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SOME INEQUALITIES INVOLVING THE RATIO OF GAMMA FUNCTIONS

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ABSTRACT. In this paper, we present and prove some inequalities involving the ratios $\frac{\Gamma_k(t)}{\Gamma_p(t)}$ and $\frac{\Gamma_k(t)}{\Gamma_q(t)}$. Our approach makes use of the series representations of the functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$.

1. INTRODUCTION

We begin by recalling some basic definitions related to the Gamma function.

The classical Euler's Gamma function $\Gamma(t)$ is defined by,

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx, \qquad t > 0. \tag{1}$$

The p-Gamma function $\Gamma_p(t)$, also known as the p-analogue of the Gamma function is defined as (see [3], [2])

$$\Gamma_p(t) = \frac{p! p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}, \quad p \in N, \quad t > 0. \tag{2}$$

The p-psi function $\psi_p(t)$ is defined as the logarithmic derivative of the p-Gamma function. That is,

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma_p'(t)}{\Gamma_p(t)}, \qquad t > 0. \tag{3}$$

The q-Gamma function, $\Gamma_q(t)$ is defined as (see [5])

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1), \quad t > 0. \tag{4}$$

The q-psi function, $\psi_q(t)$ is also defined as,

$$\psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma_q'(t)}{\Gamma_q(t)}, \qquad t > 0. \tag{5}$$

The k-Gamma function, $\Gamma_k(t)$ is defined as (see [1], [6])

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\, dx, \quad k > 0, \quad t > 0. \tag{6}$$

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The k-psi function, $\psi_k(t)$ is similarly defined as follows.

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad t > 0. \tag{7}$$

In a recent paper [4], Krasniqi and Shabani proved the following result.

$$\frac{p^{-t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)} \tag{8}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in [2], Krasniqi, Mansour and Shabani proved the following result.

$$\frac{(1-q)^t e^{-\gamma t}\Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_q(\alpha+1)} \tag{9}$$

for $t \in (0,1)$, where α is a positive real number such that $\alpha+t > 1$ and $q \in (0,1)$.

Further, Nantomah [7] established the following related results.

$$\frac{k^{-\frac{t}{k}}e^{-t\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha)}{\Gamma_k(\alpha)} \le \frac{\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)} \le \frac{k^{\frac{1-t}{k}}e^{(1-t)\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+1)}{\Gamma_k(\alpha+1)} \tag{10}$$

for $t \in (0,1)$, $k \ge 1$ where α is a positive real number.

Our objective is to establish and prove some results similar to (8), (9) and (10).

2. Preliminaries

We present the following auxiliary results.

Lemma 2.1. *The function $\psi_p(t)$ as defined in (3) has the following series representation.*

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t} \tag{11}$$

Proof. See [4].

Lemma 2.2. *The function $\psi_q(t)$ as defined in (5) has the following series representation.*

$$\psi_q(t) = -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{t+n}}{1-q^{t+n}} \tag{12}$$

Proof. See [2].

Lemma 2.3. *The function $\psi_k(t)$ as defined in (7) also has the following series representation.*

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{13}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [6]

Lemma 2.4. *Let $t>0$. Then,*

$$-\frac{\ln k-\gamma}{k}+\ln p+\frac{1}{t}+\psi_k(t)-\psi_p(t)>0$$

Proof. Using the series representations in equations (11) and (13) we have,

$$-\frac{\ln k-\gamma}{k}+\ln p+\frac{1}{t}+\psi_k(t)-\psi_p(t)=\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}+\sum_{n=0}^{p}\frac{1}{(n+t)}>0$$

Lemma 2.5. *Let α be a positive real number such that $\alpha+t>0$. Then,*

$$-\frac{\ln k-\gamma}{k}+\ln p+\frac{1}{\alpha+t}+\psi_k(\alpha+t)-\psi_p(\alpha+t)>0$$

Proof. Follows directly from Lemma 2.4 by replacing t with $\alpha+t$.

Lemma 2.6. *Let $t>0$. Then,*

$$-\frac{\ln k-\gamma}{k}-\ln(1-q)+\frac{1}{t}+\psi_k(t)-\psi_q(t)>0$$

Proof. Using the series representations in equations (12) and (13) we have,

$$-\frac{\ln k-\gamma}{k}-\ln(1-q)+\frac{1}{t}+\psi_k(t)-\psi_q(t)=\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}-\ln q\sum_{n=0}^{\infty}\frac{q^{x+n}}{1-q^{x+n}}>0$$

Lemma 2.7. *Let α be a positive real number such that $\alpha+t>0$. Then,*

$$-\frac{\ln k-\gamma}{k}-\ln(1-q)+\frac{1}{\alpha+t}+\psi_k(\alpha+t)-\psi_q(\alpha+t)>0$$

Proof. Follows directly from Lemma 2.6 by replacing t with $\alpha+t$.

3. Main Results

We now state and prove the results of this paper.

Theorem 3.1. *Define a function Ω by*

$$\Omega(t)=\frac{(\alpha+t)e^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{p^{-t}\Gamma_p(\alpha+t)},\quad t\in(0,\infty),\ k>0,\ p\in N. \tag{14}$$

where α is a positive real number. Then Ω is increasing on $t\in(0,\infty)$ and the inequality

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)p^t\Gamma_p(\alpha)}<\frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)}<\frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)p^{t-1}\Gamma_p(\alpha+1)} \tag{15}$$

holds for every $t\in(0,1)$.

Proof. Let $u(t) = \ln \Omega(t)$ for every $t \in (0, \infty)$. Then,

$$u(t) = \ln \frac{(\alpha+t)e^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{p^{-t}\Gamma_p(\alpha+t)}$$
$$= \ln(\alpha+t) + t\ln p - t(\frac{\ln k-\gamma}{k}) + \ln\Gamma_k(\alpha+t) - \ln\Gamma_p(\alpha+t)$$

Then,

$$u'(t) = -\frac{\ln k-\gamma}{k} + \ln p + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_p(\alpha+t) > 0. \quad \text{(by Lemma 2.5)}$$

That implies u is increasing on $t \in (0, \infty)$. Hence Ω is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have,

$$\Omega(0) < \Omega(t) < \Omega(1) \qquad \text{yielding,}$$

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)p^t\Gamma_p(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)p^{t-1}\Gamma_p(\alpha+1)}.$$

Corollary 3.2. *If $t \in [1, \infty)$, then the following inequality holds.*

$$\frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)p^{t-1}\Gamma_p(\alpha+1)} \leq \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)}$$

Proof. If $t \in [1, \infty)$, then we have $\Omega(1) \leq \Omega(t)$ yielding the result.

Theorem 3.3. *Define a function ϕ by*

$$\phi(t) = \frac{(\alpha+t)e^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{(1-q)^t\Gamma_q(\alpha+t)}, \quad t \in (0,\infty),\ k>0,\ q \in (0,1). \tag{16}$$

where α is a positive real number. Then ϕ is increasing on $t \in (0, \infty)$ and the inequality

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)(1-q)^{-t}\Gamma_q(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)(1-q)^{1-t}\Gamma_q(\alpha+1)} \tag{17}$$

holds for every $t \in (0, 1)$.

Proof. Let $v(t) = \ln \phi(t)$ for every $t \in (0, \infty)$. Then,

$$v(t) = \ln \frac{(\alpha+t)e^{-t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+t)}{(1-q)^t\Gamma_q(\alpha+t)}$$
$$= \ln(\alpha+t) - t\ln(1-q) - t(\frac{\ln k-\gamma}{k}) + \ln\Gamma_k(\alpha+t) - \ln\Gamma_q(\alpha+t)$$

Then,

$$v'(t) = -\frac{\ln k - \gamma}{k} - \ln(1-q) + \frac{1}{\alpha+t} + \psi_k(\alpha+t) - \psi_q(\alpha+t) > 0. \text{ (by Lemma 2.7)}$$

That implies v is increasing on $t \in (0,\infty)$. Hence ϕ is increasing on $t \in (0,\infty)$ and for every $t \in (0,1)$ we have,

$$\phi(0) < \phi(t) < \phi(1) \qquad \text{yielding,}$$

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)(1-q)^{-t}\Gamma_q(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)(1-q)^{1-t}\Gamma_q(\alpha+1)}.$$

Corollary 3.4. *If $t \in [1,\infty)$, then the following inequality holds.*

$$\frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)(1-q)^{1-t}\Gamma_q(\alpha+1)} \leq \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)}$$

Proof. If $t \in [1,\infty)$, then we have $\phi(1) \leq \phi(t)$ yielding the result.

Remark 3.5. This paper is a corrected version of the paper [8].

References

[1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pachhammer k-symbol*, Divulgaciones Matemtícas, **15**(2)(2007), 179-192.
[2] V. Krasniqi, T. Mansour and A. Sh. Shabani, *Some Monotonicity Properties and Inequalities for Γ and ζ Functions*, Mathematical Communications, **15**(2)(2010), 365-376.
[3] V. Krasniqi and F. Merovci, *Logarithmically completely monotonic functions involving the generalized gamma function*, Le Matematiche, **LXV**(2010), 15-23.
[4] V. Krasniqi, A. Sh. Shabani, *Convexity Properties and Inequalities for a Generalized Gamma Function*, Applied Mathematics E-Notes, **10**(2010), 27-35.
[5] T. Mansour, *Some inequalities for the q-Gamma Function*, J. Ineq. Pure Appl. Math., **9**(1)(2008), Art. 18.
[6] F. Merovci, *Power Product Inequalities for the Γ_k Function*, Int. Journal of Math. Analysis, **4**(21)(2010), 1007-1012.
[7] K. Nantomah, *Inequalities for the Psi and k-Gamma functions*, arXiv.org, Available online at http://arxiv.org/abs/1602.05117v1
[8] K. Nantomah and M. M. Iddrisu, *Some Inequalities involving the Ratio of Gamma Functions*, Int. Journal of Math. Analysis, **8**(12)(2014), 555-560.

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