \mathring{I}$ and $\lambda \in \mathbb{C}\setminus\{0,1\}$.

This proves the first inequality in (3.5).

The last part is obvious by Hölder's inequality

$$cd + uv \leq \begin{cases} \max\{c,u\}(d+v) \\ \\ (c^p+u^p)^{1/p}(d^q+v^q)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

□

Remark 2. *For $p=q=2$ we have for $\lambda \in [0,1]$ and $x \in [a,b]$ with $a<b$ that*

$$\begin{aligned}
&\left|f(x) - (1-\lambda)f(a) - \lambda f(b) - \frac{\phi+\Phi}{2}[x-(1-\lambda)a-\lambda b]\right| \\
&\leq \frac{1}{2}|\Phi-\phi|\left[(1-\lambda)(x-a) + \lambda(b-x)\right] \\
&\leq |\Phi-\phi|\left(\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right)^{1/2}\left(\frac{1}{4}+\left(x-\frac{a+b}{2}\right)^2\right)^{1/2}.
\end{aligned} \tag{3.8}$$

Corollary 5. *With the assumptions of Theorem 3 for the function f, we have for any $x,\ a,\ b \in \mathring{I}$ that*

$$\left|f(x) - \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right]\right| \leq |\Phi-\phi|\frac{|(b-x)(x-a)|}{|b-a|} \tag{3.9}$$

and

$$\left|f(x)-\frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right]-(\phi+\Phi)\left(x-\frac{a+b}{2}\right)\right| \le |\Phi-\phi|\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]|b-a|. \tag{3.10}$$

Proof. Follows by Theorem 3 on taking $\lambda=\frac{x-a}{b-a}$ and $\lambda=\frac{b-x}{b-a}$, respectively. □

Corollary 6. *With the assumptions of Theorem 3 for the function f, we have for any $a,\ b\in\mathring{I}$ and $\lambda\in[0,1]$ that*

$$|f((1-\lambda)a+\lambda b)-(1-\lambda)f(a)-\lambda f(b)|\le|\Phi-\phi|(1-\lambda)\lambda|b-a| \tag{3.11}$$

and

$$\left|f((1-\lambda)b+\lambda a)-(1-\lambda)f(a)-\lambda f(b)-(\phi+\Phi)(b-a)\left(\frac{1}{2}-\lambda\right)\right| \le|\Phi-\phi|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]|b-a|. \tag{3.12}$$

Remark 3. *If we take $\lambda=\frac{1}{2}$ in either of the inequalities from Corollary 6 we get*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{4}|\Phi-\phi||b-a| \tag{3.13}$$

for any $a,\ b\in\mathring{I}$. The constant $\frac{1}{4}$ is best possible in (3.13).

Indeed, if we consider the absolutely continuous function $f(x)=\left|x-\frac{a+b}{2}\right|$, then $f'(x)=1$ for $x>\frac{a+b}{2}$ and $f'(x)=-1$ for $x<\frac{a+b}{2}$. Taking $\phi=-1$ and $\Phi=1$ in (3.13) we obtain in both terms the same quantity $\frac{1}{2}|b-a|$ that proves the sharpness of the constant $\frac{1}{4}$.

If the function f is real-valued locally absolutely continuous function on $\mathring{I}$ and

$$-\infty<k\le f'(x)\le K<\infty \text{ for almost every } x\in\mathring{I}, \tag{3.14}$$

then we have from (3.5) that

$$\begin{aligned}&\left|f(x)-(1-\lambda)f(a)-\lambda f(b)-\frac{k+K}{2}[x-(1-\lambda)a-\lambda b]\right| \\ &\le\frac{1}{2}(K-k)[|1-\lambda||x-a|+|\lambda||b-x|] \\ &\le\frac{1}{2}(K-k)\begin{cases}\max\{|1-\lambda|,|\lambda|\}(|x-a|+|b-x|), \\ \\ (|1-\lambda|^p+|\lambda|^p)^{1/p}(|x-a|^q+|b-x|^q)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ (|1-\lambda|+|\lambda|)\max\{|x-a|,|b-x|\},\end{cases}\end{aligned} \tag{3.15}$$

for any $x,\ a,\ b\in\mathring{I}$ and $\lambda\in\mathbb{R}\setminus\{0,1\}$.

Now, if $m, M \in \mathbb{R}$ with $m < M$ and $f : [m, M] \to \mathbb{R}$ is a convex function, then for any x, a, $b \in [m, M]$ we have from (3.15) for $k = f'_+(m)$ and $K = f'_-(M)$ that

$$\left| f(x) - (1-\lambda) f(a) - \lambda f(b) - \frac{f'_+(m) + f'_-(M)}{2} [x - (1-\lambda) a - \lambda b] \right| \tag{3.16}$$
$$\leq \frac{1}{2} \left(f'_-(M) - f'_+(m) \right) [|1-\lambda| |x-a| + |\lambda| |b-x|]$$
$$\leq \frac{1}{2} \left(f'_-(M) - f'_+(m) \right) \begin{cases} \max\{|1-\lambda|, |\lambda|\} (|x-a| + |b-x|), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} (|x-a|^q + |b-x|^q)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ (|1-\lambda| + |\lambda|) \max\{|x-a|, |b-x|\}, \end{cases}$$

for any $\lambda \in \mathbb{R} \setminus \{0, 1\}$.

For real-valued functions we have the following result as well:

Theorem 4. *Let $f : I \to \mathbb{R}$ be a locally absolutely continuous function on $\mathring{I}$ and with the property that there exists real numbers k, K such that the condition (3.14) is valid. Then for all a, $b \in \mathring{I}$ with $a < b$, $x \in [a, b]$ and $\lambda \in [0, 1]$ we have*

$$f(x) \geq (1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a) k - \lambda (b-x) K. \tag{3.17}$$

Proof. From (2.1) we have

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + (1-\lambda)(x-a) k - \lambda (b-x) K + S_\lambda (x, a, b; k, K), \tag{3.18}$$

for all a, $b \in \mathring{I}$ with $a < b$, $x \in [a, b]$ and $\lambda \in [0, 1]$, where the remainder $S_\lambda (x, a, b; k, K)$ is given by

$$S_\lambda (x, a, b; k, K) := (1-\lambda)(x-a) \int_0^1 [f'((1-s) a + sx) - k] \, ds + \lambda (b-x) \int_0^1 [K - f'((1-s) x + sb)] \, ds. \tag{3.19}$$

Since for all $x \in [a, b]$ we have $f'((1-s) a + sx) \geq k$ and $K \geq f'((1-s) x + sb)$ for almost every $s \in [0, 1]$, then $S_\lambda (x, a, b; k, K) \geq 0$ for every $\lambda \in [0, 1]$ and by (3.18) we get the desired result (3.17). $\square$

Corollary 7. *With the assumptions of Theorem 4 for the function f we have*

$$f(x) \geq \frac{1}{b-a} [(b-x) f(a) + (x-a) f(b)] - \frac{(b-x)(x-a)}{b-a} (K-k) \tag{3.20}$$

and

$$f(x) \geq \frac{1}{b-a} [(x-a) f(a) + (b-x) f(b)] + \frac{1}{b-a} \left[(x-a)^2 k - (b-x)^2 K \right], \tag{3.21}$$

for any $x \in [a, b]$.

We also have

Corollary 8. *With the assumptions of Theorem 4 for the function f we have*

$$f\left(\left(1-\lambda\right)a+\lambda b\right)\geq\left(1-\lambda\right)f\left(a\right)+\lambda f\left(b\right)-\left(1-\lambda\right)\lambda\left(b-a\right)\left(K-k\right) \tag{3.22}$$

and

$$f\left(\lambda a+\left(1-\lambda\right)b\right)\geq\left(1-\lambda\right)f\left(a\right)+\lambda f\left(b\right)+\left(b-a\right)\left[\left(1-\lambda\right)^{2}k-\lambda^{2}K\right] \tag{3.23}$$

for any $a,\ b\in\mathring{I}$ with $a<b$ and $\lambda\in[0,1]$.

4. Some Examples

Let $a,\ b,\ x\in[m,M]\subset(0,\infty)$, then by writing the inequality (3.16) for the convex function $f\left(t\right)=-\ln t,\ t>0$, we have

$$\begin{aligned}&\left|\left(1-\lambda\right)\ln a+\lambda\ln b-\ln x+\frac{m+M}{2mM}\left[x-\left(1-\lambda\right)a-\lambda b\right]\right| \\ &\leq\frac{M-m}{2mM}\left[\left(1-\lambda\right)\left|x-a\right|+\lambda\left|b-x\right|\right] \\ &\leq\frac{M-m}{2mM}\begin{cases}\max\left\{1-\lambda,\lambda\right\}\left(\left|x-a\right|+\left|b-x\right|\right), \\ \\ \left(\left(1-\lambda\right)^{p}+\lambda^{p}\right)^{1/p}\left(\left|x-a\right|^{q}+\left|b-x\right|^{q}\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{\left|x-a\right|,\left|b-x\right|\right\},\end{cases}\end{aligned} \tag{4.1}$$

for any $\lambda\in[0,1]$.

If we take $x=\left(1-\lambda\right)a+\lambda b$ in (4.1), where $a,\ b\in[m,M]$ and $\lambda\in[0,1]$, then we get

$$0\leq\ln\left(\left(1-\lambda\right)a+\lambda b\right)-\left(1-\lambda\right)\ln a-\lambda\ln b\leq\frac{M-m}{mM}\left(1-\lambda\right)\lambda \tag{4.2}$$

that is equivalent to

$$1\leq\frac{A_{\lambda}\left(a,b\right)}{G_{\lambda}\left(a,b\right)}\leq\exp\left[\frac{M-m}{mM}\left(1-\lambda\right)\lambda\right] \tag{4.3}$$

for any $a,\ b\in[m,M]\subset(0,\infty)$ and $\lambda\in[0,1]$, where $A_{\lambda}\left(a,b\right):=\left(1-\lambda\right)a+\lambda b$ is the weighted arithmetic mean and $G_{\lambda}\left(a,b\right):=a^{1-\lambda}b^{\lambda}$ is the weighted geometric mean.

If we take in (4.1) $x=G_{\lambda}\left(a,b\right)$, then we get

$$\begin{aligned}0&\leq A_{\lambda}\left(a,b\right)-G_{\lambda}\left(a,b\right) \\ &\leq\frac{M-m}{m+M}\left[\left(1-\lambda\right)\left|G_{\lambda}\left(a,b\right)-a\right|+\lambda\left|b-G_{\lambda}\left(a,b\right)\right|\right] \\ &\leq\frac{M-m}{m+M}\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left|b-a\right|\end{aligned} \tag{4.4}$$

for any $a,\ b\in[m,M]\subset(0,\infty)$ and $\lambda\in[0,1]$.

If we take $x = \lambda a + (1-\lambda) b$ in (4.1), where a, $b \in [m, M]$ and $\lambda \in [0,1]$, then we get

$$\left|\ln\left(\frac{G_\lambda(a,b)}{A_{1-\lambda}(a,b)}\right) - \frac{m+M}{2mM}(b-a)\left(\lambda-\frac{1}{2}\right)\right| \le \frac{M-m}{mM}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]|b-a|. \tag{4.5}$$

Let c, d, $y \in [k, K] \subset \mathbb{R}$, then by writing the inequality (3.16) for the convex function $f(y) = \exp y$, $y \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\exp(y) - (1-\lambda)\exp c - \lambda \exp d - \frac{\exp k + \exp K}{2}[y-(1-\lambda)c - \lambda d]\right| \\
&\le \frac{1}{2}(\exp K - \exp k)\left[|1-\lambda||y-c| + |\lambda||d-y|\right] \\
&\le \frac{1}{2}(\exp K - \exp k)\begin{cases} \max\{|1-\lambda|, |\lambda|\}(|y-c|+|d-y|), \\[1ex] (|1-\lambda|^p + |\lambda|^p)^{1/p}(|y-c|^q + |d-y|^q)^{1/q}, \\ \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[1ex] \max\{|y-c|, |d-y|\}(|1-\lambda|+|\lambda|), \end{cases}
\end{aligned} \tag{4.6}$$

for any $\lambda \in \mathbb{R}\setminus\{0,1\}$.

If a, b, $x \in [m, M] \subset (0,\infty)$, then by taking $c = \ln a$, $d = \ln b$, $y = \ln x$, $k = \ln m$ and $K = \ln M$ we get

$$\begin{aligned}
&\left|x - (1-\lambda)a - \lambda b - \frac{m+M}{2}\ln\left(\frac{x}{a^{1-\lambda}b^\lambda}\right)\right| \\
&\le \frac{1}{2}(M-m)\left[|1-\lambda|\left|\ln\frac{x}{a}\right| + |\lambda|\left|\ln\frac{b}{x}\right|\right] \\
&\le \frac{1}{2}(M-m)\begin{cases} \max\{|1-\lambda|, |\lambda|\}\left(\left|\ln\frac{x}{a}\right| + \left|\ln\frac{b}{x}\right|\right), \\[1ex] (|1-\lambda|^p + |\lambda|^p)^{1/p}\left(\left|\ln\frac{x}{a}\right|^q + \left|\ln\frac{b}{x}\right|^q\right)^{1/q}, \\ \ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[1ex] \max\left\{\left|\ln\frac{x}{a}\right|, \left|\ln\frac{b}{x}\right|\right\}(|1-\lambda|+|\lambda|), \end{cases}
\end{aligned} \tag{4.7}$$

for any $\lambda \in \mathbb{R}\setminus\{0,1\}$.

Now, if a, b, $x \in [m, M] \subset (0,\infty)$ and $\lambda \in [0,1]$ then by taking $x = a^{1-\lambda}b^\lambda = G_\lambda(a,b)$ in the first part of (4.7) we get

$$0 \le A_\lambda(a,b) - G_\lambda(a,b) \le (M-m)(1-\lambda)\lambda|\ln b - \ln a|. \tag{4.8}$$

Also, if a, b, $x \in [m, M] \subset (0,\infty)$ and $\lambda \in [0,1]$ then by taking $x = (1-\lambda)a + \lambda b = A_\lambda(a,b)$ in the first part of (4.7) we get

$$1 \le \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \le \exp\left[\frac{M-m}{M+m}\left[(1-\lambda)\left|\ln\frac{A_\lambda(a,b)}{a}\right| + \lambda\left|\ln\frac{A_\lambda(a,b)}{b}\right|\right]\right]. \tag{4.9}$$

Since

$$\begin{aligned}&\left[(1-\lambda)\left|\ln\frac{A_\lambda(a,b)}{a}\right|+\lambda\left|\ln\frac{A_\lambda(a,b)}{b}\right|\right]\\&\leq\max\{1-\lambda,\lambda\}\left[\left|\ln\frac{A_\lambda(a,b)}{a}\right|+\left|\ln\frac{A_\lambda(a,b)}{b}\right|\right]\\&\leq\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)|\ln b-\ln a|,\end{aligned}$$

hence by (4.9) we get

$$1\leq\frac{A_\lambda(a,b)}{G_\lambda(a,b)}\leq\exp\left[\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\frac{M-m}{m+M}|\ln b-\ln a|\right] \tag{4.10}$$

for any a, $b\in[m,M]\subset(0,\infty)$ and $\lambda\in[0,1]$.

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