Received 22/02/16

INEQUALITIES FOR OPERATOR NONCOMMUTATIVE PERSPECTIVES OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS WITH APPLICATIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some inequalities for operator perspectives of continuously differentiable functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

1. Introduction

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset I$. Then by using the continuous functional calculus, we can define the perspective $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset I$.

It is well known that (see [7] and [6] or [8]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A) \to \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is convex.

In the recent paper [1] we established the following reverse inequality for the perspective $\mathcal{P}_f(B,A)$.

Let $f : [m,M] \to \mathbb{R}$ be a *convex function* on the real interval $[m,M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA \le B \le MA, \tag{1.1}$$

then we have

$$\begin{aligned} 0 &\le \frac{1}{M-m}\left[f(m)(MA-B) + f(M)(B-mA)\right] - \mathcal{P}_f(B,A) \\ &\le \frac{f'_-(M) - f'_+(m)}{M-m}\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]A. \end{aligned} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60,.26D15, 26D10.

Key words and phrases. Operator noncommutative perspectives, Young's inequality, Convex functions, Arithmetic mean-Geometric mean inequality.

Let $f : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \le f''(t) \le D \text{ for any } t \in \mathring{J}. \tag{1.3}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well [2]

$$\begin{aligned} &\frac{1}{2}d\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right) \\ &\le \frac{1}{M-m}\left[f(m)(MA - B) + f(M)(B - mA)\right] - \mathcal{P}_f(B, A) \\ &\le \frac{1}{2}D\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right). \end{aligned} \tag{1.4}$$

If $d > 0$, then the first inequality in (1.4) is better than the same inequality in (1.2).

Motivated by the above results, we give in this paper some integral representations for the perspective $\mathcal{P}_f(B, A)$ of continously differentiable functions f and apply them in obtaining various norm and vector inequalities including the cases of *weighted operator geometric mean* and *operator relative entropy.*

2. Some Identities

We have the following lemma that is of interest in itself:

Lemma 1. *Let $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$ the interior of I. If T is a selfadjoint operator such that the spectrum $\operatorname{Sp}(T) \subset \mathring{I}$, then for any $a \in I$ and $\mu \in \mathbb{C}$ we have*

$$\begin{aligned} f(T) &= f(a) I + \mu (T - aI) \\ &\quad + (T - aI) \int_0^1 \left[f'((1-t) aI + tT) - \mu I\right] dt. \end{aligned} \tag{2.1}$$

In particular, for any $x \in H$, $\|x\| = 1$ we have

$$\begin{aligned} f(T) &= f(\langle Tx, x\rangle) I + \mu (T - \langle Tx, x\rangle I) \\ &\quad + (T - \langle Tx, x\rangle I) \int_0^1 \left[f'((1-t)\langle Tx, x\rangle I + tT) - \mu I\right] dt. \end{aligned} \tag{2.2}$$

Proof. We have, by the change of variable $[0, 1] \ni t \longmapsto s = (1-t) a + tb$ that

$$f(b) - f(a) = \int_a^b f'(s)\, ds = (b-a)\int_0^1 f'((1-t) a + tb)\, dt$$

giving that

$$f(b) = f(a) + (b-a)\int_0^1 f'((1-t) a + tb)\, dt \tag{2.3}$$

for any a, $b \in \mathring{I}$.

Fix $a \in I$, and take the real numbers m, M such that $\operatorname{Sp}(T) \subseteq [m, M] \subset \mathring{I}$. If $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the operator T, then by the spectral representation theorem (SRT) we have

$$f(T)=\int_{m-0}^{M} f(\lambda)\,dE_\lambda := \lim_{\varepsilon\to 0+}\int_{m-\varepsilon}^{M} f(\lambda)\,dE_\lambda,$$

where the integral is taken in the Riemann-Stieltjes sense.

Let $\varepsilon>0$ small enough such that $[m-\varepsilon,M]\subset \mathring{I}$, then by integrating the equality (2.3) on the interval $[m-\varepsilon,M]$ and using the Fubini theorem, we have

$$\begin{aligned}\int_{m-\varepsilon}^{M} f(\lambda)\,dE_\lambda &\\ &= f(a)\int_{m-\varepsilon}^{M} dE_\lambda + \int_{m-\varepsilon}^{M}\left(\int_0^1 (\lambda-a)\,f'((1-t)\,aI+t\lambda)\,dt\right)dE_\lambda \\ &= f(a)\int_{m-\varepsilon}^{M} dE_\lambda + \int_0^1\left(\int_{m-\varepsilon}^{M}(\lambda-a)\,f'((1-t)\,aI+t\lambda)\,dE_\lambda\right)dt\end{aligned} \tag{2.4}$$

for any $a\in\mathring{I}$.

Taking the limit over $\varepsilon\to 0+$ in (2.4) we get by SRT that

$$\begin{aligned}f(T) &= f(a)\,I+\int_0^1 (T-aI)\,f'((1-t)\,aI+tT)\,dt \\ &= f(a)\,I+(T-aI)\int_0^1[\mu I+f'((1-t)\,aI+tT)-\mu I]\,dt \\ &= f(a)\,I+\mu(T-aI)+(T-aI)\int_0^1[f'((1-t)\,aI+tT)-\mu I]\,dt\end{aligned} \tag{2.5}$$

and the identity (2.1) is obtained.

Now, since $\operatorname{Sp}(T)\subseteq[m,M]\subset\mathring{I}$ where m, M is as above, then $\langle Tx,x\rangle\in[m,M]$ for any $x\in H$, $\|x\|=1$ and by taking $a=\langle Tx,x\rangle$ in (2.1) we get the desired result (2.1). □

Corollary 1. *With the assumptions of Theorem 1 we have for any $x,y\in H$, that*

$$\begin{aligned}\langle f(T)\,x,y\rangle &= f(a)\,\langle x,y\rangle+\mu\,(\langle Tx,y\rangle-a\,\langle x,y\rangle) \\ &\quad+\int_0^1\langle f'((1-t)\,aI+tT)\,x-\mu x,Ty-ay\rangle\,dt\end{aligned} \tag{2.6}$$

and, in particular,

$$\begin{aligned}\langle f(T)\,z,z\rangle &= f(a)+\mu\,(\langle Tz,z\rangle-a) \\ &\quad+\int_0^1\langle f'((1-t)\,aI+tT)\,z-\mu z,Tz-az\rangle\,dt\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\langle f(T)\,z,z\rangle &= f(\langle Tz,z\rangle) \\ &\quad+\int_0^1\langle f'((1-t)\,\langle Tz,z\rangle\,I+tT)\,z-\mu z,Tz-\langle Tz,z\rangle\,z\rangle\,dt\end{aligned} \tag{2.8}$$

for any $z\in H$ with $\|z\|=1$.

Proof. By using the identity (2.1) we have

$$\begin{aligned}\langle f(T)x,y\rangle &= f(a)\langle x,y\rangle+\mu\langle (T-aI)x,y\rangle\\&+\left\langle \left((T-aI)\int_0^1 [f'((1-t)aI+tT)-\mu I]\,dt\right)x,y\right\rangle\\&= f(a)\langle x,y\rangle+\mu\langle (T-aI)x,y\rangle\\&+\left\langle \left(\int_0^1 [f'((1-t)aI+tT)-\mu I]\,dt\right)x,(T-aI)y\right\rangle\\&= f(a)\langle x,y\rangle+\mu\langle (T-aI)x,y\rangle\\&+\int_0^1 \langle f'((1-t)aI+tT)x-\mu x,(T-aI)y\rangle\,dt\end{aligned}$$

for any $x,y\in H$, which proves the equality (2.6).

The rest is obvious. □

Remark 1. *If T is such that $mI\le T\le MI$ with $[m,M]\subset \mathring{I}$ and $f:I\to\mathbb{C}$ is a continuously differentiable function on $\mathring{I}$, then we also have*

$$f(T)=f\left(\frac{m+M}{2}\right)I+\mu\left(T-\frac{m+M}{2}I\right)+\left(T-\frac{m+M}{2}I\right)\int_0^1\left[f'\left((1-t)\frac{m+M}{2}I+tT\right)-\mu I\right]dt, \tag{2.9}$$

$$\langle f(T)x,y\rangle=f\left(\frac{m+M}{2}\right)\langle x,y\rangle+\mu\left(\langle Tx,y\rangle-\frac{m+M}{2}\langle x,y\rangle\right)+\int_0^1\left\langle f'\left((1-t)\frac{m+M}{2}I+tT\right)x-\mu x,Ty-\frac{m+M}{2}y\right\rangle dt \tag{2.10}$$

and

$$\langle f(T)z,z\rangle=f\left(\frac{m+M}{2}\right)+\mu\left(\langle Tz,z\rangle-\frac{m+M}{2}\right)+\int_0^1\left\langle f'\left((1-t)\frac{m+M}{2}I+tT\right)z-\mu z,Tz-\frac{m+M}{2}z\right\rangle dt, \tag{2.11}$$

for any $x,y\in H$ and any $z\in H$ with $\|z\|=1$.

We say that the function defined an interval I containing the real number 1 is normalized, if $f(1)=0$. In this situation, the equality (2.1) has a simpler form,

$$f(T)=\mu(T-I)+(T-I)\int_0^1[f'((1-t)I+tT)-\mu I]\,dt, \tag{2.12}$$

the equality (2.6) becomes

$$\langle f(T)x,y\rangle=\mu(\langle Tx,y\rangle-\langle x,y\rangle)+\int_0^1\langle f'((1-t)I+tT)x-\mu x,Ty-y\rangle\,dt \tag{2.13}$$

for any $x,y\in H$, while (2.7) can be written as

$$\langle f(T)z,z\rangle=\mu(\langle Tz,z\rangle-1)+\int_0^1\langle f'((1-t)I+tT)z-\mu z,Tz-z\rangle\,dt \tag{2.14}$$

for any $z \in H$ with $\|z\| = 1$.

Corollary 2. *If T is such that $mI \leq T \leq MI$ with $[m, M] \subset \mathring{I}$ and $f : I \to \mathbb{C}$ is a continuously differentiable function on $\mathring{I}$, then we have*

$$
\begin{aligned}
f(T) &= \frac{1}{M-m} \int_m^M f(u)\, du I + \mu \left(T - \frac{m+M}{2} I \right) \\
&\quad + \frac{1}{M-m} \int_m^M \left[(T - uI) \int_0^1 \left[f'((1-t) uI + tT) - \mu I \right] dt \right] du \\
&= \frac{1}{M-m} \int_m^M f(u)\, du I + \mu \left(T - \frac{m+M}{2} I \right) \\
&\quad + \frac{1}{M-m} \int_0^1 \left[\int_m^M (T - uI) \left[f'((1-t) uI + tT) - \mu I \right] du \right] dt.
\end{aligned} \tag{2.15}
$$

The proof follows by taking the integral mean in the equality (2.1) and using Fubini's theorem.

Now, let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2} B A^{-1/2}\right) \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. We define, by the use of continuous functional calculus, the *noncommutative perspective* of f and A, B as

$$
P_f(B, A) := A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}. \tag{2.16}
$$

If $f_\nu : [0, \infty) \to [0, \infty)$, $f_\nu(t) = t^\nu$, $\nu \in [0, 1]$, then

$$
P_{f_\nu}(B, A) := A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^\nu A^{1/2} =: A \sharp_\nu B,
$$

the *weighted operator geometric mean* of the positive invertible operators A and B with the weight ν.

We define the *weighted operator arithmetic mean* by

$$
A \nabla_\nu B := (1 - \nu) A + \nu B, \ \nu \in [0, 1].
$$

It is well known that the following Young's type inequality holds:

$$
A \sharp_\nu B \leq A \nabla_\nu B
$$

for any $\nu \in [0, 1]$.

If we take the function $f = \ln$, then

$$
P_{\ln}(B, A) := A^{1/2} \ln \left(A^{-1/2} B A^{-1/2}\right) A^{1/2} =: S(A|B),
$$

the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [9], [10] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [22].

Theorem 1. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2} B A^{-1/2}\right) \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. Then for any $a \in I$ and $\mu \in \mathbb{C}$ we have*

$$
\begin{aligned}
P_f(B, A) &= f(a) A + \mu (B - aA) \\
&\quad + \left(B A^{-1} - aI\right) \int_0^1 P_{f' - \mu}\left((aA) \nabla_t B, A\right) dt.
\end{aligned} \tag{2.17}
$$

Proof. If we take $T = A^{-1/2}BA^{-1/2}$ in (2.1), then we have

$$\begin{aligned} (2.18)\quad & f\left(A^{-1/2}BA^{-1/2}\right) \\ & = f(a) I + \mu \left(A^{-1/2}BA^{-1/2} - aI\right) \\ & + \left(A^{-1/2}BA^{-1/2} - aI\right) \int_0^1 \left[f'\left((1-t)aI + tA^{-1/2}BA^{-1/2}\right) - \mu I\right] dt \\ & = f(a) I + \mu A^{-1/2}(B - aA) A^{-1/2} \\ & + A^{-1/2}(B - aA) A^{-1/2} \int_0^1 \left[f'\left(A^{-1/2}[(1-t)aA + tB] A^{-1/2}\right) - \mu I\right] dt. \end{aligned}$$

If we multiply both sides of (2.18) by $A^{1/2}$ we get

$$\begin{aligned} & A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2} \\ & = f(a) A + \mu (B - aA) \\ & + (B - aA) A^{-1/2} \int_0^1 \left[f'\left(A^{-1/2}[(1-t)aA + tB] A^{-1/2}\right) - \mu I\right] A^{1/2} dt \\ & = f(a) A + \mu (B - aA) \\ & + \left(BA^{-1} - aI\right) \int_0^1 A^{1/2} \left[f'\left(A^{-1/2}[(1-t)aA + tB] A^{-1/2}\right) - \mu I\right] A^{1/2} dt \end{aligned}$$

and the representation (2.17) is obtained. □

We observe that if $1 \in I$ and the function f is normalized, then the equality (2.17) has a simpler form

$$P_f(B, A) = \mu (B - A) + \left(BA^{-1} - I\right) \int_0^1 P_{f'-\mu}(A \nabla_t B, A)\, dt. \quad (2.19)$$

Corollary 3. *Let A be a positive invertible operator, B a selfadjoint operator such that*

$$mA \le B \le MA \quad (2.20)$$

for some real numbers m, M with $[m, M] \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$. Then for any $\mu \in \mathbb{C}$ we have

$$\begin{aligned} (2.21)\quad P_f(B, A) = & f\left(\frac{m+M}{2}\right) A + \mu \left(B - \frac{m+M}{2} A\right) \\ & + \left(BA^{-1} - \frac{m+M}{2} I\right) \int_0^1 P_{f'-\mu}\left(\left(\frac{m+M}{2} A\right) \nabla_t B, A\right) dt \end{aligned}$$

and

$$(2.22)\quad P_f(B,A) = \left(\frac{1}{M-m}\int_m^M f(u)\,du\right)A + \mu\left(B - \frac{m+M}{2}A\right) + \frac{1}{M-m}\int_m^M \left[\left(BA^{-1}-uI\right)\int_0^1 P_{f'-\mu}\left((uA)\nabla_t B, A\right)dt\right]du$$
$$= \left(\frac{1}{M-m}\int_m^M f(u)\,du\right)A + \mu\left(B - \frac{m+M}{2}A\right) + \frac{1}{M-m}\int_0^1 \left[\int_m^M \left(BA^{-1}-uI\right) P_{f'-\mu}\left((uA)\nabla_t B, A\right)du\right]dt.$$

From the condition (2.20) we have by multiplying both sides with $A^{-1/2}$ that $mI \le A^{-1/2}BA^{-1/2} \le MI$. If we take $T = A^{-1/2}BA^{-1/2}$ and use the inequalities (2.9) and (2.15) we get (2.21) and (2.22).

Corollary 4. *With the assumptions of Theorem 1 we have for any $x \in H$, $x \neq 0$ that*

$$(2.23)\quad P_f(B,A) = f\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)A + \mu\left(B - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right) + \left(BA^{-1} - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}I\right)\int_0^1 P_{f'-\mu}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_t B, A\right)dt$$

and

$$(2.24)\quad \langle P_f(B,A)x,x\rangle = f\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle + \int_0^1 \left\langle P_{f'-\mu}\left(\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\nabla_t B, A\right)x, BA^{-1}x - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}x\right\rangle dt.$$

Proof. Since $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$ then there exists some real numbers m, M such that $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subseteq [m,M] \subset \mathring{I}$.

Let $x \in H$, $x \neq 0$ and put

$$a = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle} = \frac{\left\langle A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}x,x\right\rangle}{\left\langle A^{1/2}x,A^{1/2}x\right\rangle}$$
$$= \frac{\left\langle \left(A^{-1/2}BA^{-1/2}\right)A^{1/2}x,A^{1/2}x\right\rangle}{\left\|A^{1/2}x\right\|^2} = \left\langle \left(A^{-1/2}BA^{-1/2}\right)u,u\right\rangle \in [m,M]$$

where $u = \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|^2} \neq 0$ and $\|u\| = 1$.

Now, by taking $a = \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}$ in (2.17) we get (2.23).

The equality (2.24) follows by (2.23) on taking the inner product $\langle P_f(B,A)x,x\rangle$ and doing the appropriate calculation in the right side. The details are omitted. □

3. Inequalities for Bounded Derivatives

Now, for ϕ, $\Phi \in \mathbb{C}$ and I an interval of real numbers, define the sets of complex-valued functions (see for instance [5])

$$\bar{U}_I(\phi,\Phi) := \left\{ g : I \to \mathbb{C} \,|\, \operatorname{Re}\left[(\Phi - g(t)) \left(\overline{g(t)} - \overline{\phi} \right) \right] \geq 0 \text{ for almost every } t \in I \right\}$$

and

$$\bar{\Delta}_I(\phi,\Phi) := \left\{ g : I \to \mathbb{C} \,|\, \left| g(t) - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \text{ for a.e. } t \in I \right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_I(\phi,\Phi)$ and $\bar{\Delta}_I(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_I(\phi,\Phi) = \bar{\Delta}_I(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - z)(\bar{z} - \phi) \right] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| z - \frac{\phi+\Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - z)(\bar{z} - \phi) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 5. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_I(\phi,\Phi) = \{ g : I \to \mathbb{C} \mid & (\operatorname{Re}\Phi - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\phi) \geq 0 \textit{ for a.e. } t \in I \}. \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi) \geq \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \geq \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_I(\phi,\Phi) := \{ g : I \to \mathbb{C} \mid & \operatorname{Re}(\Phi) \geq \operatorname{Re} g(t) \geq \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \geq \operatorname{Im} g(t) \geq \operatorname{Im}(\phi) \text{ for a.e. } t \in I \}. \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_I(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_I(\phi,\Phi) \subseteq \bar{U}_I(\phi,\Phi). \tag{3.4}$$

We need the following lemma:

Lemma 2. *Let T be a selfadjoint operator and $A \geq 0$. Then we have*

$$-A^{1/2} |T| A^{1/2} \leq A^{1/2} T A^{1/2} \leq A^{1/2} |T| A^{1/2} \tag{3.5}$$

in the operator order, where $|T|$ is the absolute value of T.

We also have

$$\left\| A^{1/2} T A^{1/2} \right\| \leq \left\| A^{1/2} |T| A^{1/2} \right\|. \tag{3.6}$$

Proof. If we use Jensen's operator inequality for the convex function $f(t) = |t|$, then we have

$$|\langle Ty, y\rangle| \leq \langle |T| y, y\rangle$$

for any $y \in H$.

If we take in this inequality $y = A^{1/2}x$, $x \in H$, then we get

$$\left|\left\langle TA^{1/2}x, A^{1/2}x\right\rangle\right| \leq \left\langle |T| A^{1/2}x, A^{1/2}x\right\rangle$$

that is equivalent to

$$\left|\left\langle A^{1/2}TA^{1/2}x, x\right\rangle\right| \leq \left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle \tag{3.7}$$

or to

$$-\left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle \leq \left\langle A^{1/2}TA^{1/2}x, x\right\rangle \leq \left\langle A^{1/2} |T| A^{1/2}x, x\right\rangle$$

for any $x \in H$, which proves the inequality (3.5).

By taking the supremum over $x \in H$, $\|x\| = 1$ in (3.7) we obtain the desired inequality (3.6). □

Theorem 2. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$ and $f : I \to \mathbb{C}$ be a continuously differentiable function on $\mathring{I}$ and such that $f' \in \bar{\Delta}_{\mathring{I}}(\phi, \Phi)$ for some ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$. Then for any $a \in I$ we have*

$$\left\|P_f(B, A) - f(a) A - \frac{\phi + \Phi}{2}(B - aA)\right\| \leq \frac{1}{2}|\Phi - \phi| \left\|BA^{-1} - aI\right\| \|A\|. \tag{3.8}$$

In particular, we have

$$\left\|P_f(B, A) - f\left(\frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}\right) A - \frac{\phi + \Phi}{2}\left(B - \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}A\right)\right\| \tag{3.9}$$
$$\leq \frac{1}{2}|\Phi - \phi| \left\|BA^{-1} - \frac{\langle Bx, x\rangle}{\langle Ax, x\rangle}I\right\| \|A\|$$

for any $x \in H$, $x \neq 0$.

Proof. Using the identity (2.17) for $\mu = \frac{\phi+\Phi}{2}$ and taking the operator norm, then we have

$$\left\|P_f(B, A) - f(a) A - \frac{\phi + \Phi}{2}(B - aA)\right\| \tag{3.10}$$
$$\leq \left\|BA^{-1} - aI\right\|$$
$$\times \left\|\int_0^1 A^{1/2}\left[f'\left(A^{-1/2}\left[(1-t) aA + tB\right] A^{-1/2}\right) - \frac{\phi + \Phi}{2}I\right] A^{1/2}dt\right\|$$
$$\leq \left\|BA^{-1} - aI\right\|$$
$$\times \int_0^1 \left\|A^{1/2}\left[f'\left(A^{-1/2}\left[(1-t) aA + tB\right] A^{-1/2}\right) - \frac{\phi + \Phi}{2}I\right] A^{1/2}\right\| dt$$

for any $a \in I$.

Since $f' \in \bar{\Delta}_{\mathring{I}}(\phi, \Phi)$, then for any $s \in \mathring{I}$ we have

$$\left|f'(s) - \frac{\phi + \Phi}{2}\right| \leq \frac{1}{2}|\Phi - \phi|. \tag{3.11}$$

We observe that

$$\operatorname{Sp}\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)=\operatorname{Sp}\left((1-t)\,aI+tA^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$$

for any $a\in\mathring{I}$ and $t\in[0,1]$, and by the continuous functional calculus we get from (3.11) that

$$\left|f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right|\leq\frac{1}{2}\left|\Phi-\phi\right|I \tag{3.12}$$

for any $a\in\mathring{I}$ and $t\in[0,1]$.

Now, multiplying both sides of (3.12) by $A^{1/2}$, we get

$$A^{1/2}\left|f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right|A^{1/2}\leq\frac{1}{2}\left|\Phi-\phi\right|A$$

and by taking the norm in this inequality, we get

$$\begin{aligned}&\left\|A^{1/2}\left|f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right|A^{1/2}\right\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|A\right\|\end{aligned} \tag{3.13}$$

for any $a\in\mathring{I}$ and $t\in[0,1]$.

Using Lemma 2 we get

$$\begin{aligned}&\left\|A^{1/2}\left[f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right]A^{1/2}\right\|\\&\leq\left\|A^{1/2}\left|f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right|A^{1/2}\right\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|A\right\|\end{aligned}$$

and by (3.10) we obtain the desired result (3.1). □

From Theorem 2 we have the following particular inequalities of interest:

Corollary 6. *Let A be a positive invertible operator, B a selfadjoint operator such that the inequality (2.20) is valid for some real numbers m, M with $[m,M]\subset\mathring{I}$ and $f:I\to\mathbb{C}$ be a continuously differentiable function on $\mathring{I}$ and such that $f'\in\bar{\Delta}_{\mathring{I}}(\phi,\Phi)$ for some ϕ, $\Phi\in\mathbb{C}$, $\phi\neq\Phi$. Then for any $\mu\in\mathbb{C}$ we have*

$$\begin{aligned}&\left\|P_f(B,A)-f\left(\frac{m+M}{2}\right)A-\frac{\phi+\Phi}{2}\left(B-\frac{m+M}{2}A\right)\right\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|BA^{-1}-\frac{m+M}{2}I\right\|\left\|A\right\|\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\left\|P_f(B,A)-\left(\frac{1}{M-m}\int_m^M f(u)\,du\right)A-\frac{\phi+\Phi}{2}\left(B-\frac{m+M}{2}A\right)\right\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\left\|A\right\|\frac{1}{M-m}\int_m^M\left\|BA^{-1}-uI\right\|du.\end{aligned} \tag{3.15}$$

Remark 2. *Since $mA \leq B \leq MA$, then*

$$\left\| B - \frac{m+M}{2} A \right\| \leq \frac{1}{2} (M - m) \|A\|$$

implying that

$$\begin{aligned} \left\| BA^{-1} - \frac{m+M}{2} I \right\| &= \left\| \left(B - \frac{m+M}{2} A \right) A^{-1} \right\| \\ &\leq \left\| B - \frac{m+M}{2} A \right\| \left\| A^{-1} \right\| \\ &\leq \frac{1}{2} (M - m) \|A\| \left\| A^{-1} \right\| . \end{aligned}$$

Therefore by (3.14) we obtain the following simpler (however coarser) inequality

$$\begin{aligned} &\left\| P_f (B, A) - f \left(\frac{m+M}{2} \right) A - \frac{\phi + \Phi}{2} \left(B - \frac{m+M}{2} A \right) \right\| \\ &\leq \frac{1}{4} |\Phi - \phi| (M - m) \|A\|^2 \left\| A^{-1} \right\| . \end{aligned} \tag{3.16}$$

We observe that if $f : [m, M] \to \mathbb{R}$ is a convex function and if $f'_+ (m)$ and $f'_+ (M)$ are finite, then from the above inequalities we can state the following inequalities that provide a large number of examples:

$$\begin{aligned} &\left\| P_f (B, A) - f (a) A - \frac{f'_+ (m) + f'_+ (M)}{2} (B - aA) \right\| \\ &\leq \frac{1}{2} \left[f'_+ (M) - f'_+ (m) \right] \left\| BA^{-1} - aI \right\| \|A\| , \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\left\| P_f (B, A) - f \left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} \right) A - \frac{f'_+ (m) + f'_+ (M)}{2} \left(B - \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} A \right) \right\| \\ &\leq \frac{1}{2} \left[f'_+ (M) - f'_+ (m) \right] \left\| BA^{-1} - \frac{\langle Bx, x \rangle}{\langle Ax, x \rangle} I \right\| \|A\| \end{aligned} \tag{3.18}$$

for any $x \in H$, $x \neq 0$,

$$\begin{aligned} &\left\| P_f (B, A) - f \left(\frac{m+M}{2} \right) A - \frac{f'_+ (m) + f'_+ (M)}{2} \left(B - \frac{m+M}{2} A \right) \right\| \\ &\leq \frac{1}{2} \left[f'_+ (M) - f'_+ (m) \right] \left\| BA^{-1} - \frac{m+M}{2} I \right\| \|A\| \\ &\leq \frac{1}{4} \left[f'_+ (M) - f'_+ (m) \right] (M - m) \|A\|^2 \left\| A^{-1} \right\| \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &\left\| P_f (B, A) - \left(\frac{1}{M - m} \int_m^M f (u) \, du \right) A \right. \\ &\left. - \frac{f'_+ (m) + f'_+ (M)}{2} \left(B - \frac{m+M}{2} A \right) \right\| \\ &\leq \frac{1}{2} \left[\frac{f'_+ (M) - f'_+ (m)}{M - m} \right] \|A\| \int_m^M \left\| BA^{-1} - uI \right\| du. \end{aligned} \tag{3.20}$$

Now, by taking the inner product in the equality (2.17) we have

$$\begin{aligned}\langle P_f(B,A)x,y\rangle &= f(a)\langle Ax,y\rangle+\mu(\langle Bx,y\rangle-a\langle Ax,y\rangle)\\&\quad+\left\langle\left(BA^{-1}-aI\right)\int_0^1 P_{f'-\mu}((aA)\nabla_t B,A)\,dtx,y\right\rangle\\&= f(a)\langle Ax,y\rangle+\mu(\langle Bx,y\rangle-a\langle Ax,y\rangle)\\&\quad+\int_0^1\left\langle P_{f'-\mu}((aA)\nabla_t B,A)x,\left(BA^{-1}-aI\right)y\right\rangle dt\end{aligned}\tag{3.21}$$

for any $x,y\in H$.

We have:

Theorem 3. *Let A be a positive invertible operator, B a selfadjoint operator such that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$ and $f:I\to\mathbb{C}$ be a continuously differentiable function on $\mathring{I}$ and such that $f'\in\bar{\Delta}_{\mathring{I}}(\phi,\Phi)$ for some $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$. Then for any $a\in I$ we have*

$$\begin{aligned}&\left|\langle P_f(B,A)x,y\rangle-f(a)\langle Ax,y\rangle-\frac{\phi+\Phi}{2}(\langle Bx,y\rangle-a\langle Ax,y\rangle)\right|\\&\le\frac{1}{2}|\Phi-\phi|\,\|A\|\left\|BA^{-1}y-ay\right\|\|x\|\\&\le\frac{1}{2}|\Phi-\phi|\,\|A\|\left\|BA^{-1}-aI\right\|\|x\|\,\|y\|\end{aligned}\tag{3.22}$$

for all $x,\ y\in H$.

In particular, we have

$$\begin{aligned}&\left|\langle P_f(B,A)x,x\rangle-f\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle\right|\\&\le\frac{1}{2}|\Phi-\phi|\,\|A\|\left\|BA^{-1}x-\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}x\right\|\end{aligned}\tag{3.23}$$

for any $x\in H$, $\|x\|=1$.

Proof. By the equality (3.21) we have

$$\begin{aligned}&\left|\langle P_f(B,A)x,y\rangle-f(a)\langle Ax,y\rangle-\frac{\phi+\Phi}{2}(\langle Bx,y\rangle-a\langle Ax,y\rangle)\right|\\&\le\int_0^1\left|\left\langle A^{1/2}\left[f'\left(A^{-1/2}[(1-t)aA+tB]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right]A^{1/2}x,\right.\right.\\&\left.\left.\left(BA^{-1}-aI\right)y\right\rangle\right|dt\\&\le\left\|\left(BA^{-1}-aI\right)y\right\|\\&\times\int_0^1\left\|A^{1/2}\left[f'\left(A^{-1/2}[(1-t)aA+tB]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right]A^{1/2}x\right\|dt,\end{aligned}\tag{3.24}$$

where for the last inequality we used the Schwarz inequality.

Since

$$\left\|\left(BA^{-1}-aI\right)y\right\|\le\left\|BA^{-1}-aI\right\|\|y\|$$

and

$$\begin{aligned}&\left\|A^{1/2}\left[f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right]A^{1/2}x\right\|\\&\leq\left\|A^{1/2}\left[f'\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)-\frac{\phi+\Phi}{2}I\right]A^{1/2}\right\|\|x\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\|x\|,\end{aligned}$$

then by (3.24) we get the desired inequality (3.22). □

We notice that, if $f:[m,M]\to\mathbb{R}$ is convex, then by (3.23) we have the following reverse of Jensen's inequality for perspectives

$$\begin{aligned}0&\leq\left\langle P_f\left(B,A\right)x,x\right\rangle-f\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle\\&\leq\frac{1}{2}\left[f'_+\left(M\right)-f'_+\left(m\right)\right]\|A\|\left\|BA^{-1}x-\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}x\right\|\end{aligned}\tag{3.25}$$

for any $x\in H$, $\|x\|=1$, provided $mA\leq B\leq MA$.

Corollary 7. *With the assumptions of Corollary 6 we have*

$$\begin{aligned}&\left|\left\langle P_f\left(B,A\right)x,y\right\rangle-f\left(\frac{m+M}{2}\right)\langle Ax,y\rangle\right.\\&\left.-\frac{\phi+\Phi}{2}\left(\langle Bx,y\rangle-\frac{m+M}{2}\langle Ax,y\rangle\right)\right|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\|A\|\left\|BA^{-1}y-\frac{m+M}{2}y\right\|\|x\|\\&\leq\frac{1}{2}\left|\Phi-\phi\right|\|A\|\left\|BA^{-1}-\frac{m+M}{2}I\right\|\|x\|\|y\|\\&\leq\frac{1}{4}\left|\Phi-\phi\right|\left(M-m\right)\|A\|^2\left\|A^{-1}\right\|\|x\|\|y\|\end{aligned}\tag{3.26}$$

for any $x,y\in H$.

4. Applications for Operator Geometric Mean

If we consider the continuous function $f_\nu:[0,\infty)\to[0,\infty)$, $f_\nu\left(t\right)=t^\nu$, $\nu\in[0,1]$, then the operator *ν-weighted arithmetic-geometric mean* can be interpreted as the perspective $\mathcal{P}_{f_\nu}\left(B,A\right)$, namely

$$\mathcal{P}_{f_\nu}\left(B,A\right)=A\sharp_\nu B.$$

For recent results on operator Young inequality see [11]-[14], [15] and [23]-[24].

Using the representation (2.17), we have for positive invertible operators A, B that

$$\begin{aligned}A\sharp_\nu B&=a^\nu A+\mu\left(B-aA\right)\\&+\left(BA^{-1}-aI\right)\\&\times\int_0^1A^{1/2}\left[\nu\left(A^{-1/2}\left[(1-t)\,aA+tB\right]A^{-1/2}\right)^{\nu-1}-\mu I\right]A^{1/2}dt,\end{aligned}\tag{4.1}$$

for any $a>0$ and $\mu\in\mathbb{R}$.

If we take in this equality $a=1$ and $\mu=\nu$, then we get the equality

$$\begin{aligned}A\sharp_{\nu}B &= A+\nu\left(B-A\right)\\ &+\nu\left(BA^{-1}-I\right)\\ &\times\int_0^1 A^{1/2}\left[\left(A^{-1/2}\left[(1-t)A+tB\right]A^{-1/2}\right)^{\nu-1}-I\right]A^{1/2}dt,\end{aligned}$$

that is equivalent to

$$\begin{aligned}(4.2)\quad &(0\leq)\,A\nabla_{\nu}B-A\sharp_{\nu}B\\ &=\nu\left(I-BA^{-1}\right)\int_0^1 A^{1/2}\left[\left(A^{-1/2}\left[(1-t)A+tB\right]A^{-1/2}\right)^{\nu-1}-I\right]A^{1/2}dt.\end{aligned}$$

Similar equalities may be obtained by utilizing the other results from the second section, however the details are omitted.

The function $f(t)=-t^{\nu}$, $\nu\in[0,1]$ is convex, and by (3.17)-(3.19) we have

$$\begin{aligned}(4.3)\quad &\left\|A\sharp_{\nu}B-a^{\nu}A-\nu\frac{m^{\nu-1}+M^{\nu-1}}{2}\left(B-aA\right)\right\|\\ &\leq\frac{1}{2}\nu\left(m^{\nu-1}-M^{\nu-1}\right)\left\|BA^{-1}-aI\right\|\left\|A\right\|,\end{aligned}$$

$$\begin{aligned}(4.4)\quad &\left\|A\sharp_{\nu}B-\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)^{\nu}A-\nu\frac{m^{\nu-1}+M^{\nu-1}}{2}\left(B-\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\right\|\\ &\leq\frac{1}{2}\nu\left(m^{\nu-1}-M^{\nu-1}\right)\left\|BA^{-1}-\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}I\right\|\left\|A\right\|\end{aligned}$$

for any $x\in H$, $x\neq 0$ and

$$\begin{aligned}(4.5)\quad &\left\|A\sharp_{\nu}B-\left(\frac{m+M}{2}\right)^{\nu}A-\nu\frac{m^{\nu-1}+M^{\nu-1}}{2}\left(B-\frac{m+M}{2}A\right)\right\|\\ &\leq\frac{1}{2}\nu\left(m^{\nu-1}-M^{\nu-1}\right)\left\|BA^{-1}-\frac{m+M}{2}I\right\|\left\|A\right\|\\ &\leq\frac{1}{4}\nu\left(m^{\nu-1}-M^{\nu-1}\right)(M-m)\left\|A\right\|^2\left\|A^{-1}\right\|,\end{aligned}$$

where A, B are positive invertible operators such that $mA\leq B\leq MA$ and $0<m<M$.

From (3.25) we also have for $\nu\in[0,1]$ that

$$\begin{aligned}(4.6)\quad &0\leq\langle Bx,x\rangle^{\nu}\langle Ax,x\rangle^{1-\nu}-\langle A\sharp_{\nu}Bx,x\rangle\\ &\leq\frac{1}{2}\nu\left(m^{\nu-1}-M^{\nu-1}\right)\left\|A\right\|\left\|BA^{-1}x-\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}x\right\|\end{aligned}$$

for any $x\in H$, $\|x\|=1$, provided $mA\leq B\leq MA$.

5. Applications for Relative Operator Entropy

Consider the logarithmic function ln . Then the relative operator entropy can be interpreted as the permanent of ln, namely

$$\mathcal{P}_{\ln}(B,A)=S(A|B).$$

For some recent results on relative operator entropy see [3]-[4], [16]-[17] and [19]-[20].

Using the identity (2.17) for $f = \ln$, we have for the invertible positive operators A, B that

$$\begin{aligned}
(5.1)\qquad & S(A|B) \\
&= \ln aA + \mu(B - aA) \\
&+ \left(BA^{-1} - aI\right)\int_0^1 A^{1/2}\left[A^{1/2}\left[(1-t)aA + tB\right]^{-1}A^{1/2} - \mu I\right]A^{1/2}dt \\
&= \ln aA + \mu(B - aA) \\
&+ (B - aA)\left(\int_0^1 \left[\left[(1-t)aA + tB\right]^{-1} - \mu A^{-1}\right]dt\right)A
\end{aligned}$$

for any $a > 0$ and $\mu \in \mathbb{R}$.

If we take in (5.1) $a = 1$ and $\mu = 1$, then we have the simpler equality

$$S(A|B) = B - A + (B - A)\left(\int_0^1 \left[\left[(1-t)A + tB\right]^{-1} - A^{-1}\right]dt\right)A. \tag{5.2}$$

If we consider the convex function $f(t) = -\ln t$ and assume that $mA \le B \le MA$ for $0 < m < M$, then by (3.17)-(3.19) we get

$$\begin{aligned}
(5.3)\qquad & \left\|S(A|B) - \ln aA - \frac{m+M}{2mM}(B - aA)\right\| \\
&\le \frac{M-m}{2mM}\left\|BA^{-1} - aI\right\|\|A\|,
\end{aligned}$$

$$\begin{aligned}
(5.4)\qquad & \left\|S(A|B) - \ln\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)A - \frac{m+M}{2mM}\left(B - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}A\right)\right\| \\
&\le \frac{M-m}{2mM}\left\|BA^{-1} - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}I\right\|\|A\|
\end{aligned}$$

for any $x \in H$, $x \neq 0$, and

$$\begin{aligned}
(5.5)\qquad & \left\|S(A|B) - \ln\left(\frac{m+M}{2}\right)A - \frac{m+M}{2mM}\left(B - \frac{m+M}{2}A\right)\right\| \\
&\le \frac{M-m}{2mM}\left\|BA^{-1} - \frac{m+M}{2}I\right\|\|A\| \le \frac{(M-m)^2}{4mM}\|A\|^2\left\|A^{-1}\right\|.
\end{aligned}$$

From (3.25) we also have

$$\begin{aligned}
(5.6)\qquad & 0 \le \ln\left(\frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}\right)\langle Ax,x\rangle - \langle S(A|B)x,x\rangle \\
&\le \frac{M-m}{2mM}\|A\|\left\|BA^{-1}x - \frac{\langle Bx,x\rangle}{\langle Ax,x\rangle}x\right\|
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$, provided $mA \le B \le MA$.

References

[1] S. S. Dragomir, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [Online http://rgmia.org/papers/v18/v18a130.pdf].

[2] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135. [Online http://rgmia.org/papers/v18/v18a135.pdf].

[3] S. S. Dragomir, Some inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 145. [Online http://rgmia.org/papers/v18/v18a145.pdf].

[4] S. S. Dragomir, Further inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 160.[Online http://rgmia.org/papers/v18/v18a160.pdf].

[5] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373. Preprint *RGMIA Res. Rep. Coll.***14** (2011), Art. 84. [Online http://rgmia.org/papers/v14/v14a84.pdf].

[6] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, *Proc. Natl. Acad. Sci. USA*, **108** (2011), no. 18, 7313–7314.

[7] E. G. Effros, A matrix convexity approach to some celebrated quantum inequalities, *Proc. Natl. Acad. Sci. USA* **106** (2009), 1006–1008.

[8] E. G. Effros and F. Hansen, Noncomutative perspectives, *Ann. Funct. Anal.* **5** (2014), no. 2, 74–79.

[9] J. I. Fujii and E. Kamei, Uhlmann's interpolational method for operator means. *Math. Japon.* **34** (1989), no. 4, 541–547.

[10] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory. *Math. Japon.* **34** (1989), no. 3, 341–348.

[11] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[12] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.

[13] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[14] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.

[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[16] I. H. Kim, Operator extension of strong subadditivity of entropy, *J. Math. Phys.* 53(2012), 122204

[17] P. Kluza and M. Niezgoda, Inequalities for relative operator entropies, *Elec. J. Lin. Alg.* **27** (2014), Art. 1066.

[18] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.

[19] M. S. Moslehian, F. Mirzapour, and A. Morassaei, Operator entropy inequalities. *Colloq. Math.*, **130** (2013), 159–168.

[20] I. Nikoufar, On operator inequalities of some relative operator entropies, *Adv. Math.* **259** (2014), 376-383.

[21] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.

[22] M. Nakamura and H. Umegaki, A note on the entropy for operator algebras. *Proc. Japan Acad.* **37** (1961) 149–154.

[23] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.

[24] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa