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AN INEQUALITY OF OSTROWSKI-GRÜSS TYPE FOR DOUBLE INTEGRALS

HÜSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

Abstract. In this study, we establish Ostrowski-Grüss type involving functions of two independent variables for double integrals. Cubature formula is also provided.

1. Introduction

In 1935, G. Grüss [6] proved the following inequality:

(1.1)
$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi_1-\varphi_1)(\Phi_2-\varphi_2),$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi_1\leq f(x)\leq\Phi_1 \text{ and } \varphi_2\leq g(x)\leq\Phi_2 \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [8]:

Theorem 1 (Ostrowski inequality)**.** *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.3}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty, \tag{1.4}$$

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where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.5}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = ess\sup_{t\in[a,b]}|p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [3]:

Theorem 2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is L-Lipshitzian on $[a,b]$, i.e,*

$$|u(x)-u(y)|\le L|x-y| \quad \text{for all } x\in[a,b], \tag{1.6}$$

f is Riemann integrable on $[a,b]$ and there exist the real numbers m,M so that

$$m\le f(x)\le M \quad \text{for all } x\in[a,b]. \tag{1.7}$$

Then we have the inequality,

$$\left|\int_a^b f(x)du(x)-\frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2}L(M-m)(b-a).$$

From [7], if $f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt, \tag{1.8}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x\\ \frac{t-b}{b-a}, & x<t\le b.\end{cases}$$

In [4], Dragomir and Wang proved following Ostrowski-Grüss type inequality using the inequality (1.1) and Montgomery identity (1.8):

Theorem 3. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differantiable mapping in I° and let $a,b\in I^\circ$ with $a<b$. If $f\in L_1[a,b]$ and*

$$\varphi_3\le f'(x)\le\Phi_3,\ \ \forall x\in[a,b],$$

then we have the following inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\le\frac{1}{4}(b-a)(\Phi_3-\varphi_3), \tag{1.9}$$

for all $x\in[a,b]$.

Barnett and Dragomir established following Ostrowski inequality for double integrals in [1]:

Theorem 4. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous on $[a,b]\times[c,d]$, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$, and is bounded, i.e.,*

$$\|f_{xy}\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty$$

then we have the inequality

$$\begin{aligned}(1.10)\quad & \left|\int_a^b\int_c^d f(t,s)dsdt-\left[(b-a)\int_c^d f(x,s)ds+(d-c)\int_a^b f(t,y)dt\right.\right.\\ & \left.\left.-(b-a)(d-c)f(x,y)\right]\right| \\ \le\ & \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\|f_{xy}\|_\infty\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$.

In [1], the inequality (1.10) is established by the use of integral identity involving Peano kernels. In [9], Pachpatte obtained an inequality in the view (1.10) by using elementary analysis. The interested reader is also refered to ([1], [5], [9],[10],[12]-[14]) for Ostrowski type inequalities in several independent variables.

Recently, Sarikaya and Kiris have proved the following Grüss type inequality for double integrals in [11]:

Theorem 5. *Let* $f,g:[a,b]\times[c,d]\to\mathbb{R}$ *be two functions defined and integrable on* $[a,b]\times[c,d]$. *Then for*

$$\varphi\le f(x,y)\le\Phi \text{ and } \gamma\le g(x,y)\le\Gamma. \text{ for all } (x,y)\in[a,b]\times[c,d]$$

we have

$$\begin{aligned}(1.11)\quad & \left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right.\\ & \left.-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right|\\ \le\ & \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).\end{aligned}$$

In this work, using the inequality (1.11), we will obtain an Ostrowski-Grüss type inequality for functions of two independent variables.

2. Main Results

Theorem 6. *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be a continuous on* $[a,b]\times[c,d]$, $f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$ *exists on* $(a,b)\times(c,d)$. *If* f *integrable and*

$$\varphi\le f_{xy}(x,y)\le\Phi,\ \forall(x,y)\in[a,b]\times[c,d]$$

then we have the following inequality

$$\begin{aligned}
(2.1)\quad & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f(x,s)ds + \frac{1}{(b-a)} \int_a^b f(t,y)dt \right.\right. \\
& \left.\left. -f(x,y) \right] - \frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)} \left(x - \frac{a+b}{2} \right) \left(y - \frac{c+d}{2} \right) \right| \\
\leq\ & \frac{1}{4} (P-p)(\Phi - \varphi)
\end{aligned}$$

where

$$P = \max \{ (x-a)(y-c), (b-x)(d-y) \}$$

and

$$p = \min \{ (x-a)(y-d), (x-b)(y-c) \}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Proof. Define the kernel $p(x,t;y,s)$ by

$$p(x,t;y,s) := \begin{cases} (t-a)(s-c), & \text{if } (t,s) \in [a,x] \times [c,y] \\ (t-a)(s-d), & \text{if } (t,s) \in [a,x] \times (y,d] \\ (t-b)(s-c), & \text{if } (t,s) \in (x,b] \times [c,y] \\ (t-b)(s-d), & \text{if } (t,s) \in (x,b] \times (y,d]. \end{cases}$$

Then, we have

$$\begin{aligned}
(2.2)\qquad & \int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)dsdt \\
= & \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)dsdt + \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)dsdt \\
& + \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)dsdt + \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)dsdt \\
= & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us calculate the integrals I_1, I_2, I_3 and I_4. Firstly, we have the equality

$$
\begin{aligned}
(2.3) \qquad I_1 &= \int_a^x \int_c^y (t-a)(s-c) f_{ts}(t,s)\,ds\,dt \\
&= \int_a^x (t-a)\left[(y-c) f_t(t,y) - \int_c^y f_t(t,s)\,ds\right] dt \\
&= (y-c)\int_a^x (t-a) f_t(t,y)\,dt - \int_c^y \left(\int_a^x (t-a) f_t(t,s)\,dt\right) ds \\
&= (y-c)\left[(x-a) f(x,y) - \int_a^x f(t,y)\,dt\right] \\
&\quad - \int_c^y \left[(x-a) f(x,s) - \int_a^x f(t,s)\,dt\right] ds \\
&= (x-a)(y-c) f(x,y) - (y-c)\int_a^x f(t,y)\,dt \\
&\quad -(x-a)\int_c^y f(x,s)\,ds + \int_a^x \int_c^y f(t,s)\,ds\,dt.
\end{aligned}
$$

Also, similar compitations we have the equalities

$$
\begin{aligned}
(2.4) \qquad I_2 &= \int_a^x \int_y^d (t-a)(s-d) f_{ts}(t,s)\,ds\,dt \\
&= (x-a)(d-y) f(x,y) - (d-y)\int_a^x f(t,y)\,dt \\
&\quad -(x-a)\int_y^d f(x,s)\,ds + \int_a^x \int_y^d f(t,s)\,ds\,dt,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_x^b \int_c^y (t-b)(s-c) f_{ts}(t,s)\,ds\,dt \\
&= (b-x)(y-c) f(x,y) - (y-c)\int_x^b f(t,y)\,dt \\
&\quad -(b-x)\int_c^y f(x,s)\,ds + \int_x^b \int_c^y f(t,s)\,ds\,dt,
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
I_4 &= \int_x^b \int_y^d (t-b)(s-d) f_{ts}(t,s)\,ds\,dt \\
&= (b-x)(d-y) f(x,y) - (d-y)\int_x^b f(t,y)\,dt \\
&\quad -(b-x)\int_y^d f(x,s)\,ds + \int_x^b \int_y^d f(t,s)\,ds\,dt.
\end{aligned} \tag{2.6}
$$

If we substitute the equalities (2.3)-(2.6) in (2.2), then we have

$$
\begin{aligned}
&\int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,ds\,dt \\
&= (b-a)\,(d-c)\,f(x,y) - (b-a)\int_c^d f(x,s)\,ds - (d-c)\int_a^b f(t,y)\,dt + \int_a^b \int_c^d f(t,s)\,ds\,dt.
\end{aligned} \tag{2.7}
$$

Applying Theorem 5 to mappings $p(x,.;y,.)$ and $f_{ts}(.,.)$, we establish

$$
\begin{aligned}
&\left| \frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s) f_{ts}(t,s)\,ds\,dt \right. \\
&\left. - \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d p(x,t;y,s)\,ds\,dt \right) \left(\frac{1}{(b-a)\,(d-c)} \int_a^b \int_c^d f_{ts}(t,s)\,ds\,dt \right) \right| \\
&\leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma).
\end{aligned} \tag{2.8}
$$

where

$$\begin{aligned}
\Gamma &= \sup_{(t,s)\in[a,b]\times[c,d]} p(x,t;y,s) \\
&= \max\left\{ \sup_{(t,s)\in[a,x]\times[c,y]} (t-a)(s-c), \sup_{(t,s)\in[a,x]\times(y,d]} (t-a)(s-d), \right. \\
&\qquad \left. \sup_{(t,s)\in(x,b]\times[c,y]} (t-b)(s-c), \sup_{(t,s)\in(x,b]\times(y,d]} (t-b)(s-d) \right\} \\
&= \max\{(x-a)(y-c),(b-x)(d-y)\} = P,
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\gamma &= \inf_{(t,s)\in[a,b]\times[c,d]} p(x,t;y,s) \\
&= \min\left\{ \inf_{(t,s)\in[a,x]\times[c,y]} (t-a)(s-c), \inf_{(t,s)\in[a,x]\times(y,d]} (t-a)(s-d), \right. \\
&\qquad \left. \inf_{(t,s)\in(x,b]\times[c,y]} (t-b)(s-c), \inf_{(t,s)\in(x,b]\times(y,d]} (t-b)(s-d) \right\} \\
&= \min\{(x-a)(y-d),(x-b)(y-c)\} = p.
\end{aligned} \tag{2.10}$$

Also, we have the equalities

$$\begin{aligned}
&\int_a^b \int_c^d p(x,t;y,s)dsdt \\
&= \int_a^x \int_c^y (t-a)(s-c)dsdt + \int_a^x \int_y^d (t-a)(s-d)dsdt \\
&\quad + \int_x^b \int_c^y (t-b)(s-c)dsdt + \int_x^b \int_y^d (t-b)(s-d)dsdt \\
&= \frac{(x-a)^2(y-c)^2}{4} - \frac{(x-a)^2(d-y)^2}{4} - \frac{(b-x)^2(y-c)^2}{4} + \frac{(b-x)^2(d-y)^2}{4} \\
&= \frac{\left[(x-a)^2-(b-x)^2\right]\left[(y-c)^2-(d-y)^2\right]}{4} \\
&= (b-a)(d-c)\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right)
\end{aligned} \tag{2.11}$$

and

$$\int_a^b \int_c^d f_{ts}(t,s)dsdt = f(b,d)-f(b,c)-f(a,d)+f(a,c). \tag{2.12}$$

If we put the equalities (2.7) and (2.9)-(2.12) in (2.8), then we obtain the desired inequality (2.1). □

Corollary 1. *With the assumptions in Theorem 6, if* $|f_{xy}(x,y)| \leq M$ *for all* $(x,y) \in [a,b] \times [c,d]$ *and some positive constant* M*, then we have*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f(x,s)ds + \frac{1}{(b-a)} \int_a^b f(t,y)dt \right.\right. \\
&\left.\left. -f(x,y) \right] - \frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{(b-a)(d-c)} \left(x - \frac{a+b}{2} \right) \left(y - \frac{c+d}{2} \right) \right| \\
\leq\ & \frac{1}{2} (P-p) M
\end{aligned}$$

where

$$P = \max \{ (x-a)(y-c), (b-x)(d-y) \}$$

and

$$p = \min \{ (x-a)(y-d), (x-b)(y-c) \}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Corollary 2. *Under assumptions of Theorem 6 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, we have the following inequality*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right. \\
&\left. - \left[\frac{1}{(d-c)} \int_c^d f\left(\frac{a+b}{2}, s \right) ds + \frac{1}{(b-a)} \int_a^b f\left(t, \frac{c+d}{2} \right) dt - f\left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right] \right| \\
\leq\ & \frac{1}{8} (b-a)(d-c)(\Phi - \varphi).
\end{aligned}$$

Corollary 3. *Under assumption of Theorem 6 with* $x = b$ *and* $y = d$*, we get the inequality*

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt - \left[\frac{1}{(d-c)} \int_c^d f(b,s)ds + \frac{1}{(b-a)} \int_a^b f(t,d)dt \right.\right. \\
&\left.\left. -f(b,d) \right] - \frac{f(b,d)-f(b,c)-f(a,d)+f(a,c)}{4} \right| \\
\leq\ & \frac{1}{4} (b-a)(d-c)(\Phi - \varphi).
\end{aligned}$$

3. Applications for Cubature Formulae

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$ $(i = 0, ..., n-1)$, and $l_j := y_{j+1} - y_j$ $(j = 0, ..., m-1)$,

$$\upsilon(h) := \max \{ h_i | \ i = 0, ..., n-1 \},$$

$$\mu(l) := \max\left\{ l_j \middle| \ j = 0, ..., m-1 \right\}.$$

Then, the following theorem holds.

Theorem 7. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be as in Theorem 6 and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$ be intermediate points. Then we have the cubature formula:*

$$\begin{aligned}
(3.1)\qquad & \int_a^b \int_c^d f(t,s)dsdt \\
= \ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)ds + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(t, \eta_j)dt \\
& - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j f(\xi_i, \eta_j) \\
& + \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) \right] \\
& \times \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right) \\
& + R(\xi, \eta, I_n, J_m, f).
\end{aligned}$$

where the remainer term $R(\xi, \eta, I_n, J_m, f)$ satisfies the estimation

$$(3.2)\qquad |R(\xi, \eta, I_n, J_m, f)| \leq \frac{1}{4} \upsilon(h)\mu(l) \max_{i,j} \left(P_{ij} - p_{ij} \right) (\Phi - \varphi)$$

where

$$P_{ij} = \max\left\{ (\xi_i - x_i)(\eta_j - y_j), (x_{i+1} - \xi_i)(y_{j+1} - \eta_j) \right\},$$

and

$$p_{ij} = \min\left\{ (\xi_i - x_i)(\eta_j - y_{j+1}), (\xi_i - x_{i+1})(\eta_j - y_j) \right\}.$$

Proof. Aplying Theorem 6 on the bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we get

$$\begin{aligned}
(3.3)\qquad & \left| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(t,s)dsdt - \left[h_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)ds + l_j \int_{x_i}^{x_{i+1}} f(t, \eta_j)dt - h_i l_j f(\xi_i, \eta_j) \right] \right. \\
& - \left[f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j) \right] \\
& \left. \times \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right) \left(\eta_j - \frac{y_j + y_{j+1}}{2} \right) \right| \\
\leq \ & \frac{1}{4} h_i l_j \left(P_{ij} - p_{ij} \right) \left(\Phi_{ij} - \varphi_{ij} \right)
\end{aligned}$$

where

$$\Phi_{ij} := \sup_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} |f_{ts}(t,s)|, \quad \varphi_{ij} := \inf_{(t,s)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]} |f_{ts}(t,s)|$$

for all $i = 0,1,...,n-1$; $j = 0,1,...,m-1$.

Summing the inequality (3.3) over i from 0 to $n-1$ and j from 0 to $m-1$ and using the generalized triangle inequality, we get

$$\begin{aligned} |R(\xi,\eta,I_n,J_m,f)| &\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} h_i l_j \left(P_{ij}-p_{ij}\right)\left(\Phi_{ij}-\varphi_{ij}\right) \\ &\leq \frac{1}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\max_{ij}\left(\Phi_{ij}-\varphi_{ij}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} 1 \\ &= \frac{nm}{4}\upsilon(h)\mu(l)\max_{i,j}\left(P_{ij}-p_{ij}\right)\left(\Phi-\varphi\right). \end{aligned}$$

This completes the proof. □

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Düzce University, Faculty of Science and Arts, Department of Mathematics, Konuralp Campus, Düzce, Turkey
E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com