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ON WEIGHTED GRÜSS TYPE INEQUALITIES FOR DOUBLE INTEGRALS

HÜSEYIN BUDAK AND MEHMET ZEKI SARIKAYA

ABSTRACT. In this study, we obtain some new inequalities of weighted Grüss type for functions of two independent variables. The results presented here would provide extensions of those given in earlier works.

1. Introduction

In 1935, G. Grüss [3] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\leq f(x)\leq\Phi \ \text{ and } \ \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [1] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{1.3}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.4}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty=ess\sup_{t\in[a,b]}|p(t)|$.

The following result of weighted Grüss type was proved by Dragomir [2]:

Theorem 1. *Let f and g be two functions defined and integrable on $[a,b]$. If (1.2) holds, where φ,Φ,γ and Γ are given real constant, and $w:[a,b]\to[0,\infty)$ is*

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integrable and $\int_a^b w(x)dx > 0$, *then*

$$\left|\int_a^b w(x)dx.\int_a^b f(x)g(x)dx - \int_a^b f(x)w(x)dx.\int_a^b g(x)w(x)dx\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma)\left(\int_a^b w(x)dx\right)^2 \tag{1.5}$$

and the constant $\frac{1}{4}$ *is the best possible.*

In the last years, many authors were interested in the generalization of Grüss type inequalities for mapping of one variable, we can mention the works [2], [4]-[8].

Recently, Sarikaya and Kiris have proved the following Grüss type inequality for double integrals in [9]:

Theorem 2. *Let* $f, g : [a,b]\times[c,d] \to \mathbb{R}$ *be two functions defined and integrable on* $[a,b]\times[c,d]$. *Then for*

$$\varphi \leq f(x,y) \leq \Phi \text{ and } \gamma \leq g(x,y) \leq \Gamma \text{ for all } (x,y) \in [a,b]\times[c,d]$$

we have

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx - \left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right| \leq \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma). \tag{1.6}$$

In this study, we establish some new inequalities of weighted Grüss type involving functions of two independent variables for double integrals.

2. Main Results

Throughout this work, we assume that the weight function $h : [a,b]\times[c,d] \to [0,\infty)$, is integrable, nonnegative and satisfies

$$m(a,b;c,d) = \int_a^b\int_c^d h(x,y)dydx < \infty.$$

Definition 1. *Consider a function* $f : V \to \mathbb{R}$ *defined on a subset* V *of* $\mathbb{R}^n$, $n \in \mathbb{N}$. *Let* $L = (L_1, L_2, ..., L_n)$ *where* $L_i \geq 0$, $i = 1,2,...,n$. *We say that* f *is L-Lipschitzian function if*

$$|f(x) - f(y)| \leq \sum_{i=1}^{n} L\,|x_i - y_i|$$

for all $x, y \in V$ [9].

Theorem 3. *Let h be as above and let $f, g : [a, b] \to \mathbb{R}$ be two functions defined and integrable on $[a, b] \times [c, d]$. If*

$$\varphi \le f(x, y) \le \Phi \text{ and } \gamma \le g(x, y) \le \Gamma \text{ for all } x \in [a, b] \times [c, d], \tag{2.1}$$

then we have

$$\begin{aligned} &\left| \left(\int_a^b \int_c^d h(x, y) dy dx \right) \int_a^b \int_c^d f(x, y) g(x, y) dy dx \right. \\ &\left. - \left(\int_a^b \int_c^d f(x, y) h(x, y) dy dx \right) \left(\int_a^b \int_c^d g(x, y) h(x, y) dy dx \right) \right| \\ &\le \tfrac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) \left(\int_a^b \int_c^d h(x, y) dy dx \right)^2 . \end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by smaller one.

Proof. For mappings $f, g : [a, b] \times [c, d] \to \mathbb{R}$, we have the identity

$$\begin{aligned} &\int_a^b \int_c^d \int_a^b \int_c^d [f(x, s) - f(t, y)] [g(x, s) - g(t, y)] h(t, y) h(x, s) ds dt dy dx \\ &= \int_a^b \int_c^d \int_a^b \int_c^d [f(x, s) g(x, s) h(t, y) h(x, s) - f(x, s) g(t, y) h(t, y) h(x, s) \\ &f(t, y) g(x, s) h(t, y) h(x, s) + f(t, y) g(t, y) h(t, y) h(x, s)] \, ds dt dy dx \\ &= 2 \int_a^b \int_c^d h(t, y) dy dt \int_a^b \int_c^d f(x, s) g(x, s) ds dx \\ &-2 \left(\int_a^b \int_c^d f(x, s) h(x, s) ds dx \right) \left(\int_a^b \int_c^d g(t, y) h(t, y) dy dt \right) . \end{aligned} \tag{2.3}$$

Appling Cauchy-Buniakowski-Schwarz's inequality, we have the inequality

$$\left[\frac{1}{2[m(a,b;c,d)]^2}\int_a^b\int_c^d\int_a^b\int_c^d[f(x,s)-f(t,y)][g(x,s)-g(t,y)]h(t,y)h(x,s)dsdtdydx\right]^2 \tag{2.4}$$

$$\leq\left(\frac{1}{2[m(a,b;c,d)]^2}\int_a^b\int_c^d\int_a^b\int_c^d[f(x,s)-f(t,y)]h(t,y)h(x,s)dsdtdydx\right)$$

$$\times\left(\frac{1}{2[m(a,b;c,d)]^2}\int_a^b\int_c^d\int_a^b\int_c^d[g(x,s)-g(t,y)]h(t,y)h(x,s)dsdtdydx\right)$$

$$=\left[\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f^2(x,s)h(x,s)dsdx-\left(\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right)^2\right]$$

$$\times\left[\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g^2(x,s)h(x,s)dsdx-\left(\int_a^b\int_c^d g(x,s)h(x,s)dsdx\right)^2\right].$$

It is easy to observe that

$$\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f^2(x,s)h(x,s)dsdx-\left(\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right)^2$$

$$=\left(\Phi-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right)$$

$$\times\left(\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx-\varphi\right)$$

$$-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d[\Phi-f(x,s)][f(x,s)-\varphi]h(x,s)dsdx.$$

Since $[\Phi-f(x,s)][f(x,s)-\varphi]\geq 0$ for each $(x,s)\in[a,b]\times[c,d]$, then we get

$$\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f^2(x,s)h(x,s)dsdx-\left(\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right)^2$$

$$\leq\left(\Phi-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right) \tag{2.5}$$

$$\left(\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx-\varphi\right).$$

Similarly, we obtain

$$\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g^2(x,s)h(x,s)dsdx-\left(\int_a^b\int_c^d g(x,s)h(x,s)dsdx\right)^2$$

$$\leq\left(\Gamma-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g(x,s)h(x,s)dsdx\right)\tag{2.6}$$

$$\left(\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g(x,s)h(x,s)dsdx-\gamma\right).$$

Using (2.5) and (2.6) in (2.4), we get the following inequality
(2.7)

$$\left[\frac{1}{2(b-a)^2(d-c)^2}\int_a^b\int_c^d\int_a^b\int_c^d\left[f(x,s)-f(t,y)\right]\left[g(x,s)-g(t,y)\right]dsdtdydx\right]^2$$

$$\leq\left(\Phi-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx\right)\left(\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d f(x,s)h(x,s)dsdx-\varphi\right)$$

$$\times\left(\Gamma-\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g(x,s)h(x,s)dsdx\right)\left(\frac{1}{m(a,b;c,d)}\int_a^b\int_c^d g(x,s)h(x,s)dsdx-\gamma\right).$$

Now, using the elementary inequality for real numbers

$$4pq\leq(p+q)^2,\quad p,q\in\mathbb{R}$$

we get

$$\left[\frac{1}{2[m(a,b;c,d)]^2}\int_a^b\int_c^d\int_a^b\int_c^d\left[f(x,s)-f(t,y)\right]\left[g(x,s)-g(t,y)\right]h(t,y)h(x,s)dsdtdydx\right]^2$$

$$\leq\frac{1}{16}(\Phi-\varphi)^2(\Gamma-\gamma)^2$$

which completes the proof. To prove the sharpness of (2.2), let choose $h(x,y)=1$ and

$$f(x,y)=g(x,y)=\begin{cases}1, & a\leq x<\frac{a+b}{2},\ c\leq y<\frac{c+d}{2}\\ -1, & a\leq x<\frac{a+b}{2},\ \frac{c+d}{2}\leq y\leq d\\ -1, & \frac{a+b}{2}\leq x\leq b,\ c\leq y<\frac{c+d}{2}\\ 1, & \frac{a+b}{2}\leq x\leq b,\ \frac{c+d}{2}\leq y\leq d\end{cases}$$

then

$$\int_a^b\int_c^d h(x,y)dydx=(b-a)(d-c),$$

$$\int_a^b\int_c^d f(x,y)g(x,y)dydx=(b-a)\left(d-c\right),$$

$$\int_a^b\int_c^d f(x,y)h(x,y)dydx = \int_a^b\int_c^d g(x,y)h(x,y)dydx = 0$$

and

$$(\Phi-\varphi) = (\Gamma-\gamma) = 2$$

which the equality (2.2) is realized. This implies the constant $\frac{1}{4}$ is the best possible. □

Remark 1. *If we choose $h(x,y)=1$ in (2.2), then the inequality (2.2) reduces the inequality (1.6).*

The following inequality of weighted Gruss type for Lipschitzian mappings holds:

Theorem 4. *Let $f,g:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ satisfies L-Lipschitzian conditions. That is, for (x,s) and (t,y) belong to $\Delta:=[a,b]\times[c,d]$, then we have*

$$|f(x,s)-f(t,y)| \leq L_1|x-t| + L_2|s-y|$$

$$|g(x,s)-g(t,y)| \leq L_3|x-t| + L_4|s-y|$$

where L_1, L_2, L_3 and L_4 are positive constants. Then, we have the following inequality:

$$\begin{aligned}
&\left|\left(\int_a^b\int_c^d h(x,y)dydx\right)\int_a^b\int_c^d f(x,y)g(x,y)dydx\right. \\
&\left.-\left(\int_a^b\int_c^d f(x,y)h(x,y)dydx\right)\left(\int_a^b\int_c^d g(x,y)h(x,y)dydx\right)\right| \\
&\leq L_1L_3\left[m(a,b;c,d)\int_a^b\int_c^d x^2h(x,y)dydx - \left(\int_a^b\int_c^d xh(x,y)dydx\right)^2\right] \\
&+L_2L_4\left[m(a,b;c,d)\int_a^b\int_c^d y^2h(x,y)dydx - \left(\int_a^b\int_c^d yh(x,y)dydx\right)^2\right] \\
&+\frac{L_1L_4+L_2L_3}{2}\int_a^b\int_c^d A(x,y)dxdy
\end{aligned} \tag{2.8}$$

where

$$A(x,y) = \int_a^b\int_c^d |x-t|\,|y-s|\,h(t,y)h(x,s)dsdt.$$

Proof. Since f,g are L-Lipschitzian, we have

$$\begin{aligned}
&|[f(x,s)-f(t,y)][g(x,s)-g(t,y)]| \\
\leq\ & L_1L_3(x-t)^2 + L_2L_4(s-y)^2 + (L_1L_4+L_2L_3)|x-t|\,|s-y|
\end{aligned}$$

for all (x,s), $(t,y) \in \Delta := [a,b] \times [c,d]$. Then by (2.3) we have that

$$\left| \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)] [g(x,s) - g(t,y)] h(t,y) h(x,s) ds dt dy dx \right|$$

$$\leq \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d |[f(x,s) - f(t,y)] [g(x,s) - g(t,y)]| h(t,y) h(x,s) ds dt dy dx$$

$$\leq \frac{1}{2} \int_a^b \int_c^d \int_a^b \int_c^d \left[L_1 L_3 (x-t)^2 + L_2 L_4 (s-y)^2 \right.$$

$$\left. + (L_1 L_4 + L_2 L_3) |x-t| |s-y| \right] h(t,y) h(x,s) ds dt dy dx$$

$$= L_1 L_3 \left[m(a,b;c,d) \int_a^b \int_c^d x^2 h(x,y) dy dx - \left(\int_a^b \int_c^d x h(x,y) dy dx \right)^2 \right]$$

$$+ L_2 L_4 \left[m(a,b;c,d) \int_a^b \int_c^d y^2 h(x,y) dy dx - \left(\int_a^b \int_c^d y h(x,y) dy dx \right)^2 \right]$$

$$+ \frac{L_1 L_4 + L_2 L_3}{2} \int_a^b \int_c^d A(x,y) dx dy$$

which completes the proof. □

Remark 2. *If we choose $h(x,y) = 1$ in the inequality (2.8), then we have the inequality*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) dy dx \right.$$

$$\left. - \left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx \right) \left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx \right) \right|$$

$$\leq \frac{(b-a)^2}{12} L_1 L_3 + \frac{(d-c)^2}{12} L_2 L_4 + \frac{(b-a)(d-c)}{18} (L_1 L_4 + L_2 L_3)$$

which was given by Sarikaya and Kiriş in [9].

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Düzce University, Faculty of Science and Arts, Department of Mathematics, Konuralp Campus, Düzce, Turkey
E-mail address: hsyn.budak@gmail.com

E-mail address: sarikayamz@gmail.com