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A COMPANION OF OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS OF BOUNDED VARIATION AND SOME APPLICATIONS

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Abstract. In this paper, we establish a companion of Ostrowski type inequalities for mappings of bounded variation and the quadrature formula is also provided.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty , \tag{1.1}$$

for all $x\in[a,b]$[20]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions. Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b (f) := \sup\left\{ \sum(P) : P\in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [13], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

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Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

Dragomir gave the following trapezoid inequality in [10]:

Theorem 2. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)dt\right|\leq\frac{1}{2}(b-a)\bigvee_a^b(f). \tag{1.3}$$

The constant $\frac{1}{2}$ is the best possible.

We introduce the notation $I_n: a=x_0<x_1<...<x_n=b$ for a division of the interval $[a,b]$ with $h_i:=x_{i+1}-x_i$ and $v(h)=\max\{h_i:i=0,1,...,n-1\}$. Then we have

$$\int_a^b f(t)dt=A_T(f,I_n)+R_T(f,I_n) \tag{1.4}$$

where

$$A_T(f,I_n):=\sum_{i=0}^{n}\frac{f(x_i)+f(x_{i+1})}{2}h_i \tag{1.5}$$

and the remainder term satisfies

$$|R_T(f,I_n)|\leq\frac{1}{2}v(h)\bigvee_a^b(f). \tag{1.6}$$

In [14], Dragomir proved the following companion Ostrowski type inequalities related functions of bounded variation:

Theorem 3. *Assume that the function $f:[a,b]\to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right] \\
\leq\ & \begin{cases}
\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f), \\
\left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
\times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^b(f)\right]^{\beta}\right]^{\frac{1}{\beta}} & ,\ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\
\left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}
\end{cases}
\end{aligned}
\tag{1.7}
$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ where $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$. The constant $\frac{1}{4}$ is the best possible in the first branch of second inequality in (1.7).

For recent results concerning the above Ostrowski's inequality and other related results see [1]-[26].

In this work, we obtain a new companion of Ostrowski type integral inequalities for functions of bounded variation. Then we give some applications for our results.

2. Main Results

Now, we give a new companion of Ostrowski type integral inequalities for functions of bounded variation:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then, we have the inequality*

$$
\begin{aligned}
&\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right| \\
\leq\ & \max\left\{\left|x-\frac{3a+b}{4}\right|,\left(\frac{a+b}{2}-x\right),\frac{x-a}{2}\right\}\bigvee_a^b(f)
\end{aligned}
\tag{2.1}
$$

where $x\in\left[a,\frac{a+b}{2}\right]$ and $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$.

Proof. Consider the kernel $P(x,t)$ defined by Qayyum et al. in [21]

$$P(x,t)=\begin{cases} t-a, & t\in\left[x,\frac{a+x}{2}\right] \\ t-\frac{3a+b}{4}, & t\in\left(\frac{a+x}{2},x\right] \\ t-\frac{a+b}{2}, & t\in(x,a+b-x] \\ t-\frac{a+3b}{4}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right] \\ t-b, & t\in\left[\frac{a+2b-x}{2},b\right]. \end{cases}$$

Integrating by parts, we get

$$\begin{aligned} &\int_a^b P(x,t)df(t) \\ &= \frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt. \end{aligned} \tag{2.2}$$

It is well known that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right|\leq \sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{2.3}$$

On the other hand, by using (2.3), we get

$$\begin{aligned} &\left|\int_a^b P(x,t)df(t)\right| \\ &\leq \left|\int_a^{\frac{a+x}{2}}(t-a)\,df(t)\right|+\left|\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)df(t)\right|+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)\right| \\ &\quad+\left|\int_{a+b-x}^{\frac{a+b-x}{2}}\left(t-\frac{a+3b}{4}\right)df(t)\right|+\left|\int_{\frac{a+b-x}{2}}^{b}(t-b)\,df(t)\right| \end{aligned}$$

$$
\begin{aligned}
&\le \sup_{t\in\left[a,\frac{a+x}{2}\right]}|t-a|\bigvee_{a}^{\frac{a+x}{2}}(f)+\sup_{t\in\left[\frac{a+x}{2},x\right]}\left|t-\frac{3a+b}{4}\right|\bigvee_{\frac{a+x}{2}}^{x}(f)\\
&\quad+\sup_{t\in[x,a+b-x]}\left|t-\frac{a+b}{2}\right|\bigvee_{x}^{a+b-x}(f)+\sup_{t\in\left[a+b-x,\frac{a+2b-x}{2}\right]}\left|t-\frac{a+3b}{4}\right|\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f)\\
&\quad+\sup_{t\in\left[\frac{a+2b-x}{2},b\right]}|t-b|\bigvee_{\frac{a+2b-x}{2}}^{b}(f)\\
&= \frac{x-a}{2}\bigvee_{a}^{\frac{a+x}{2}}(f)+\max\left\{\left|x-\frac{3a+b}{4}\right|,\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right\}\bigvee_{\frac{a+x}{2}}^{x}(f)\\
&\quad+\left(\frac{a+b}{2}-x\right)\bigvee_{x}^{a+b-x}(f)+\max\left\{\left|x-\frac{3a+b}{4}\right|,\frac{1}{2}\left(\frac{a+b}{2}-x\right)\right\}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f)\\
&\quad+\frac{x-a}{2}\bigvee_{\frac{a+2b-x}{2}}^{b}(f)\\
&\le \max\left\{\left|x-\frac{3a+b}{4}\right|,\left(\frac{a+b}{2}-x\right),\frac{x-a}{2}\right\}\bigvee_{a}^{b}(f).
\end{aligned}
$$

This completes the proof. □

Remark 1. *If we choose $x=a$ in Theorem 4, the inequality (2.1) reduces the inequality (1.3).*

Corollary 1. *Under the assumption of Theorem 4 with $x=\frac{a+b}{2}$, then we have the following inequality*

(2.4)
$$\left|\frac{b-a}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\int_a^b f(t)dt\right|\le\frac{1}{4}(b-a)\bigvee_{a}^{b}(f).$$

The constant $\frac{1}{4}$ is the best possible.

Proof. For proof of the sharpness of the constant, assume that (2.4) holds with a constant $A>0$, that is,

(2.5)
$$\left|\frac{b-a}{4}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\int_a^b f(t)dt\right|\le A(b-a)\bigvee_{a}^{b}(f).$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}1, & \text{if } x\in\left\{\frac{a+b}{2},\frac{3a+b}{4},\frac{a+3b}{4}\right\}\\[2mm] 0, & \text{if } x\in[a,b]\,/\left\{\frac{a+b}{2},\frac{3a+b}{4},\frac{a+3b}{4}\right\}\end{cases}$$

then f is of bounded variation on $[a,b]$, and

$$2f\left(\frac{a+b}{2}\right)+f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)=4,\ \int_a^b f(t)dt=0,\ \text{and}\ \bigvee_a^b(f)=4,$$

giving in (2.5), $1\leq 4A$, thus $A\geq\frac{1}{4}$. $\square$

Corollary 2. *Under the assumption of Theorem 4 with $x=\frac{3a+b}{4}$, then we get the inequality*

$$\begin{aligned}&\left|\frac{b-a}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right.\right.\\&\left.\left.+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]-\int_a^b f(t)dt\right|\\\leq\ &\frac{1}{8}(b-a)\bigvee_a^b(f)\end{aligned}\tag{2.6}$$

The constant $\frac{1}{8}$ is the best possible.

Proof. For proof of the sharpness of the constant, assume that (3.4) holds with a constant $B>0$, that is,

$$\begin{aligned}&\left|\frac{b-a}{4}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right.\right.\\&\left.\left.+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]-\int_a^b f(t)dt\right|\\\leq\ &B(b-a)\bigvee_a^b(f).\end{aligned}\tag{2.7}$$

If we choose $f:[a,b]\to\mathbb{R}$ with

$$f(x)=\begin{cases}1, & \text{if } x\in\left\{\frac{3a+b}{4},\frac{a+3b}{4},\frac{7a+b}{8},\frac{a+7b}{8}\right\}\\[2ex] 0, & \text{if } x\in[a,b]\,/\left\{\frac{3a+b}{4},\frac{a+3b}{4},\frac{7a+b}{8},\frac{a+7b}{8}\right\}\end{cases}$$

then f is of bounded variation on $[a,b]$, and

$$f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)+f\left(\frac{7a+b}{8}\right)+f\left(\frac{a+7b}{8}\right)=4,$$

$$\int_a^b f(t)dt=0,\ \text{and}\ \bigvee_a^b(f)=8,$$

giving in (2.7), $1\leq 8B$, thus $B\geq\frac{1}{8}$. $\square$

Corollary 3. *Let f is defined as in Theorem 4, and, additionally, if $f(x) = f(a+b-x)$, then we have*

(2.8)
$$\left|\frac{b-a}{4}\left[2f(x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right|$$
$$\leq \max\left\{\left|x-\frac{3a+b}{4}\right|,\left(\frac{a+b}{2}-x\right),\frac{x-a}{2}\right\}\bigvee_a^b(f).$$

Corollary 4. *If we choose $x=a$ in Corollary 3, then we have the inequality*
$$\left|\frac{3f(a)+f(b)}{4}(b-a)-\int_a^b f(t)dt\right|\leq\frac{1}{2}(b-a)\bigvee_a^b(f).$$
The constant $\frac{1}{2}$ is the best possible.

The sharpness of the constant can be proved similarly Corollary 1 and Corollary 2, it is omitted.

Corollary 5. *Under the assumption of Theorem 4. Suppose that $f\in C^1[a,b]$, then we have*
$$\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right|$$
$$\leq \max\left\{\left|x-\frac{3a+b}{4}\right|,\left(\frac{a+b}{2}-x\right),\frac{x-a}{2}\right\}\|f'\|_1$$
for all $x\in\left[a,\frac{a+b}{2}\right]$. Here as subsequently $\|.\|_1$ is the L_1-norm
$$\|f'\|_1:=\int_a^b f'(t)dt.$$

Corollary 6. *Under the assumption of Theorem 4. Let $f:[a,b]\to\mathbb{R}$ be a Lipschitzian with the constant $L>0$. Then*
$$\left|\frac{b-a}{4}\left[f(x)+f(a+b-x)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]-\int_a^b f(t)dt\right|$$
$$\leq \max\left\{\left|x-\frac{3a+b}{4}\right|,\left(\frac{a+b}{2}-x\right),\frac{x-a}{2}\right\}(b-a)L$$
for all $x\in\left[a,\frac{a+b}{2}\right]$.

3. Application to Quadrature Formula

We now introduce the intermediate points $\xi_i\in[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ in the division $I_n: a=x_0<x_1<...<x_n=b$. Let $h_i:=x_{i+1}-x_i$ and $v(h)=$

$\max\{h_i : i = 0, 1, ..., n-1\}$ and define the sum

$$A(f, I_n, \xi) := \frac{1}{4}\sum_{i=0}^{n} h_i \left[f(\xi_i) + f\left(x_i + x_{i+1} - \xi_i\right) + f\left(\frac{x_i + \xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) \right]. \tag{3.1}$$

Then the following Theorem holds:

Theorem 5. *Let f be as Theorem 4. Then*

$$\int_a^b f(t)dt = A(f, I_n, \xi) + R(f, I_n, \xi) \tag{3.2}$$

where $A(f, I_n, \xi)$ is defined as above and the remainder term $R(f, I_n, \xi)$ satisfies

$$\begin{aligned} &|R(f, I_n, \xi)| \\ &\leq \max_{i\in\{0,1,...,n-1\}} \left[\max\left\{ \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right), \frac{\xi_i - x_i}{2} \right\} \right] \bigvee_a^b (f). \end{aligned} \tag{3.3}$$

Proof. Application of Theorem 4 to the interval $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ gives

$$\begin{aligned} &\left| \frac{h_i}{4} \left[f(\xi_i) + f\left(x_i + x_{i+1} - \xi_i\right) + f\left(\frac{x_i + \xi_i}{2}\right) \right.\right. \\ &\left.\left. + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) \right] - \int_{x_i}^{x_{i+1}} f(t)dt \right| \\ \leq\ & \max\left\{ \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right), \frac{\xi_i - x_i}{2} \right\} \bigvee_{x_i}^{x_{i+1}} (f) \end{aligned} \tag{3.4}$$

for all $i \in \{0, 1, ..., n-1\}$. Summing the inequality (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we have

$$\begin{aligned} &|R(f, I_n, \xi)| \\ \leq\ & \sum_{i=0}^{n} \max\left\{ \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right), \frac{\xi_i - x_i}{2} \right\} \bigvee_{x_i}^{x_{i+1}} (f) \\ \leq\ & \max_{i\in\{0,1,...,n-1\}} \left[\max\left\{ \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right), \frac{\xi_i - x_i}{2} \right\} \right] \sum_{i=0}^{n} \bigvee_{x_i}^{x_{i+1}} (f) \\ =\ & \max_{i\in\{0,1,...,n-1\}} \left[\max\left\{ \left| \xi_i - \frac{3x_i + x_{i+1}}{4} \right|, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right), \frac{\xi_i - x_i}{2} \right\} \right] \bigvee_a^b (f) \end{aligned}$$

which completes the proof. □

Remark 2. *If we choose $\xi_i = x_i$ in Theorem 5, we get (1.4) with (1.5) and (1.6).*

Corollary 7. *If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$ *in Theorem 5, then we have*

$$\int_a^b f(t)dt = A(f, I_n) + R(f, I_n)$$

where

$$A(f, I_n) := \frac{1}{4}\sum_{i=0}^{n} h_i \left[2f\left(\frac{x_i + x_{i+1}}{2}\right) + f\left(\frac{3x_i + x_{i+1}}{2}\right) + f\left(\frac{x_i + 3x_{i+1}}{2}\right)\right]$$

and the remainder term $R(f, I_n)$ *satisfies*

$$|R(f, I_n)| \leq \frac{1}{4} v(h) \bigvee_a^b (f).$$

Corollary 8. *If we choose* $\xi_i = \frac{3x_i + x_{i+1}}{2}$ *in Theorem 5, then we have*

$$\int_a^b f(t)dt = A(f, I_n) + R(f, I_n)$$

where

$$\begin{aligned} & A(f, I_n) \\ : & = \frac{1}{4}\sum_{i=0}^{n} h_i \left[f\left(\frac{3x_i + x_{i+1}}{2}\right) + f\left(\frac{x_i + 3x_{i+1}}{2}\right)\right. \\ & \left. + f\left(\frac{7x_i + x_{i+1}}{8}\right) + f\left(\frac{x_i + 7x_{i+1}}{8}\right)\right] \end{aligned}$$

and the remainder term $R(f, I_n)$ *satisfies*

$$|R(f, I_n)| \leq \frac{1}{8} v(h) \bigvee_a^b (f).$$

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