Received 25/02/16

NEW INEQUALITIES FOR OPERATOR NONCOMMUTATIVE PERSPECTIVES RELATED TO CONVEX FUNCTIONS

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Abstract. In this paper we obtain some new inequalities for operator noncommutative perspectives related to convex functions. Applications for weighted operator geometric mean and relative operator entropy are also given.

1. Introduction

If $\Phi : I \to \mathbb{R}$ is a convex function on the real interval I and T is a selfadjoint operator on the complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$ with the spectrum $\operatorname{Sp}(T) \subset \mathring{I}$ the interior of I, then we have the following

$$\langle \Phi(T) x, x \rangle \geq \Phi(\langle Tx, x \rangle) \tag{1.1}$$

for any $x \in H$ with $\|x\| = 1$.

For various Jensen's type inequalities for functions of selfadjoint operators, see the recent monograph [2] and the references therein.

In the recent paper [7] we showed that if A is a positive invertible operator and B is a selfadjoint operator such that $\operatorname{Sp}\left(A^{-1/2} B A^{-1/2}\right) \subset \mathring{I}$, then

$$\frac{\left\langle A^{1/2} \Phi\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} x, x \right\rangle}{\langle Ax, x \rangle} \geq \Phi\left(\frac{\langle Bx, x \rangle}{\langle Ax, x \rangle}\right), \tag{1.2}$$

for any $x \in H$, $x \neq 0$. This result can be reformulated in terms of perspective as follows.

Let Φ be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2} B A^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_\Phi(B, A)$ by setting

$$\mathcal{P}_\Phi(B, A) := A^{1/2} \Phi\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_\Phi(B, A) = A \Phi\left(B A^{-1}\right)$$

provided $\operatorname{Sp}\left(B A^{-1}\right) \subset \mathring{I}$.

It is well known that (see [9] and [8] or [10]), if Φ is an *operator convex function* defined in the positive half-line, then the mapping

$$(B, A) \to \mathcal{P}_\Phi(B, A)$$

defined in pairs of positive definite operators, is convex.

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60,.26D15, 26D10.

Key words and phrases. Young's Inequality, Convex functions, Arithmetic mean-Geometric mean inequality.

RGMIA Res. Rep. Coll. 19 (2016), Art. 26

Let $\Phi : [m, M] \to \mathbb{R}$ be a *convex function* on the real interval $[m, M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA \leq B \leq MA, \tag{1.3}$$

then we have

$$\begin{aligned} 0 &\leq \frac{\Phi(m)+\Phi(M)}{2}A + \frac{\Phi(M)-\Phi(m)}{M-m}\left(B - \frac{m+M}{2}A\right) - \mathcal{P}_{\Phi}(B, A) \\ &\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right) \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]A. \end{aligned} \tag{1.4}$$

This result has been obtained in an equivalent form in the recent paper [3].

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq \Phi''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.5}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.3) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well

$$\begin{aligned} &\frac{1}{2}d\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right) \\ &\leq \frac{\Phi(m)+\Phi(M)}{2}A + \frac{\Phi(M)-\Phi(m)}{M-m}\left(B - \frac{m+M}{2}A\right) - \mathcal{P}_{\Phi}(B, A) \\ &\leq \frac{1}{2}D\left(MA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - mA^{1/2}\right). \end{aligned} \tag{1.6}$$

This result has been obtained in an equivalent form in the recent paper [4].

If $d > 0$, then the first inequality in (1.6) is better than the same inequality in (1.4).

In this paper we obtain some new inequalities for operator noncommutative perspectives of convex functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

2. The Results

Let $t \in \mathbb{R}$ and consider the convex function $f_t : \mathbb{R} \to [0, \infty)$ defined by $f_t(x) := |x - t|$. We define the permanent

$$\mathcal{P}_{|\cdot|,t}(B, A) := \mathcal{P}_{f_t}(B, A) = A^{1/2}\left|A^{-1/2}(B - tA)A^{-1/2}\right|A^{1/2}.$$

We have:

Theorem 1. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I, A a positive invertible operator and B a selfadjoint operator such that*

$$Am \leq B \leq MA \tag{2.1}$$

with $[m, M] \subset \mathring{I}$, *for some real numbers* m, M *with* $m < M$. *Then we have*

$$\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(A-\frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right) \tag{2.2}$$
$$\leq \frac{\Phi(m)+\Phi(M)}{2}A+\frac{\Phi(M)-\Phi(m)}{M-m}\left(B-\frac{m+M}{2}A\right)-\mathcal{P}_f(B,A)$$
$$\leq \left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(A+\frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right).$$

Proof. Recall the following result obtained by Dragomir in 2006 [1] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$n\min_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right] \tag{2.3}$$
$$\leq \frac{1}{P_n}\sum_{j=1}^{n}p_j\Phi(x_j)-\Phi\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j\right)$$
$$\leq n\max_{j\in\{1,2,\dots,n\}}\{p_j\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi(x_j)-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_j\right)\right],$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (2.3) that

$$2\min\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right] \tag{2.4}$$
$$\leq \nu\Phi(x)+(1-\nu)\Phi(y)-\Phi[\nu x+(1-\nu)y]$$
$$\leq 2\max\{\nu,1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2}-\Phi\left(\frac{x+y}{2}\right)\right]$$

for any $x, y \in \mathbb{R}$ and $\nu \in [0, 1]$.

Let $t \in [m, M]$ and take $x = M$, $y = m$ and $\nu = \frac{t-m}{M-m}$. Then

$$\nu x+(1-\nu)y=\frac{t-m}{M-m}M+\frac{M-t}{M-m}m=t$$

and by (2.4) we get

$$2\min\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \tag{2.5}$$
$$\leq \frac{t-m}{M-m}\Phi(M)+\frac{M-t}{M-m}\Phi(m)-\Phi(t)$$
$$\leq 2\max\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$

for any $t \in [m, M]$.

Since

$$\min\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}=\frac{1}{2}-\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|$$

and

$$\max\left\{\frac{t-m}{M-m},\frac{M-t}{M-m}\right\}=\frac{1}{2}+\frac{1}{M-m}\left|x-\frac{m+M}{2}\right|$$

then by (2.5) we get

$$\begin{aligned}(2.6)\qquad &\left(1-\frac{2}{M-m}\left|x-\frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\\ &\le\frac{t-m}{M-m}\Phi(M)+\frac{M-t}{M-m}\Phi(m)-\Phi(t)\\ &\le\left(1+\frac{2}{M-m}\left|x-\frac{m+M}{2}\right|\right)\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\end{aligned}$$

for any $t\in[m,M]$.

Using the continuous functional calculus for a selfadjoint operator X with $\operatorname{Sp}(X)\subseteq[m,M]$ we have from (2.6) that

$$\begin{aligned}(2.7)\qquad &\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(I-\frac{2}{M-m}\left|X-\frac{m+M}{2}I\right|\right)\\ &\le\Phi(M)\frac{X-mI}{M-m}+\Phi(m)\frac{MI-X}{M-m}-\Phi(X)\\ &\le\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(I+\frac{2}{M-m}\left|X-\frac{m+M}{2}I\right|\right)\end{aligned}$$

and since

$$\begin{aligned}&\Phi(M)\frac{X-mI}{M-m}+\Phi(m)\frac{MI-X}{M-m}\\ &=\frac{\Phi(m)+\Phi(M)}{2}I+\Phi(m)\left(\frac{MI-X}{M-m}-\frac{1}{2}I\right)+\Phi(M)\left(\frac{X-mI}{M-m}-\frac{1}{2}I\right)\\ &=\frac{\Phi(m)+\Phi(M)}{2}I-\Phi(m)\left(\frac{X-\frac{m+M}{2}I}{M-m}\right)+\Phi(M)\left(\frac{X-\frac{m+M}{2}I}{M-m}\right)\\ &=\frac{\Phi(m)+\Phi(M)}{2}I+\frac{\Phi(M)-\Phi(m)}{M-m}\left(X-\frac{m+M}{2}I\right),\end{aligned}$$

then by (2.7) we get

$$\begin{aligned}(2.8)\qquad &\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(I-\frac{2}{M-m}\left|X-\frac{m+M}{2}I\right|\right)\\ &\le\frac{\Phi(m)+\Phi(M)}{2}I+\frac{\Phi(M)-\Phi(m)}{M-m}\left(X-\frac{m+M}{2}I\right)-\Phi(X)\\ &\le\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]\left(I+\frac{2}{M-m}\left|X-\frac{m+M}{2}\right|\right),\end{aligned}$$

which is an inequality of interest in itself.

If the condition (2.1) is valid, then by multiplying both sides by $A^{-1/2}$ we get

$$mI\le A^{-1/2}BA^{-1/2}\le MI.$$

Now, if we take $X = A^{-1/2}BA^{-1/2}$ in (2.4), then we get

$$\begin{aligned}
&\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(I - \frac{2}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}I\right|\right) \\
&\leq \frac{\Phi(m)+\Phi(M)}{2}I + \frac{\Phi(M)-\Phi(m)}{M-m}\left(A^{-1/2}BA^{-1/2} - \frac{m+M}{2}I\right) \\
&- \Phi\left(A^{-1/2}BA^{-1/2}\right) \\
&\leq \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(I + \frac{2}{M-m}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}\right|\right).
\end{aligned} \tag{2.9}$$

By multiplying both sides of (2.9) with $A^{1/2}$ we get

$$\begin{aligned}
&\left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(A - \frac{2}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}I\right|A^{1/2}\right) \\
&\leq \frac{\Phi(m)+\Phi(M)}{2}A + \frac{\Phi(M)-\Phi(m)}{M-m}\left(BA - \frac{m+M}{2}A\right) \\
&- A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \\
&\leq \left[\frac{\Phi(m)+\Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\
&\times \left(A + \frac{2}{M-m}A^{1/2}\left|A^{-1/2}BA^{-1/2} - \frac{m+M}{2}\right|A^{1/2}\right)
\end{aligned}$$

and the inequality (2.2) is proved. □

For the function $f_{c,2} : \mathbb{R} \to [0,\infty)$, $f_{c,2}(x) = (x-c)^2$ we consider the perspective

$$\mathcal{P}_{f_{c,2}}(B,A) := A^{1/2} f_{c,2}\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} \tag{2.10}$$

where A is a positive invertible operator and B a selfadjoint operator on the Hilbert space H.

We observe that

$$\begin{aligned}
\mathcal{P}_{f_{c,2}}(B,A) &= A^{1/2}\left(A^{-1/2}BA^{-1/2} - cI\right)^2 A^{1/2} \\
&= A^{1/2}\left(A^{-1/2}(B-cA)A^{-1/2}\right)^2 A^{1/2} \\
&= (B-cA)A^{-1}(B-cA) = (cA-B)A^{-1}(cA-B).
\end{aligned} \tag{2.11}$$

We have:

Theorem 2. *With the assumptions of Theorem 2 and if we define the function*

$$\Psi_\Phi(t;m,M) = \frac{\Phi(M)-\Phi(t)}{M-t} - \frac{\Phi(t)-\Phi(m)}{t-m} \geq 0,\ t \in (m,M)$$

then we have

$$\begin{aligned}
(2.12)\qquad & \frac{\Phi(m)+\Phi(M)}{2}A+\frac{\Phi(M)-\Phi(m)}{M-m}\left(B-\frac{m+M}{2}A\right)-\mathcal{P}_f(B,A) \\
&\le \frac{1}{M-m}\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)\left[\frac{1}{4}(M-m)^2A-\mathcal{P}_{f_{\frac{m+M}{2},2}}(B,A)\right] \\
&\le \frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\left[\frac{1}{4}(M-m)^2A-\mathcal{P}_{f_{\frac{m+M}{2},2}}(B,A)\right] \\
&\le \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]A
\end{aligned}$$

and

$$\begin{aligned}
(2.13)\qquad & \frac{\Phi(m)+\Phi(M)}{2}A+\frac{\Phi(M)-\Phi(m)}{M-m}\left(B-\frac{m+M}{2}A\right)-\mathcal{P}_f(B,A) \\
&\le \frac{1}{4}(M-m)\mathcal{P}_{\Psi_\Phi(\cdot;m,M)}(B,A)\le\frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_\Phi(t;m,M)A \\
&\le \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]A,
\end{aligned}$$

where

$$\mathcal{P}_{f_{\frac{m+M}{2},2}}(B,A)=\left(B-\frac{m+M}{2}A\right)A^{-1}\left(B-\frac{m+M}{2}A\right)$$

and

$$\begin{aligned}
&\mathcal{P}_{\Psi_\Phi(\cdot;m,M)}(B,A) \\
&=A^{1/2}\left(\Phi(M)I-\Phi\left(A^{-1/2}BA^{-1/2}\right)\right)\left(MI-A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2} \\
&-A^{1/2}\left(\Phi\left(A^{-1/2}BA^{-1/2}\right)-\Phi(m)I\right)\left(A^{-1/2}BA^{-1/2}-mI\right)^{-1}A^{1/2}.
\end{aligned}$$

Proof. By denoting

$$\Delta_\Phi(t;m,M):=\frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m}-\Phi(t)\ge 0\quad t\in[m,M]$$

we have

$$\begin{aligned}
(2.14)\qquad \Delta_\Phi(t;m,M) &= \frac{(t-m)\Phi(M)+(M-t)\Phi(m)-(M-m)\Phi(t)}{M-m} \\
&= \frac{(t-m)\Phi(M)+(M-t)\Phi(m)-(M-t+t-m)\Phi(t)}{M-m} \\
&= \frac{(t-m)\left[\Phi(M)-\Phi(t)\right]-(M-t)\left[\Phi(t)-\Phi(m)\right]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Psi_\Phi(t;m,M).
\end{aligned}$$

Since Φ is a convex function, then we have

$$\begin{aligned}\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M) &= \sup_{t\in(m,M)}\left[\frac{\Phi(M)-\Phi(t)}{M-t}-\frac{\Phi(t)-\Phi(m)}{t-m}\right]\\ &\le \sup_{t\in(m,M)}\left[\frac{\Phi(M)-\Phi(t)}{M-t}\right]+\sup_{t\in(m,M)}\left[-\frac{\Phi(t)-\Phi(m)}{t-m}\right]\\ &= \sup_{t\in(m,M)}\left[\frac{\Phi(M)-\Phi(t)}{M-t}\right]-\inf_{t\in(m,M)}\left[\frac{\Phi(t)-\Phi(m)}{t-m}\right]\\ &= \Phi'_-(M)-\Phi'_+(m).\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{(M-t)(t-m)}{M-m}\Psi_{\Phi}(t;m,M)\\ &\le\begin{cases}\frac{(M-t)(t-m)}{M-m}\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M)\\ \\ \Psi_{\Phi}(t;m,M)\sup_{t\in(m,M)}\left[\frac{(M-t)(t-m)}{M-m}\right]\end{cases}\\ &=\begin{cases}\frac{(M-t)(t-m)}{M-m}\left[\Phi'_-(M)-\Phi'_+(m)\right]\\ \\ \frac{1}{4}(M-m)\Psi_{\Phi}(t;m,M)\end{cases}\\ &\le\frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]\end{aligned}$$

for any $t\in(m,M)$.

By making use of (2.14) we get

$$\begin{aligned}0&\le\frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m}-\Phi(t)\\ &\le\begin{cases}\frac{(M-t)(t-m)}{M-m}\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M)\\ \\ \frac{1}{4}(M-m)\Psi_{\Phi}(t;m,M)\end{cases}\\ &\le\begin{cases}\frac{(M-t)(t-m)}{M-m}\left[\Phi'_-(M)-\Phi'_+(m)\right]\\ \\ \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_{\Phi}(t;m,M)\end{cases}\\ &\le\frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]\end{aligned}$$

and since, as in the proof of Theorem 1,

$$\begin{aligned}&\frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m}-\Phi(t)\\ &=\frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(t-\frac{m+M}{2}\right)-\Phi(t),\end{aligned}$$

and

$$(M-t)(t-m)=\frac{1}{4}(M-m)^2-\left(t-\frac{m+M}{2}\right)^2$$

then we have the following inequality of interest in itself

$$
\begin{aligned}
0 &\le \frac{\Phi(m)+\Phi(M)}{2}+\frac{\Phi(M)-\Phi(m)}{M-m}\left(t-\frac{m+M}{2}\right)-\Phi(t) \\
&\le \begin{cases} \frac{1}{M-m}\left[\frac{1}{4}(M-m)^2-\left(t-\frac{m+M}{2}\right)^2\right]\sup_{t\in(m,M)}\Psi_\Phi(t;m,M) \\ \\ \frac{1}{4}(M-m)\Psi_\Phi(t;m,M) \end{cases} \\
&\le \begin{cases} \frac{1}{M-m}\left[\frac{1}{4}(M-m)^2-\left(t-\frac{m+M}{2}\right)^2\right]\left[\Phi'_-(M)-\Phi'_+(m)\right] \\ \\ \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_\Phi(t;m,M) \end{cases} \\
&\le \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right]
\end{aligned}
$$

for any $t\in(m,M)$.

Now, by employing a similar argument to the one in Theorem 1 for the operator $A^{-1/2}BA^{-1/2}$ we deduce the desired results (2.12). The details are omitted. □

3. Applications for Operator Geometric Mean

Assume that A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators [20]

$$A\nabla_\nu B:=(1-\nu)A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean,* where $\nu\in[0,1]$. When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The definition of $A\sharp_\nu B$ can be extended accordingly for any real number ν.

The following inequality is well as the operator *Young inequality* or operator *ν-weighted arithmetic-geometric mean inequality*:

$$A\sharp_\nu B\le A\nabla_\nu B \text{ for all } \nu\in[0,1]. \tag{3.1}$$

For recent results on operator Young inequality see [13]-[16], [17] and [25]-[26].

For $x\neq y$ and $p\in\mathbb{R}\setminus\{-1,0\}$, we define the *$p$-logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y):=\left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

In fact the singularities at $p=-1,\ 0$ are removable and L_p can be defined for $p=-1,\ 0$ so as to make $L_p(x,y)$ a continuous function of p. In the limit as $p\to 0$ we obtain the *identric mean* $I(x,y)$, given by

$$I(x,y):=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}, \tag{3.2}$$

and in the case $p\to -1$ the *logarithmic mean* $L(x,y)$, given by

$$L(x,y):=\frac{y-x}{\ln y-\ln x}.$$

In each case we define the mean as x when $y = x$, which occurs as the limiting value of $L_p(x, y)$ for $y \to x$.

If we consider the continuous function $f_\nu : [0, \infty) \to [0, \infty)$, $f_\nu(x) = x^\nu$ then the operator *ν-weighted arithmetic-geometric mean* can be interpreted as the perspective $\mathcal{P}_{f_\nu}(B, A)$, namely

$$\mathcal{P}_{f_\nu}(B, A) = A\sharp_\nu B.$$

Consider the convex function $f = -f_\nu$. Then by applying the inequalities (2.2) we have

$$\begin{aligned} &\left[\left(\frac{m+M}{2}\right)^\nu - \frac{m^\nu + M^\nu}{2}\right]\left(A - \frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B, A)\right) \\ &\le A\sharp_\nu B - \frac{m^\nu + M^\nu}{2}A - \frac{M^\nu - m^\nu}{M-m}\left(B - \frac{m+M}{2}A\right) \\ &\le \left[\left(\frac{m+M}{2}\right)^\nu - \frac{m^\nu + M^\nu}{2}\right]\left(A + \frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B, A)\right) \end{aligned} \tag{3.3}$$

for any $\nu \in [0, 1]$ and A, B positive invertible operators that satisfy condition (2.1).

Consider the function

$$\Psi_\nu(t; m, M) = \frac{t^\nu - m^\nu}{t - m} - \frac{M^\nu - t^\nu}{M - t},\ t \in (m, M),$$

then by the inequalities (2.12) and (2.13) we have

$$\begin{aligned} &A\sharp_\nu B - \frac{m^\nu + M^\nu}{2}A - \frac{M^\nu - m^\nu}{M-m}\left(B - \frac{m+M}{2}A\right) \\ &\le \frac{1}{M-m}\sup_{t\in(m,M)}\Psi_\nu(t; m, M)\left[\frac{1}{4}(M-m)^2 A - \mathcal{P}_{f_{\frac{m+M}{2},2}}(B, A)\right] \\ &\le \nu\frac{M^{1-\nu} - m^{1-\nu}}{(M-m)M^{1-\nu}m^{1-\nu}}\left[\frac{1}{4}(M-m)^2 A - \mathcal{P}_{f_{\frac{m+M}{2},2}}(B, A)\right] \\ &\le \frac{1}{4}\nu\frac{M^{1-\nu} - m^{1-\nu}}{(M-m)M^{1-\nu}m^{1-\nu}}A \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\frac{\Phi(m) + \Phi(M)}{2}A + \frac{\Phi(M) - \Phi(m)}{M-m}\left(B - \frac{m+M}{2}A\right) - \mathcal{P}_f(B, A) \\ &\le \frac{1}{4}(M-m)\mathcal{P}_{\Psi_\nu(\cdot;m,M)}(B, A) \le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_\nu(t; m, M)A \\ &\le \frac{1}{4}\nu\frac{M^{1-\nu} - m^{1-\nu}}{(M-m)M^{1-\nu}m^{1-\nu}}A, \end{aligned} \tag{3.5}$$

for any $\nu \in [0, 1]$, where

$$\begin{aligned} &\mathcal{P}_{\Psi_\Phi(\cdot;m,M)}(B, A) \\ &= A^{1/2}\left(\left(A^{-1/2}BA^{-1/2}\right)^\nu - m^\nu I\right)\left(A^{-1/2}BA^{-1/2} - mI\right)^{-1}A^{1/2} \\ &- A^{1/2}\left(M^\nu I - \left(A^{-1/2}BA^{-1/2}\right)^\nu I\right)\left(MI - A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2} \end{aligned}$$

and A, B positive invertible operators that satisfy condition (2.1).

4. Applications for Relative Operator Entropy

Kamei and Fujii [11], [12] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}, \tag{4.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [24].

For some recent results on relative operator entropy see [5]-[6], [18]-[19] and [21]-[22].

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the permanent of ln, namely

$$\mathcal{P}_{\ln}(B,A) = S(A|B).$$

If we use the inequalities (2.2) for the convex function $f = -\ln$ we have

$$\begin{aligned} &\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(A - \frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right) \\ &\leq S(A|B) - \left(\ln\sqrt{mM}\right)A - \frac{\ln M - \ln m}{M-m}\left(B - \frac{m+M}{2}A\right) \\ &\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(A + \frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right), \end{aligned} \tag{4.2}$$

and A, B are positive invertible operators that satisfy condition (2.1).

Consider the function

$$\begin{aligned} \Psi_{-\ln}(t;m,M) &= \frac{\ln t - \ln m}{t-m} + \frac{\ln t - \ln M}{M-t} \\ &= \ln\left[\left(\frac{t}{m}\right)^{\frac{1}{t-m}}\left(\frac{t}{M}\right)^{\frac{1}{M-t}}\right] \geq 0,\ t \in (m,M). \end{aligned}$$

Then by the inequalities (2.12) and (2.13) we have

$$\begin{aligned} &S(A|B) - \left(\ln\sqrt{mM}\right)A - \frac{\ln M - \ln m}{M-m}\left(B - \frac{m+M}{2}A\right) \\ &\leq \frac{1}{M-m}\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M)\left[\frac{1}{4}(M-m)^2 A - \mathcal{P}_{f_{\frac{m+M}{2}},2}(B,A)\right] \\ &\leq \frac{1}{Mm}\left[\frac{1}{4}(M-m)^2 A - \mathcal{P}_{f_{\frac{m+M}{2}},2}(B,A)\right] \leq \frac{1}{4}\frac{(M-m)^2}{Mm}A \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &S(A|B) - \left(\ln\sqrt{mM}\right)A - \frac{\ln M - \ln m}{M-m}\left(B - \frac{m+M}{2}A\right) \\ &\leq \frac{1}{4}(M-m)\mathcal{P}_{\Psi_{-\ln}(\cdot;m,M)}(B,A) \leq \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M)A \\ &\leq \frac{1}{4}\frac{(M-m)^2}{Mm}A, \end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
(4.5)\qquad & \mathcal{P}_{\Psi_{-\ln}(\cdot;m,M)}(B,A) \\
&= A^{1/2}\left(\ln\left(A^{-1/2}BA^{-1/2}\right)-\ln mI\right)\left(A^{-1/2}BA^{-1/2}-mI\right)^{-1}A^{1/2} \\
&- A^{1/2}\left(\ln MI-\ln\left(A^{-1/2}BA^{-1/2}\right)\right)\left(MI-A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2}
\end{aligned}
$$

and A, B are positive invertible operators that satisfy condition (2.1).

If we consider the entropy function $\eta(t)=-t\ln t$, then it is well known that for any positive invertible operators A, B we have

$$
S(A|B)=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}. \tag{4.6}
$$

The function $f(t)=t\ln t=-\eta(t)$, $t>0$, is convex, then the perspective of this function is

$$
\mathcal{P}_{(\cdot)\ln(\cdot)}(B,A)=-A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=-S(B|A),
$$

where for the last equality we used (4.6) for A replacing B.

From the inequality (2.2) we have for the convex function $f(t)=t\ln t$, $t>0$ that

$$
\begin{aligned}
(4.7)\qquad & \left[\frac{m\ln m+M\ln M}{2}-\frac{m+M}{2}\ln\left(\frac{m+M}{2}\right)\right] \\
&\times\left(A-\frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right) \\
&\le S(B|A)+\frac{m\ln m+M\ln M}{2}A+\frac{M\ln M-m\ln m}{M-m}\left(B-\frac{m+M}{2}A\right) \\
&\le\left[\frac{m\ln m+M\ln M}{2}-\frac{m+M}{2}\ln\left(\frac{m+M}{2}\right)\right] \\
&\times\left(A+\frac{2}{M-m}\mathcal{P}_{|\cdot|,\frac{m+M}{2}}(B,A)\right),
\end{aligned}
$$

where A, B are positive invertible operators that satisfy condition (2.1).

Similar results may be stated by employing the inequalities (2.12) and (2.13), however the details are not presented here.

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