

# IMPROVEMENT IN COMPANION OF OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE OF BOUNDED VARIATION AND APPLICATION

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ABSTRACT. The main aim of this paper is to obtain a improved and generalized version of companion of Ostrowski type integral inequalities for mappings whose first derivatives are of bounded variation. Some previous results are also recaptured as a special case. Efficient quadrature formulae are also provided.

## 1. INTRODUCTION

In 1938, Ostrowski [14] established a following interesting integral inequality associated with differentiable mappings.

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  whose derivative  $f' : (a, b) \rightarrow \mathbb{R}$  is bounded on  $(a, b)$ , i.e.  $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$ . Then, we have the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all  $x \in [a, b]$ .

The constant  $\frac{1}{4}$  is the best possible.

Ostrowski inequality has applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory. During the past few years, many authors have studied on Ostrowski type inequalities for function of bounded variation, see for example ([1],[2],[3],[5]-[12]). Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications.

The following definitions will be frequently used to prove our results.

**Definition 1.** *Let  $P : a = x_0 < x_1 < \dots < x_n = b$  be any partition of  $[a, b]$  and let  $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$ , then  $f$  is said to be of bounded variation if the sum*

$$\sum_{i=1}^m |\Delta f(x_i)|$$

*is bounded for all such partitions.*

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**Definition 2.** Let  $f$  be of bounded variation on  $[a, b]$ , and  $\sum \Delta f(P)$  denotes the sum  $\sum_{i=1}^n |\Delta f(x_i)|$  corresponding to the partition  $P$  of  $[a, b]$ . The number

$$\bigvee_a^b(f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of  $f$  on  $[a, b]$ . Here  $P([a, b])$  denotes the family of partitions of  $[a, b]$ .

In [9], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

**Theorem 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a mapping of bounded variation on  $[a, b]$ . Then

$$\left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \left[ \frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

holds for all  $x \in [a, b]$ . The constant  $\frac{1}{2}$  is the best possible.

In [13], Liu proved following Ostrowski type inequalities for the functions with first derivatives of bounded variation:

**Theorem 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'$  is a continuous function of bounded variation on  $[a, b]$ . Then for any  $x \in [a, b]$  and  $\theta \in [0, 1]$  we have

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \left[ (1-\theta) f(x) + \theta \frac{f(a)+f(b)}{2} - (1-\theta) \left( x - \frac{a+b}{2} \right) f'(x) \right] \right| \\ & \leq \frac{1}{16} \left[ 4(x-b)^2 - 4\theta(b-a)(b-x) + \theta^2(b-a)^2 \right. \\ & \quad \left. \left| 4(x-b)^2 - 4\theta(b-a)(b-x) - \theta^2(b-a)^2 \right| \right] \bigvee_a^b(f') \end{aligned}$$

for  $a \leq x \leq \frac{a+b}{2}$  and

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \left[ (1-\theta) f(x) + \theta \frac{f(a)+f(b)}{2} - (1-\theta) \left( x - \frac{a+b}{2} \right) f'(x) \right] \right| \\ & \leq \frac{1}{16} \left[ 4(x-a)^2 - 4\theta(b-a)(x-a) + \theta^2(b-a)^2 \right. \\ & \quad \left. \left| 4(x-a)^2 - 4\theta(b-a)(x-a) - \theta^2(b-a)^2 \right| \right] \bigvee_a^b(f') \end{aligned}$$

for  $\frac{a+b}{2} \leq x \leq b$ .

In [7], authors gave the following Ostrowski type inequality:

**Theorem 4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'$  is a continuous function of bounded variation on  $[a, b]$ . Then we have the inequality

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\ & \quad \left. + \frac{1}{2} \left( x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\ & \leq \frac{1}{16} \left[ \frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4 \left| x - \frac{3a+b}{4} \right| \right] \bigvee_a^b(f') \end{aligned}$$

for any  $x \in [a, \frac{a+b}{2}]$ .

In [4], Budak and Sarikaya obtained following Ostrowski type inequality in weighted form for the mappings whose first derivatives are of bounded variation:

**Theorem 5.** Let  $w : [a, b] \rightarrow \mathbb{R}$  be nonnegative and continuous and let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable mapping on  $[a, b]$ . If  $f'$  is of bounded variation on  $[a, b]$ , then we have the weighted inequality

$$\begin{aligned} & \left| \left( \int_a^b (u-x) w(u) du \right) f'(x) + \left( \int_a^b w(u) du \right) f(x) - \int_a^b w(t) f(t) dt \right| \\ & \leq \left( \int_a^x (u-x) w(u) du \right) \bigvee_a^x(f') + \left( \int_x^b (u-x) w(u) du \right) \bigvee_x^b(f') \end{aligned}$$

for any  $x \in [a, b]$ .

Recently, Qayyum et. al [15], proved Ostrowski inequality using 5-step quadratic kernel. In this paper, we establish a new companion of Ostrowski type integral inequalities for functions whose first derivatives are of bounded variation. At the end, we apply our results for new efficient quadrature rules. The results presented here would provide extensions of those given in [7].

## 2. DERIVATION OF COMPANION OF OSTROWSKI TYPE INTEGRAL INEQUALITIES

Before we prove our results for the 5-step quadratic kernel, we give the following lemma for 5-step linear kernel.

**Lemma 1.** Consider the kernel  $P(x, t)$  defined by Qayyum et al. in [15]

$$(2.1) \quad P(x, t) = \begin{cases} \frac{1}{2} (t-a)^2, & t \in [x, \frac{a+x}{2}] \\ \frac{1}{2} (t - \frac{3a+b}{4})^2, & t \in (\frac{a+x}{2}, x] \\ \frac{1}{2} (t - \frac{a+b}{2})^2, & t \in (x, a+b-x] \\ \frac{1}{2} (t - \frac{a+3b}{4})^2, & t \in (a+b-x, \frac{a+2b-x}{2}] \\ \frac{1}{2} (t-b)^2, & t \in [\frac{a+2b-x}{2}, b] \end{cases}$$

for all  $x \in [a, \frac{a+b}{2}]$ , then the following identity

$$\begin{aligned}
 (2.2) \quad & \int_a^b P(x, t) df'(t) \\
 = & \int_a^b f(t) dt - \frac{b-a}{4} \left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right. \\
 & + \left( x - \frac{5a+3b}{8} \right) \{ f'(a+b-x) - f'(x) \} \\
 & \left. + \frac{1}{2} \left( x - \frac{3a+b}{4} \right) \left\{ f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right) \right\} \right].
 \end{aligned}$$

holds.

*Proof.* By using (2.1), we have

$$\begin{aligned}
 (2.3) \quad & \int_a^b P(x, t) df'(t) \\
 = & \frac{1}{2} \int_a^{\frac{a+x}{2}} (t-a)^2 df'(t) + \frac{1}{2} \int_{\frac{a+x}{2}}^x \left( t - \frac{3a+b}{4} \right)^2 df'(t) + \frac{1}{2} \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 df'(t) \\
 & + \frac{1}{2} \int_{a+b-x}^{\frac{a+2b-x}{2}} \left( t - \frac{a+3b}{4} \right)^2 df'(t) + \frac{1}{2} \int_{\frac{a+2b-x}{2}}^b (t-b)^2 df'(t).
 \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
 & \int_a^b P(x, t) df'(t) \\
 = & \frac{(x-a)^2}{8} f'\left(\frac{a+x}{2}\right) - \frac{x-a}{2} f\left(\frac{a+x}{2}\right) + \int_a^{\frac{a+x}{2}} f(t) dt \\
 & + \frac{1}{2} \left( x - \frac{3a+b}{4} \right)^2 f'(x) - \frac{1}{8} \left( x - \frac{a+b}{2} \right)^2 f'\left(\frac{a+x}{2}\right) \\
 & - \left( x - \frac{3a+b}{4} \right) f(x) - \frac{1}{2} \left( x - \frac{a+b}{2} \right) f\left(\frac{a+x}{2}\right) + \int_{\frac{a+x}{2}}^x f(t) dt \\
 & + \frac{1}{2} \left( \frac{a+b}{2} - x \right)^2 f'(a+b-x) - \frac{1}{2} \left( \frac{a+b}{2} - x \right)^2 f'(x)
 \end{aligned}$$

$$\begin{aligned}
& - \left( \frac{a+b}{2} - x \right) f(a+b-x) - \left( \frac{a+b}{2} - x \right) f(x) + \int_x^{a+b-x} f(t) dt \\
& + \frac{1}{8} \left( \frac{a+b}{2} - x \right)^2 f' \left( \frac{a+2b-x}{2} \right) - \frac{1}{2} \left( \frac{3a+b}{4} - x \right)^2 f'(a+b-x) \\
& - \frac{1}{2} \left( \frac{a+b}{2} - x \right) f \left( \frac{a+2b-x}{2} \right) + \left( \frac{3a+b}{4} - x \right) f(a+b-x) + \int_{a+b-x}^{\frac{a+2b-x}{2}} f(t) dt \\
& - \frac{(x-a)^2}{8} f' \left( \frac{a+2b-x}{2} \right) - \frac{x-a}{2} f \left( \frac{a+2b-x}{2} \right) + \int_{\frac{a+2b-x}{2}}^b f(t) dt
\end{aligned}$$

After simplification, we get our required identity (2.2).  $\square$

Now using above identity, we state and prove the following theorem.

**Theorem 6.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'$  is a continuous function of bounded variation on  $[a, b]$ . Then we have the inequality*

$$\begin{aligned}
(2.4) \quad & \left| \int_a^b f(t) dt - \frac{b-a}{4} \left[ f(x) + f(a+b-x) + f \left( \frac{a+x}{2} \right) + f \left( \frac{a+2b-x}{2} \right) \right. \right. \\
& + \left( x - \frac{5a+3b}{8} \right) \{ f'(a+b-x) - f'(x) \} \\
& \left. \left. + \frac{1}{2} \left( x - \frac{3a+b}{4} \right) \left\{ f' \left( \frac{a+2b-x}{2} \right) - f' \left( \frac{a+x}{2} \right) \right\} \right] \right| \\
& \leq \frac{1}{2} \max \left\{ \left( x - \frac{3a+b}{4} \right)^2, \frac{1}{4} \left( \frac{a+b}{2} - x \right)^2, \frac{(x-a)^2}{4} \right\} \bigvee_a^b(f'),
\end{aligned}$$

where  $x \in [a, \frac{a+b}{2}]$  and  $\bigvee_a^b(f')$  denotes the total variation of  $f'$  on  $[a, b]$ .

*Proof.* It is well known that if  $g, f : [a, b] \rightarrow \mathbb{R}$  are such that  $g$  is continuous on  $[a, b]$  and  $f$  is of bounded variation on  $[a, b]$ , then  $\int_a^b g(t) df(t)$  exists and

$$(2.5) \quad \left| \int_a^b g(t) df(t) \right| \leq \sup_{t \in [a, b]} |g(t)| \bigvee_a^b(f).$$

On the other hand, by using (2.5), we get

$$\begin{aligned}
& \left| \int_a^b P(x, t) df'(t) \right| \\
& \leq \frac{1}{2} \left[ \left| \int_a^{\frac{a+x}{2}} (t-a)^2 df'(t) \right| + \left| \int_{\frac{a+x}{2}}^x \left( t - \frac{3a+b}{4} \right)^2 df'(t) \right| + \left| \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 df'(t) \right| \right. \\
& \quad \left. + \left| \int_{a+b-x}^{\frac{a+2b-x}{2}} \left( t - \frac{a+3b}{4} \right)^2 df'(t) \right| + \left| \int_{\frac{a+2b-x}{2}}^b (t-b)^2 df'(t) \right| \right] \\
& \leq \frac{1}{2} \left[ \sup_{t \in [a, \frac{a+x}{2}]} (t-a)^2 \bigvee_a^{\frac{a+x}{2}}(f') + \sup_{t \in [\frac{a+x}{2}, x]} \left( t - \frac{3a+b}{4} \right)^2 \bigvee_{\frac{a+x}{2}}^x(f') \right. \\
& \quad + \sup_{t \in [x, a+b-x]} \left( t - \frac{a+b}{2} \right)^2 \bigvee_x^{a+b-x}(f') + \sup_{t \in [a+b-x, \frac{a+2b-x}{2}]} \left( t - \frac{a+3b}{4} \right)^2 \bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f') \\
& \quad \left. + \sup_{t \in [\frac{a+2b-x}{2}, b]} (t-b)^2 \bigvee_{\frac{a+2b-x}{2}}^b(f') \right] \\
& = \frac{1}{2} \left[ \frac{(x-a)^2}{4} \bigvee_a^{\frac{a+x}{2}}(f') + \max \left\{ \left( x - \frac{3a+b}{4} \right)^2, \frac{1}{4} \left( \frac{a+b}{2} - x \right)^2 \right\} \bigvee_{\frac{a+x}{2}}^x(f') \right. \\
& \quad + \left( \frac{a+b}{2} - x \right)^2 \bigvee_x^{a+b-x}(f') + \max \left\{ \left( x - \frac{3a+b}{4} \right)^2, \frac{1}{4} \left( \frac{a+b}{2} - x \right)^2 \right\} \bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f') \\
& \quad \left. + \frac{(x-a)^2}{4} \bigvee_{\frac{a+2b-x}{2}}^b(f') \right] \\
& \leq \frac{1}{2} \max \left\{ \left( x - \frac{3a+b}{4} \right)^2, \left( \frac{a+b}{2} - x \right)^2, \frac{(x-a)^2}{4} \right\} \bigvee_a^b(f').
\end{aligned}$$

This completes the proof.  $\square$

**Remark 1.** If we choose  $x = a$  in Theorem 6, we get the result proved by Budak et al. [7].

**Corollary 1.** *Under the assumption of Theorem 6 with  $x = \frac{a+b}{2}$ , then we have the following inequality*

$$(2.6) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{4} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] - \frac{b-a}{32} \left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right\} \right| \leq \frac{b-a}{32} \bigvee_a^b(f').$$

**Corollary 2.** *Under the assumption of Theorem 6 with  $x = \frac{3a+b}{4}$ , then we get the inequality*

$$(2.7) \quad \left| \int_a^b f(t) dt - \frac{b-a}{4} \left[ f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) + f\left(\frac{7a+b}{8}\right) + f\left(\frac{a+7b}{8}\right) \right] + \frac{b-a}{32} \left\{ f'\left(\frac{a+3b}{4}\right) - f'\left(\frac{3a+b}{4}\right) \right\} \right| \leq \frac{b-a}{32} \bigvee_a^b(f').$$

Under assumption of of Theorem 6, we have the following corollaries:

**Corollary 3.** *Let  $f \in C^2[a, b]$ . Then we have the inequality*

$$(2.8) \quad \left| \int_a^b f(t) dt - \frac{b-a}{4} \left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right] + \left(x - \frac{5a+3b}{8}\right) \{f'(a+b-x) - f'(x)\} + \frac{1}{2} \left(x - \frac{3a+b}{4}\right) \left\{ f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right) \right\} \right| \leq \frac{1}{2} \max \left\{ \left(x - \frac{3a+b}{4}\right)^2, \frac{1}{4} \left(\frac{a+b}{2} - x\right)^2, \frac{(x-a)^2}{4} \right\} \|f''\|_{[a,b],1}$$

for all  $x \in [a, \frac{a+b}{2}]$ , where  $\|\cdot\|_{[a,b],1}$  is the  $L_1$ -norm, namely

$$\|f''\|_{[a,b],1} = \int_a^b |f''(t)| dt.$$

**Corollary 4.** *Let  $f' : [a, b] \rightarrow \mathbb{R}$  be a Lipschitzian mapping with the constants  $L > 0$ . Then, we have the inequality*

$$\begin{aligned}
 (2.9) \quad & \left| \int_a^b f(t)dt - \frac{b-a}{4} \left[ f(x) + f(a+b-x) + f\left(\frac{a+x}{2}\right) + f\left(\frac{a+2b-x}{2}\right) \right. \right. \\
 & \quad \left. \left. + \left(x - \frac{5a+3b}{8}\right) \{f'(a+b-x) - f'(x)\} \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} \left(x - \frac{3a+b}{4}\right) \left\{ f'\left(\frac{a+2b-x}{2}\right) - f'\left(\frac{a+x}{2}\right) \right\} \right] \right| \\
 & \leq \frac{L(b-a)}{2} \max \left\{ \left(x - \frac{3a+b}{4}\right)^2, \frac{1}{4} \left(\frac{a+b}{2} - x\right)^2, \frac{(x-a)^2}{4} \right\}
 \end{aligned}$$

for all  $x \in [a, \frac{a+b}{2}]$ .

*Proof.* As  $f'$  is  $L$ -Lipschitzian on  $[a, b]$ , it is also of bounded variation. If  $P([a, b])$  denotes the family of divisions on  $[a, b]$ , then

$$\begin{aligned}
 \bigvee_a^b(f') &= \sup_{P \in P([a, b])} \sum_{i=0}^{n-1} |f'(x_{i+1}) - f'(x_i)| \\
 &\leq L \sup_{P \in P([a, b])} \sum_{i=0}^{n-1} |x_{i+1} - x_i| \\
 &= L(b-a)
 \end{aligned}$$

and the required result (2.9) is proved.  $\square$

### 3. DERIVATION OF EFFICIENT QUADRATURE RULE

Our obtained inequalities has many applications but in this paper, we apply our result only for efficient quadrature rule.

Let us consider the arbitrary division  $I_n : a = x_0 < x_1 < \dots < x_n = b$  with  $h_i := x_{i+1} - x_i$  and  $v(h) := \max \{ |h_i| \mid i = 0, \dots, n-1 \}$ . Then the following Theorem holds:



**Theorem 7.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'$  is a continuous function of bounded variation on  $[a, b]$  and  $\xi_i \in \left[x_i, \frac{x_i + x_{i+1}}{2}\right]$  ( $i = 0, \dots, n-1$ ). Then we have the quadrature formula:

$$\begin{aligned}
& \int_a^b f(t) dt \\
= & \frac{1}{4} \sum_{i=0}^{n-1} \left[ f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f\left(\frac{x_i + \xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) \right] h_i \\
& + \frac{1}{4} \sum_{i=0}^{n-1} \left( \xi_i - \frac{5x_i + 3x_{i+1}}{8} \right) \left\{ f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{a + \xi_i}{2}\right) \right\} h_i \\
& + \frac{1}{8} \sum_{i=0}^{n-1} \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right) \left\{ f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{x_i + \xi_i}{2}\right) \right\} h_i \\
& + R(I_n, f, \xi).
\end{aligned}$$

The remainder term  $R(I_n, f, \xi)$  satisfies

$$\begin{aligned}
(3.1) \quad & |R(I_n, f, \xi)| \\
& \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2, \frac{1}{4} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^2, \frac{(\xi_i - x_i)^2}{4} \right\} \right\} \bigvee_a^b(f').
\end{aligned}$$

*Proof.* Applying Theorem 6 to interval  $[x_i, x_{i+1}]$ , we have

$$\begin{aligned}
(3.2) \quad & \left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{h_i}{4} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i) + f\left(\frac{x_i + \xi_i}{2}\right) + f\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) \right. \\
& + \left( \xi_i - \frac{5x_i + 3x_{i+1}}{8} \right) \left\{ f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{a + \xi_i}{2}\right) \right\} \\
& + \left. \frac{1}{2} \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right) \left\{ f'\left(\frac{x_i + 2x_{i+1} - \xi_i}{2}\right) - f'\left(\frac{x_i + \xi_i}{2}\right) \right\} \right] \Big| \\
& \leq \frac{1}{2} \max \left\{ \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2, \frac{1}{4} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^2, \frac{(\xi_i - x_i)^2}{4} \right\} \bigvee_{x_i}^{x_{i+1}}(f').
\end{aligned}$$

Summing the inequality (3.2) over  $i$  from 0 to  $n-1$ , then we have

$$\begin{aligned}
& |R(I_n, f, \xi)| \\
& \leq \frac{1}{2} \sum_{i=0}^{n-1} \max \left\{ \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2, \frac{1}{4} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^2, \frac{(\xi_i - x_i)^2}{4} \right\} \bigvee_{x_i}^{x_{i+1}} (f') \\
& \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2, \frac{1}{4} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^2, \frac{(\xi_i - x_i)^2}{4} \right\} \right\} \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f') \\
& \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ \left( \xi_i - \frac{3x_i + x_{i+1}}{4} \right)^2, \frac{1}{4} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right)^2, \frac{(\xi_i - x_i)^2}{4} \right\} \right\} \bigvee_a^b (f').
\end{aligned}$$

This completes the proof of the Theorem.  $\square$

**Corollary 5.** *Under the assumption of Theorem 7 with  $\xi_i = x_i$ , we have*

$$\int_a^b f(t) dt = \frac{1}{2} \sum_{i=0}^{n-1} \left[ f(\xi_i) + f(x_{i+1}) - \frac{f'(x_{i+1}) - f'(x_i)}{4} h_i \right] h_i + R(I_n, f)$$

where remainder term  $R(I_n, f)$  satisfies

$$|R(I_n, f)| \leq \frac{(v(h))^2}{8} \bigvee_a^b (f').$$

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