Received 22/02/16

INEQUALITIES FOR OPERATOR NONCOMMUTATIVE PERSPECTIVES OF CONVEX FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some inequalities for operator noncommutative perspectives of convex functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

1. Introduction

If $\Phi : I \to \mathbb{R}$ is a convex function on the real interval I and T is a selfadjoint operator on the complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with the spectrum $\operatorname{Sp}(T) \subset \mathring{I}$ the interior of I, then we have the following ∞

$$\langle \Phi(T)x, x\rangle \geq \Phi(\langle Tx, x\rangle) \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$.

For various Jensen type inequalities for functions of selfadjoint operators, see the recent monograph [1] and the references therein.

Let Φ be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_\Phi(B, A)$ by setting

$$\mathcal{P}_\Phi(B, A) := A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_\Phi(B, A) = A\Phi\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

It is well known that (see [7] and [6] or [8]), if Φ is an *operator convex function* defined in the positive half-line, then the mapping

$$(B, A) \to \mathcal{P}_\Phi(B, A)$$

defined in pairs of positive definite operators, is convex.

In the recent paper [2] we established the following reverse inequality for the perspective $\mathcal{P}_\Phi(B, A)$.

Let $\Phi : [m, M] \to \mathbb{R}$ be a *convex function* on the real interval $[m, M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA \leq B \leq MA, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60,.26D15, 26D10.

Key words and phrases. Young's Inequality, Convex functions, Arithmetic mean-Geometric mean inequality.

then we have

$$0 \leq \frac{1}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]-\mathcal{P}_{\Phi}\left(B,A\right) \tag{1.3}$$
$$\leq \frac{\Phi'_{-}\left(M\right)-\Phi'_{+}\left(m\right)}{M-m}\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right)$$
$$\leq \frac{1}{4}\left(M-m\right)\left[\Phi'_{-}\left(M\right)-\Phi'_{+}\left(m\right)\right]A.$$

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq \Phi''\left(t\right) \leq D \text{ for any } t \in \mathring{J}. \tag{1.4}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.2) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well

$$\frac{1}{2}d\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right) \tag{1.5}$$
$$\leq \frac{1}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]-\mathcal{P}_{\Phi}\left(B,A\right)$$
$$\leq \frac{1}{2}D\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right).$$

If $d > 0$, then the first inequality in (1.5) is better than the same inequality in (1.3).

In this paper we obtain some new inequalities for operator noncommutative perspectives of convex functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

2. Operator Inequalities for Perspectives

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $\Phi : I \to \mathbb{R}$ is a convex function on I. Then Φ is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $\Phi'_{-}(x) \leq \Phi'_{+}(x) \leq \Phi'_{-}(y) \leq \Phi'_{+}(y)$ which shows that both Φ'_{-} and Φ'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $\Phi : I \to \mathbb{R}$, the subdifferential of Φ denoted by $\partial\Phi$ is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$\Phi\left(x\right) \geq \Phi\left(a\right)+\left(x-a\right)\varphi\left(a\right) \text{ for any } x, a \in I. \tag{2.1}$$

It is also well known that if Φ is convex on I, then $\partial\Phi$ is nonempty, Φ'_{-}, $\Phi'_{+} \in \partial\Phi$ and if $\varphi \in \partial\Phi$, then

$$\Phi'_{-}\left(x\right) \leq \varphi\left(x\right) \leq \Phi'_{+}\left(x\right) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If Φ is differentiable and convex on $\mathring{I}$, then $\partial\Phi = \{\Phi'\}$.

We have:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I, A a positive invertible operator and B a selfadjoint operator such that*

$$Am \leq B \leq MA \tag{2.2}$$

with $[m, M] \subset \mathring{I}$, *for some real numbers* m, M *with* $m < M$.

Then for any $\varphi \in \partial\Phi$ *and any* $t \in \mathring{I}$ *we have*

$$\mathcal{P}_{\Phi}(B, A) \geq \Phi(t) A + \varphi(t)(B - tA). \tag{2.3}$$

In particular,

$$\mathcal{P}_{\Phi}(B, A) \geq \Phi\left(\frac{m+M}{2}\right) A + \varphi\left(\frac{m+M}{2}\right)\left(B - \frac{m+M}{2} A\right). \tag{2.4}$$

Proof. From (2.1) we have

$$\Phi(x) \geq \Phi(t) + (x - t)\varphi(t) \tag{2.5}$$

for any $x \in [m, M]$ and $t \in \mathring{I}$.

Using the continuous functional calculus for a selfadjoint operator X with $\operatorname{Sp}(X) \subseteq [m, M] \subset \mathring{I}$ we have from (2.5) in the operator order that

$$\Phi(X) \geq \Phi(t) I + \varphi(t)(X - tI) \tag{2.6}$$

for any $t \in \mathring{I}$.

If the condition (2.2) is valid, then by multiplying both sides by $A^{-1/2}$ we get

$$mI \leq A^{-1/2} B A^{-1/2} \leq MI.$$

Now, if we take $X = A^{-1/2} B A^{-1/2}$ in (2.6), then we get

$$\Phi\left(A^{-1/2} B A^{-1/2}\right) \geq \Phi(t) I + \varphi(t)\left(A^{-1/2} B A^{-1/2} - tI\right) \tag{2.7}$$

for any $t \in \mathring{I}$.

By multiplying both sides of (2.7) with $A^{1/2}$ we get

$$\begin{aligned} A^{1/2}\Phi\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} &\geq \Phi(t) A + \varphi(t) A^{1/2}\left(A^{-1/2} B A^{-1/2} - tI\right) A^{1/2} \\ &= \Phi(t) A + \varphi(t)(B - tA) \end{aligned} \tag{2.8}$$

for any $t \in \mathring{I}$ and the inequality (2.3) is proved. □

Corollary 1. *With the assumptions of Theorem 1, we have for any* $x \in H \setminus \{0\}$ *that*

$$\mathcal{P}_{\Phi}(B, A) \geq \Phi\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right) A + \varphi\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right)\left(B - \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle} A\right). \tag{2.9}$$

In particular,

$$\frac{\langle \mathcal{P}_{\Phi}(B, A) x, x\rangle}{\langle Ax, x\rangle} \geq \Phi\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right). \tag{2.10}$$

Proof. For $x \in H \setminus \{0\}$ we have

$$\begin{aligned} t_{A,B} = \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle} &= \frac{\left\langle A^{1/2}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} x, x\right\rangle}{\left\langle A^{1/2} x, A^{1/2} x\right\rangle} \\ &= \frac{\left\langle \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} x, A^{1/2} x\right\rangle}{\left\langle A^{1/2} x, A^{1/2} x\right\rangle} = \frac{\left\langle \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} x, A^{1/2} x\right\rangle}{\left\|A^{1/2} x\right\|^2}. \end{aligned}$$

If we put

$$u = \frac{A^{1/2} x}{\left\|A^{1/2} x\right\|} \neq 0,$$

then $\|u\| = 1$ and

$$t_{A,B} = \left\langle \left(A^{-1/2} B A^{-1/2} \right) u, u \right\rangle \in [m, M] \subset \mathring{I}.$$

By taking $t = t_{A,B}$ in (2.3) we get (2.9).

The inequality (2.9) is equivalent to

$$\begin{aligned} \langle \mathcal{P}_\Phi (B, A) y, y \rangle \geq \Phi \left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \right) \langle Ay, y \rangle \\ + \varphi \left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \right) \left(\langle By, y \rangle - \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \langle Ay, y \rangle \right) \end{aligned} \tag{2.11}$$

for any $y \in H$.

This is an inequality of interest in itself.

In particular, if we take in (2.11) $y = x$, then we get the desired result (2.10). □

We also have:

Corollary 2. *With the assumptions of Theorem 1, we have*

$$\begin{aligned} \mathcal{P}_\Phi (B, A) \geq 2 \left(\frac{1}{M - m} \int_m^M \Phi (t) \, dt \right) A \\ - \frac{1}{M - m} \left[\Phi (M) (MA - B) + \Phi (m) (B - mB) \right]. \end{aligned} \tag{2.12}$$

Proof. If we take the integral mean in the inequality (2.3), then we get

$$\begin{aligned} \mathcal{P}_\Phi (B, A) \geq \left(\frac{1}{M - m} \int_m^M \Phi (t) \, dt \right) A \\ + \left(\frac{1}{M - m} \int_m^M \varphi (t) \, dt \right) B - \left(\frac{1}{M - m} \int_m^M t \varphi (t) \, dt \right) A. \end{aligned} \tag{2.13}$$

Observe that

$$\frac{1}{M - m} \int_m^M \varphi (t) \, dt = \frac{\Phi (M) - \Phi (m)}{M - m}$$

and

$$\begin{aligned} \frac{1}{M - m} \int_m^M t \varphi (t) \, dt &= \frac{1}{M - m} \left[t \Phi (t) |_m^M - \int_m^M \Phi (t) \, dt \right] \\ &= \frac{M \Phi (M) - m \Phi (m)}{M - m} - \frac{1}{M - m} \int_m^M \Phi (t) \, dt \end{aligned}$$

and by (2.13) we get

$$\begin{aligned}\mathcal{P}_{\Phi}(B,A) &\geq \left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A+\frac{\Phi(M)-\Phi(m)}{M-m}B\\&-\left(\frac{M\Phi(M)-m\Phi(m)}{M-m}-\frac{1}{M-m}\int_m^M\Phi(t)\,dt\right)A\\&=2\left(\frac{1}{M-m}\int_m^M\Phi(t)\,dt\right)A\\&-\frac{1}{M-m}\left[\Phi(M)(MA-B)+\Phi(m)(B-mB)\right]\end{aligned}$$

that proves the desired result (2.12). □

The following reverse of inequality (2.3) is as follows:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuously differentiable convex function on $\mathring{I}$, A a positive invertible operator and B a selfadjoint operator such that the condition (2.2) is valid with $[m,M]\subset \mathring{I}$, for some real numbers m, M with $m<M$.*

Then for any $t\in\mathring{I}$ we have

$$\begin{aligned}\mathcal{P}_{\Phi}(B,A) &\leq \Phi(t)A+\mathcal{P}_{\Phi'\ell}(B,A)-t\mathcal{P}_{\Phi'}(B,A)\\&\leq \Phi(t)A+\Phi'(t)(B-tA)+\left[\Phi'_-(M)-\Phi'_+(m)\right]\mathcal{P}_{|\cdot|,t}(B,A),\end{aligned}\tag{2.14}$$

where ℓ is the identity function, i.e. $\ell(t)=t$ and

$$\mathcal{P}_{|\cdot|,t}(B,A):=A^{1/2}\left|A^{-1/2}(B-tA)A^{-1/2}\right|A^{1/2}.$$

In particular, we have

$$\begin{aligned}\mathcal{P}_{\Phi}(B,A) &\leq \Phi\left(\frac{m+M}{2}\right)A+\mathcal{P}_{\Phi'\ell}(B,A)-\frac{m+M}{2}\mathcal{P}_{\Phi'}(B,A)\\&\leq \Phi\left(\frac{m+M}{2}\right)A+\Phi'\left(\frac{m+M}{2}\right)\left(B-\frac{m+M}{2}A\right)\\&+\left[\Phi'_-(M)-\Phi'_+(m)\right]\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\\&\leq \Phi\left(\frac{m+M}{2}\right)A+\Phi'\left(\frac{m+M}{2}\right)\left(B-\frac{m+M}{2}A\right)\\&+\frac{1}{2}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right].\end{aligned}\tag{2.15}$$

Proof. By the gradient inequality we have

$$\Phi'(x)(x-t)+\Phi(t)\geq\Phi(x)\tag{2.16}$$

for any $x\in[m,M]$ and $t\in\mathring{I}$

Using the continuous functional calculus for a selfadjoint operator X with $\operatorname{Sp}(X)\subseteq[m,M]\subset\mathring{I}$ we have from (2.16) in the operator order that

$$\Phi'(X)(X-tI)+\Phi(t)I\geq\Phi(X)\tag{2.17}$$

for any $t\in\mathring{I}$.

Now, if we take $X=A^{-1/2}BA^{-1/2}$ in (2.17), then we get

$$\Phi'\left(A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-tI\right)+\Phi(t)I\geq\Phi\left(A^{-1/2}BA^{-1/2}\right)\tag{2.18}$$

for any $t \in \mathring{I}$.

If we multiply both sides of (2.18) by $A^{1/2}$, then we obtain

$$\begin{aligned} &A^{1/2}\Phi'\left(A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-tI\right)A^{1/2}+\Phi(t)A \\ &\geq A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \end{aligned} \tag{2.19}$$

for any $t \in \mathring{I}$.

Since

$$\begin{aligned} &A^{1/2}\Phi'\left(A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-tI\right)A^{1/2} \\ &= \mathcal{P}_{\Phi'\ell}(B,A) - t\mathcal{P}_{\Phi'}(B,A), \end{aligned}$$

then by (2.19) we get the first inequality in (2.14).

Now, observe also that

$$\begin{aligned} &A^{1/2}\Phi'\left(A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-tI\right)A^{1/2}+\Phi(t)A \\ &= A^{1/2}\left[\Phi'\left(A^{-1/2}BA^{-1/2}\right)-\Phi'(t)I\right]\left(A^{-1/2}BA^{-1/2}-tI\right)A^{1/2} \\ &+ \Phi'(t)(B-tA)+\Phi(t)A \end{aligned}$$

for any $t \in \mathring{I}$.

Since Φ' is nondecreasing on $\mathring{I}$ we have for any $x \in [m,M]$ and $t \in \mathring{I}$ that

$$\begin{aligned} 0 &\leq (\Phi'(x)-\Phi'(t))(x-t) = |(\Phi'(x)-\Phi'(t))(x-t)| \\ &= |\Phi'(x)-\Phi'(t)|\,|x-t| \leq \left[\Phi'_-(M)-\Phi'_+(m)\right]|x-t|, \end{aligned}$$

which, as above, implies in the operator order that

$$\begin{aligned} &A^{1/2}\left[\Phi'\left(A^{-1/2}BA^{-1/2}\right)-\Phi'(t)I\right]\left(A^{-1/2}BA^{-1/2}-tI\right)A^{1/2} \\ &\leq \left[\Phi'_-(M)-\Phi'_+(m)\right]A^{1/2}\left|A^{-1/2}BA^{-1/2}-tI\right|A^{1/2} \\ &= \left[\Phi'_-(M)-\Phi'_+(m)\right]A^{1/2}\left|A^{-1/2}(B-tA)A^{-1/2}\right|A^{1/2}. \end{aligned}$$

This proves the second inequality in (2.14).

We need to prove only the last part of (2.15).

Since $x \in [m,M]$, then $\left|x-\frac{m+M}{2}\right| \leq \frac{1}{2}(M-m)$ that implies in the operator order

$$\left|A^{-1/2}BA^{-1/2}-\frac{m+M}{2}I\right| \leq \frac{1}{2}(M-m)I,$$

which by multiplication on both sides with $A^{1/2}$ gives that

$$\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A) \leq \frac{1}{2}(M-m)A.$$

□

Corollary 3. *With the assumptions of Theorem 1, we have for any $x \in H \setminus \{0\}$ that*

$$\mathcal{P}_{\Phi}(B,A) \leq \Phi\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)A + \mathcal{P}_{\Phi'\ell}(B,A) - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\mathcal{P}_{\Phi'}(B,A) \tag{2.20}$$
$$\leq \Phi\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)A + \Phi'\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\left(B - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)$$
$$+ \left[\Phi'_-(M) - \Phi'_+(m)\right]\mathcal{P}_{|\cdot|,\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}}(B,A).$$

In particular

$$\langle \mathcal{P}_{\Phi}(B,A)x,x\rangle \tag{2.21}$$
$$\leq \Phi\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle + \langle \mathcal{P}_{\Phi'\ell}(B,A)x,x\rangle - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\langle \mathcal{P}_{\Phi'}(B,A)x,x\rangle$$
$$\leq \Phi\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle + \left[\Phi'_-(M) - \Phi'_+(m)\right]\left\langle \mathcal{P}_{|\cdot|,\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}}(B,A)x,x\right\rangle$$

for any $x \in H \setminus \{0\}$.

We also have:

Corollary 4. *With the assumptions of Theorem 1, we have*

$$\mathcal{P}_{\Phi}(B,A) \leq \left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A + \mathcal{P}_{\Phi'\ell}(B,A) - \frac{m+M}{2}\mathcal{P}_{\Phi'}(B,A) \tag{2.22}$$
$$\leq 2\left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A$$
$$- \frac{1}{M-m}\left[\Phi(M)(MA-B) + \Phi(m)(B-mB)\right]$$
$$+ \left[\Phi'_-(M) - \Phi'_+(m)\right]\frac{1}{M-m}\int_m^M \mathcal{P}_{|\cdot|,t}(B,A)\,dt.$$

Proof. If we take the integral mean in (2.14), then we get

$$\mathcal{P}_{\Phi}(B,A) \leq \left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A + \mathcal{P}_{\Phi'\ell}(B,A) - \frac{m+M}{2}\mathcal{P}_{\Phi'}(B,A) \tag{2.23}$$
$$\leq \left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A + \frac{1}{M-m}\int_m^M \Phi'(t)(B-tA)\,dt$$
$$+ \left[\Phi'_-(M) - \Phi'_+(m)\right]\frac{1}{M-m}\int_m^M \mathcal{P}_{|\cdot|,t}(B,A)\,dt.$$

Since, as in the proof of Corollary 2, we have

$$\frac{1}{M-m}\int_m^M \Phi'(t)(B-tA)\,dt = \left(\frac{1}{M-m}\int_m^M \Phi(t)\,dt\right)A$$
$$- \frac{1}{M-m}\left[\Phi(M)(MA-B) + \Phi(m)(B-mB)\right],$$

then by (2.23) we get the last part of (2.22). □

3. Applications for Operator Geometric Mean

Assume that A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators [18]

$$A \nabla_\nu B := (1-\nu) A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A \sharp_\nu B := A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

The definition $A \sharp_\nu B$ can be extended accordingly for any real number ν.

The following inequality is well as the operator *Young inequality* or operator *ν-weighted arithmetic-geometric mean inequality*:

$$A \sharp_\nu B \le A \nabla_\nu B \text{ for all } \nu \in [0,1]. \tag{3.1}$$

For recent results on operator Young inequality see [11]-[14], [15] and [23]-[24].

For $x \neq y$ and $p \in \mathbb{R} \setminus \{-1, 0\}$, we define the *$p$-logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)} \right]^{1/p}.$$

In fact the singularities at $p = -1$, 0 are removable and L_p can be defined for $p = -1$, 0 so as to make $L_p(x,y)$ a continuous function of p. In the limit as $p \to 0$ we obtain the *identric mean* $I(x,y)$, given by

$$I(x,y) := \frac{1}{e} \left(\frac{y^y}{x^x} \right)^{1/(y-x)}, \tag{3.2}$$

and in the case $p \to -1$ the *logarithmic mean* $L(x,y)$, given by

$$L(x,y) := \frac{y-x}{\ln y - \ln x}.$$

In each case we define the mean as x when $y = x$, which occurs as the limiting value of $L_p(x,y)$ for $y \to x$.

If we consider the continuous function $f_\nu : [0,\infty) \to [0,\infty)$, $f_\nu(x) = x^\nu$ then the operator *ν-weighted arithmetic-geometric mean* can be interpreted as the perspective $\mathcal{P}_{f_\nu}(B,A)$, namely

$$\mathcal{P}_{f_\nu}(B,A) = A \sharp_\nu B.$$

Consider the convex function $f = -f_\nu$. Then by applying the inequalities (2.3) and (2.4) we have

$$A \sharp_\nu B \le (1-\nu) t^\nu A + \nu t^{\nu-1} B = (t^\nu A) \nabla_\nu \left(t^{\nu-1} B \right), \tag{3.3}$$

for any $t > 0$ and $\nu \in [0,1]$, and

$$A \sharp_\nu B \le (1-\nu) \left(\frac{m+M}{2} \right)^\nu A + \nu \left(\frac{m+M}{2} \right)^{\nu-1} B \tag{3.4}$$

for any $\nu \in [0,1]$, provided the condition (2.2) is valid.

From (2.9) and (2.10) we have for any $x \in H \setminus \{0\}$ and $\nu \in [0,1]$ that

$$A \sharp_\nu B \le (1-\nu) \left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \right)^\nu A + \nu \left(\frac{\langle Ax, x \rangle}{\langle Bx, x \rangle} \right)^{1-\nu} B \tag{3.5}$$

and

$$\langle A\sharp_\nu Bx,x\rangle \le \langle Ax,x\rangle^{1-\nu}\langle Bx,x\rangle^{\nu}, \tag{3.6}$$

for any $\nu\in[0,1]$.

The Young's inequality (3.1) can be written as

$$\langle A\sharp_\nu Bx,x\rangle \le (1-\nu)\langle Ax,x\rangle+\nu\langle Bx,x\rangle \tag{3.7}$$

for any $x\in H$.

By utilizing the scalar arithmetic mean-geometric mean inequality we also have

$$\langle Ax,x\rangle^{1-\nu}\langle Bx,x\rangle^{\nu} \le (1-\nu)\langle Ax,x\rangle+\nu\langle Bx,x\rangle \tag{3.8}$$

for any $x\in H$.

Therefore by (3.6) and (3.8) we have the following vector inequality improving (3.7)

$$\langle A\sharp_\nu Bx,x\rangle \le \langle Ax,x\rangle^{1-\nu}\langle Bx,x\rangle^{\nu} \le (1-\nu)\langle Ax,x\rangle+\nu\langle Bx,x\rangle \tag{3.9}$$

for any $x\in H$.

From (2.12) we have

$$A\sharp_\nu B \le 2L_\nu^\nu(m,M)A-\frac{1}{M-m}\left[M^\nu\left(MA-B\right)+m^\nu\left(B-mB\right)\right] \tag{3.10}$$

for any $\nu\in(0,1)$.

From the inequality (2.14) we have for any $t>0$ and A, B positive invertible operators that

$$\begin{aligned} A\sharp_\nu B &\ge t^\nu A+\nu A\sharp_\nu B-\nu tA\sharp_{\nu-1}B \\ &\ge t^\nu A+\nu t^{\nu-1}\left(B-tA\right)+\nu\left(M^{\nu-1}-m^{\nu-1}\right)\mathcal{P}_{|\cdot|,t}\left(B,A\right), \end{aligned} \tag{3.11}$$

for any $\nu\in[0,1]$.

From the first inequality in (3.11) we have

$$A\sharp_\nu B \ge \frac{1}{1-\nu}\left(t^\nu A-\nu tA\sharp_{\nu-1}B\right)$$

for any $\nu\in(0,1)$ and $t>0$.

If A and B satisfy the condition (2.2), then by (2.15) we have

$$\begin{aligned} A\sharp_\nu B &\ge \left(\frac{m+M}{2}\right)^\nu A+\nu A\sharp_\nu B-\nu\frac{m+M}{2}A\sharp_{\nu-1}B \\ &\ge (1-\nu)\left(\frac{m+M}{2}\right)^\nu A+\nu\left(\frac{m+M}{2}\right)^{\nu-1}B \\ &+\nu\left(M^{\nu-1}-m^{\nu-1}\right)\mathcal{P}_{|\cdot|,\frac{m+M}{2}}\left(B,A\right) \\ &\ge (1-\nu)\left(\frac{m+M}{2}\right)^\nu A+\nu\left(\frac{m+M}{2}\right)^{\nu-1}B \\ &+\frac{1}{2}\nu\left(M-m\right)\left(M^{\nu-1}-m^{\nu-1}\right). \end{aligned} \tag{3.12}$$

From the last inequality in (3.12) we get

$$\begin{aligned}\frac{1}{2}\nu\left(M-m\right)\left(\frac{M^{1-\nu}-m^{1-\nu}}{m^{1-\nu}M^{1-\nu}}\right) \\ \geq\left(1-\nu\right)\left(\frac{m+M}{2}\right)^{\nu}A+\nu\left(\frac{m+M}{2}\right)^{\nu-1}B-A\sharp_{\nu}B\geq 0,\end{aligned} \tag{3.13}$$

for any $\nu\in[0,1]$, which provides a simple reverse for (3.4).

4. Applications for Relative Operator Entropy

Kamei and Fujii [9], [10] defined the *relative operator entropy* $S\left(A|B\right)$, for positive invertible operators A and B, by

$$S\left(A|B\right):=A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \tag{4.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [22].

For some recent results on relative operator entropy see [4]-[5], [16]-[17] and [19]-[20].

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the permanent of ln, namely

$$\mathcal{P}_{\ln}\left(B,A\right)=S\left(A|B\right).$$

If we use the inequalities (2.3) and (2.4) for the convex function $f=-\ln$ we have

$$S\left(A|B\right)\leq\left(\ln t\right)A-A+t^{-1}B, \tag{4.2}$$

for any $t>0$ and A, B positive invertible operators.

In particular, if A, B satisfy the condition (2.2), then

$$S\left(A|B\right)\leq\left[\ln\left(\frac{m+M}{2}\right)\right]A+\left(\frac{m+M}{2}\right)^{-1}\left(B-\frac{m+M}{2}A\right). \tag{4.3}$$

From the inequalities (2.9) and (2.10) we have

$$S\left(A|B\right)\leq\ln\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)A+\frac{\langle Ax,x\rangle}{\langle Bx,x\rangle}B-A \tag{4.4}$$

and

$$\langle S\left(A|B\right)x,x\rangle\leq\langle Ax,x\rangle\ln\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right), \tag{4.5}$$

for any $x\in H$, $x\neq 0$.

The following inequality for the relative operator entropy is well known

$$S\left(A|B\right)\leq B-A \tag{4.6}$$

for any A, B positive invertible operators.

This inequality is equivalent to

$$\langle S\left(A|B\right)x,x\rangle\leq\langle Bx,x\rangle-\langle Ax,x\rangle \tag{4.7}$$

for any $x\in H$.

We know the following elementary inequality that holds for the logarithm

$$\ln t\leq t-1\text{ for any }t>0.$$

If we take in this inequality $t = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle} > 0$, $x \in H$, $x \neq 0$ and multiply with $\langle Ax, x\rangle > 0$, then we get

$$\langle Ax, x\rangle \ln\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right) \leq \langle Bx, x\rangle - \langle Ax, x\rangle \tag{4.8}$$

for any $x \in H$, $x \neq 0$.

Therefore, by (4.5) and (4.8) we have

$$\langle S(A|B)x, x\rangle \leq \langle Ax, x\rangle \ln\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right) \leq \langle Bx, x\rangle - \langle Ax, x\rangle$$

for any $x \in H$, $x \neq 0$ that is an improvement of (4.7).

From (2.12) we also have

$$\begin{aligned} S(A|B) &\leq 2\left[\ln I(m, M)\right] A \\ &\quad - \frac{1}{M-m}\left[\ln M(MA - B) + \ln M(B - mB)\right], \end{aligned} \tag{4.9}$$

where $I(m, M)$ is the identric mean defined in (3.2) and

$$\frac{1}{M-m}\int_m^M \ln t dt = \ln I(m, M).$$

From the inequality (2.14) we have

$$\begin{aligned} S(A|B) &\geq (\ln t) A + A - tAB^{-1}A \\ &\geq (\ln t) A - A + t^{-1}B - \frac{M-m}{mM}\mathcal{P}_{|\cdot|, t}(B, A), \end{aligned} \tag{4.10}$$

for any $t > 0$, where A, B are positive invertible operators satisfying the condition (2.2)

From (2.15) we also have

$$\begin{aligned} S(A|B) &\geq \left[\ln\left(\frac{m+M}{2}\right)\right] A + A - \frac{m+M}{2} AB^{-1}A \\ &\geq \left[\ln\left(\frac{m+M}{2}\right)\right] A + \left(\frac{m+M}{2}\right)^{-1}\left(B - \frac{m+M}{2}A\right) \\ &\quad - \frac{M-m}{mM}\mathcal{P}_{|\cdot|, \frac{m+M}{2}}(B, A) \\ &\geq \left[\ln\left(\frac{m+M}{2}\right)\right] A + \left(\frac{m+M}{2}\right)^{-1}\left(B - \frac{m+M}{2}A\right) \\ &\quad - \frac{1}{2}\frac{(M-m)^2}{mM}, \end{aligned} \tag{4.11}$$

provided A, B are positive invertible operators satisfying the condition (2.2).

From the last part of (4.11) we get

$$\begin{aligned} &\frac{1}{2}\frac{(M-m)^2}{mM} \\ &\geq \left[\ln\left(\frac{m+M}{2}\right)\right] A + \left(\frac{m+M}{2}\right)^{-1}\left(B - \frac{m+M}{2}A\right) - S(A|B) \\ &\geq 0 \end{aligned} \tag{4.12}$$

that provides a simple reverse of (4.3).

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa