Received 25/02/16

ON HERMITE-HADAMARD TYPE INEQUALITIES FOR s-CONVEX MAPPINGS VIA FRACTIONAL INTEGRALS OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

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Abstract. In this paper, we obtain some Hermite-Hadamard type inequalities for $s-$ convex function via fractional integrals with respect to another function which generalize the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 2. [4] *Let s be a real numbers, $s\in(0,1]$. A function $f:[0,\infty)\to[0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if f*

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y)$$

for all $x,y\in[0,\infty)$ and $\lambda\in[0,1]$.

An $s-$convex function was introduced in Breckner's paper [4] and a number of properties and connections with $s-$convexity in the first sense are discussed in paper [13]. Of course, $s-$convexity means just convexity when $s=1$.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[17, p.137], [10]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 10, 11, 17, 22, 23]) and the references cited therein.

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D15, 26A33.

Key words and phrases. Hermite-Hadamard's inequalities, Riemann-Liouville fractional integral, integral inequalities.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [12, 15, 16, 18].

Definition 3. *Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al.[20] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

with $\alpha > 0$.

Definition 4. *Let $f \in L_1[a,b]$. The Hadamard fractional integrals $\mathbf{J}_{a+}^{\alpha}f$ and $\mathbf{J}_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\mathbf{J}_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$\mathbf{J}_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)dt, \quad x < b$$

respectively.

Definition 5. *Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left-sides $(I_{a^+;g}^{\alpha}f(x))$ and right-sides $(I_{b^-;g}^{\alpha}f(x))$ fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha < 0$ are defined by*

$$I_{a^+;g}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)}{[g(x)-g(t)]^{1-\alpha}}dt, \quad x > a$$

and

$$I_{b^-;g}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)}{[g(t)-g(x)]^{1-\alpha}}dt, \quad x < b$$

respectively.

In [14], Jleli and Samet gave the following equality:

Lemma 1. *Let $\alpha > 0$ and let $\Xi_{\alpha,g} : [0,1] \to \mathbb{R}$ be a function defined by*

$$\begin{aligned}\Xi_{\alpha,g}(t) &= [g(ta+(1-t)b)-g(a)]^{\alpha} - [g(tb+(1-t)a)-g(a)]^{\alpha} \\ &\quad + [g(b)-g(tb+(1-t)a)]^{\alpha} - [g(b)-g(ta+(1-t)b)]^{\alpha}.\end{aligned}$$

If $f \in C^1(I^\circ)$, then

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4[g(b)-g(a)]^{\alpha}}\left(I^{\alpha}_{a^+;g}F(b)+I^{\alpha}_{b^-;g}F(a)\right) \\ &= \frac{b-a}{4[g(b)-g(a)]^{\alpha}}\int_0^1 \Xi_{\alpha,g}(t) f'(ta+(1-t)b)dt.\end{aligned}$$

For some recent results connected with fractional integral inequalities see ([3],[5]-[9],[19],[21],[24]-[27])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for$s-$ convex function involving fractional integrals with respect to another function. The results presented in this paper provide extensions of those given in earlier works.

2. Main Results

First, we will give following notation:

$$\begin{aligned}H_1(\alpha,s;g) &= \int_0^1 \frac{[t^s+(1-t)^s]\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}dt, \\ H_2(\alpha,s;g) &= \int_0^1 \frac{[t^s+(1-t)^s]\, g'((1-t)a+tb)}{[g((1-t)a+tb)-g(a)]^{1-\alpha}}dt.\end{aligned}$$

For $g(t)=t$, we have

$$H_1(\alpha,s;g) = H_2(\alpha,s;g) = (b-a)^{\alpha-1}\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]$$

where $\beta(x,y)$ is the Beta function.

For $\alpha>0$ and $s\in(0,1]$, we give following operator

$$L_g^{\alpha,s}(x,y) = \int_a^{\frac{a+b}{2}} |x-u|^s\,|g(y)-g(u)|^{\alpha}\,du - \int_{\frac{a+b}{2}}^b |x-u|^s\,|g(y)-g(u)|^{\alpha}\,du, \quad x,y\in[a,b]$$

Particularly, for $g(t)=t$, we have

$$L_g^{\alpha,s}(b,b) = -L_g^{\alpha,s}(a,a) = (b-a)^{\alpha+s+1}\frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}$$

and

$$L_g^{\alpha,s}(a,b) = -L_g^{\alpha,s}(b,a) = (b-a)^{\alpha+s+1}\left[\beta\left(\frac{1}{2};s+1,\alpha+1\right)-\beta\left(\frac{1}{2};\alpha+1,s+1\right)\right]$$

where $\beta(z;x,y)$ is the incomplete Beta function.

Let $f : I^{\circ} \to \mathbb{R}$ be a function such that $a, b \in I^{\circ}$ and $0 < a < b < \infty$. We suppose that $f \in L^{\infty}(a,b)$ in such a way that $I^{\alpha}_{a^+;g}f(x)$ and $I^{\alpha}_{b^-;g}f(x)$ are well defined. We define the function

$$\begin{aligned} \widetilde{f}(x) &= f(a+b-x),\ x \in [a,b] \\ F(x) &= f(x) + \widetilde{f}(x),\ x \in [a,b]. \end{aligned}$$

Theorem 2. *Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and $\alpha < 0$. If f is a s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then the following Hermite-Hadamard type inequality for fractional integrals hold:*

(2.1)
$$\begin{aligned} 2^{s-1} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}} \left[\frac{I^{\alpha}_{a^+;g}F(b) + I^{\alpha}_{b^-;g}F(a)}{2}\right] \\ &\le \left[\frac{f(a)+f(b)}{2}\right] \frac{\alpha\,(b-a)}{\left[g(b)-g(a)\right]^{\alpha}} \left[\frac{H_1(\alpha,s;g)+H_2(\alpha,s;g)}{2}\right]. \end{aligned}$$

Proof. Since f is a s-convex mapping in the second sense on $[a,b]$, we have for $x, y \in [a,b]$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2^s} \tag{2.2}$$

Now, for $t \in [0,1]$, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Then we have

$$2^s f\left(\frac{a+b}{2}\right) \le f\left(ta+(1-t)b\right) + f\left((1-t)a+tb\right) \tag{2.3}$$

Multiplying both sides of (2.3) by

$$\frac{b-a}{\Gamma(\alpha)} \frac{g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}}$$

and integraing the resulting inequality with respect to t over $(0,1)$, we get

$$\begin{aligned} &\frac{2^s\,(b-a)}{\Gamma(\alpha)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}} dt \\ &\le \frac{b-a}{\Gamma(\alpha)} \int_0^1 \frac{f\left(ta+(1-t)b\right) g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}} dt \\ &\quad + \frac{b-a}{\Gamma(\alpha)} \int_0^1 \frac{f\left((1-t)a+tb\right) g'\left((1-t)a+tb\right)}{\left[g(b)-g\left((1-t)a+tb\right)\right]^{1-\alpha}} dt. \end{aligned}$$

Using the change of variable $\tau = (1-t)a + tb$, we have

$$\frac{2^s}{\Gamma(\alpha)} f\left(\frac{a+b}{2}\right) \frac{\left[g(b)-g(a)\right]^{\alpha}}{\alpha} \le I^{\alpha}_{a^+;g}\widetilde{f}(b) + I^{\alpha}_{a^+;g}f(b)$$

i.e.

$$\frac{2^s}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)[g(b)-g(a)]^\alpha \leq I_{a^+;g}^\alpha F(b). \tag{2.4}$$

Similarly, multiplying both sides of (2.3) by

$$\frac{b-a}{\Gamma(\alpha)} \frac{g'((1-t)a+tb)}{[g((1-t)a+tb)-g(a)]^{1-\alpha}}$$

and integrating the resulting inequality with respect to t over $(0,1)$, we obtain

$$\frac{2^s}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)[g(b)-g(a)]^\alpha \leq I_{b^-;g}^\alpha F(a). \tag{2.5}$$

Summing the inequalities (2.4) and (2.5), we get

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2[g(b)-g(a)]^\alpha}\left[\frac{I_{a^+;g}^\alpha F(b)+I_{b^-;g}^\alpha F(a)}{2}\right]$$

This completes the proof of first inequality in (2.1).

For the proof of the second inequality in (2.1), since f is s-convex in the second sence, we have

$$f(ta+(1-t)b) \leq t^s f(a)+(1-t)^s f(b)$$

and

$$f((1-t)a+tb) \leq (1-t)^s f(a)+t^s f(b).$$

By adding these inequalities we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq [t^s+(1-t)^s][f(a)+f(b)]. \tag{2.6}$$

Multiplying both sides of (2.6) by

$$\frac{b-a}{\Gamma(\alpha)} \frac{g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}}$$

and integraing the resulting inequality with respect to t over $(0,1)$, we have

$$\begin{aligned}
&\frac{b-a}{\Gamma(\alpha)} \int_0^1 \frac{f(ta+(1-t)b)\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}} dt \\
&+\frac{b-a}{\Gamma(\alpha)} \int_0^1 \frac{f((1-t)a+tb)\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}} dt \\
\leq\ & [f(a)+f(b)] \frac{b-a}{\Gamma(\alpha)} \int_0^1 \frac{[t^s+(1-t)^s]\, g'((1-t)a+tb)}{[g(b)-g((1-t)a+tb)]^{1-\alpha}} dt.
\end{aligned}$$

Then, we get

$$I_{a^+;g}^\alpha \widetilde{f}(b)+I_{a^+;g}^\alpha f(b) \leq [f(a)+f(b)] \frac{b-a}{\Gamma(\alpha)} H_1(\alpha, s; g),$$

that is,

$$I_{a^+;g}^\alpha F(b) \leq [f(a)+f(b)] \frac{b-a}{\Gamma(\alpha)} H_1(\alpha, s; g). \tag{2.7}$$

Similarly, multiplying both sides of (2.6) by

$$\frac{b-a}{\Gamma(\alpha)}\frac{g'\left((1-t)a+tb\right)}{\left[g\left((1-t)a+tb\right)-g(a)\right]^{1-\alpha}}$$

and integraing the resulting inequality with respect to t over $(0,1)$, we get

$$I_{b^{-};g}^{\alpha}F(a)\leq\left[f(a)+f(b)\right]\frac{b-a}{\Gamma(\alpha)}H_2(\alpha,s;g). \tag{2.8}$$

By adding the inequalities (2.7) and (2.8), we have

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2\left[g(b)-g(a)\right]^{\alpha}}\left[\frac{I_{b^{-};g}^{\alpha}F(a)+I_{a^{+};g}^{\alpha}F(b)}{2}\right]\\ \leq\;&\left[\frac{f(a)+f(b)}{2}\right]\frac{\alpha\left(b-a\right)}{\left[g(b)-g(a)\right]^{\alpha}}\left[\frac{H_1(\alpha,s;g)+H_2(\alpha,s;g)}{2}\right].\end{aligned}$$

which completes te proof. □

Remark 1. *If we put $s=1$ in (2.1), we obtain Theorem 2.1 in* [14].

Remark 2. *If we choose $g(t)=t$ we obtain following inequality*

$$\begin{aligned}2^{s-1}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)}{2}\right]\\&\leq\alpha\left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right]\frac{f(a)+f(b)}{2}\end{aligned}$$

which proved by Set et al. in [25]

Corollary 1. *Under assumption of Theorem 2 with $g(t)=\ln t$, we have the following inequality*

$$\begin{aligned}2^{s-1}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\frac{\mathbf{J}_{a^{+}}^{\alpha}F(b)+\mathbf{J}_{b^{-}}^{\alpha}F(a)}{2}\right]\\&\leq\left[\frac{f(a)+f(b)}{2}\right]\frac{\alpha\left(b-a\right)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\frac{H_1(\alpha,s;\ln)+H_2(\alpha,s;\ln)}{2}\right].\end{aligned}$$

Theorem 3. *Let g be as above. If $f\in C^1(I^{\circ})$ and $|f'|$is a s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have the inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I_{a^{+};g}^{\alpha}F(b)+I_{b^{-};g}^{\alpha}F(a)\right)\right|\\ \leq\;&\frac{I_g^{\alpha,s}(a,b)}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\end{aligned}$$

where

$$I_g^{\alpha,s}(a,b)=L_g^{\alpha,s}(b,b)+L_g^{\alpha,s}(a,b)-L_g^{\alpha,s}(b,a)-L_g^{\alpha,s}(a,a).$$

Proof. Taking madulus in Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I_{a^{+};g}^{\alpha}F(b)+I_{b^{-};g}^{\alpha}F(a)\right)\right| \\
\leq\ & \frac{b-a}{4\left[g(b)-g(a)\right]^{\alpha}}\int_{0}^{1}\left|\Xi_{\alpha,g}(t)\right|\left|f'(ta+(1-t)b)\right|dt.
\end{aligned}$$

Since $|f'|$is a s-convex in the second sense on $[a,b]$, we get

$$|f'(ta+(1-t)b)|\leq t^{s}\left|f'(a)\right|+(1-t)^{s}\left|f'(b)\right|$$

hence,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{4\left[g(b)-g(a)\right]^{\alpha}}\left(I_{a^{+};g}^{\alpha}F(b)+I_{b^{-};g}^{\alpha}F(a)\right)\right| \\
\leq\ & \frac{b-a}{4\left[g(b)-g(a)\right]^{\alpha}}\left[\left|f'(a)\right|\int_{0}^{1}t^{s}\left|\Xi_{\alpha,g}(t)\right|dt+\left|f'(b)\right|\int_{0}^{1}(1-t)^{s}\left|\Xi_{\alpha,g}(t)\right|dt\right].
\end{aligned} \tag{2.9}$$

Here we have

$$\int_{0}^{1}t^{s}\left|\Xi_{\alpha,g}(t)\right|dt=\frac{1}{(b-a)^{s+1}}\int_{a}^{b}(b-u)^{s}\left|\varphi(u)\right|dt$$

where

$$\begin{aligned}
\varphi(u) &= \left[g\left(u\right)-g(a)\right]^{\alpha}-\left[g\left(a+b-u\right)-g(a)\right]^{\alpha} \\
&\quad+\left[g(b)-g\left(a+b-u\right)\right]^{\alpha}-\left[g(b)-g\left(u\right)\right]^{\alpha}.
\end{aligned}$$

Since g is a increasing function, φ is a non-decreasing function on $[a,b]$. Additionaly,

$$\varphi(a)=-2\left[g(b)-g\left(a\right)\right]^{\alpha}<0$$

and

$$\varphi\left(\frac{a+b}{2}\right)=0.$$

Consequently, we get

$$\begin{cases}\varphi(u)\leq 0 & \text{if } a\leq u\leq\frac{a+b}{2}\\ \varphi(u)>0 & \text{if } \frac{a+b}{2}<u\leq b.\end{cases}$$

Therefore, we have

$$\int_{a}^{b}(b-u)^{s}\left|\varphi(u)\right|dt=I_{1}+I_{2}+I_{3}+I_{4}$$

where

$$
\begin{aligned}
I_1 &= \int_a^{\frac{a+b}{2}} (b-u)^s \left[g(b)-g(u)\right]^\alpha du - \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(b)-g(u)\right]^\alpha du = L_g^{\alpha,s}(b,b) \\
I_2 &= -\int_a^{\frac{a+b}{2}} (b-u)^s \left[g(u)-g(a)\right]^\alpha du + \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(u)-g(a)\right]^\alpha du = -L_g^{\alpha,s}(b,a) \\
I_3 &= \int_a^{\frac{a+b}{2}} (b-u)^s \left[g(a+b-u)-g(a)\right]^\alpha du - \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(a+b-u)-g(a)\right]^\alpha du \\
&= -L_g^{\alpha,s}(a,a) \\
I_4 &= -\int_a^{\frac{a+b}{2}} (b-u)^s \left[g(b)-g(a+b-u)\right]^\alpha du + \int_{\frac{a+b}{2}}^b (b-u)^s \left[g(b)-g(a+b-u)\right]^\alpha du \\
&= L_g^{\alpha,s}(a,b).
\end{aligned}
$$

That is,

$$
\int_0^1 t^s \left|\Xi_{\alpha,g}(t)\right| dt = \frac{L_g^{\alpha,s}(b,b)+L_g^{\alpha,s}(a,b)-L_g^{\alpha,s}(b,a)-L_g^{\alpha,s}(a,a)}{(b-a)^{s+1}}. \tag{2.10}
$$

Similarly, it is clear that

$$
\int_0^1 (1-t)^s \left|\Xi_{\alpha,g}(t)\right| dt = \frac{L_g^{\alpha,s}(b,b)+L_g^{\alpha,s}(a,b)-L_g^{\alpha,s}(b,a)-L_g^{\alpha,s}(a,a)}{(b-a)^{s+1}}. \tag{2.11}
$$

If we put equality (2.10) and (2.11) in (2.9), we obtain the desired result. □

Remark 3. *If we put $s=1$ in (2.1), we obtain Theorem 2.5 in* [14].

Remark 4. *If we choose $g(t)=t$ we obtain following inequality*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]\right| \\
\leq\ & \frac{b-a}{2}\left\{\beta\left(\frac{1}{2}; s+1, \alpha+1\right) - \beta\left(\frac{1}{2}; \alpha+1, s+1\right)\right. \\
&\left. + \frac{2^{\alpha+s}-1}{2^{\alpha+s}(\alpha+s+1)}\right\}\left[\left|f'(a)\right| + \left|f'(b)\right|\right].
\end{aligned}
$$

This inequality given by Set et al. in [25, Theorem 4 (for $q=1$)]

Corollary 2. *Under assumption of Theorem 2 with $g(t)=\ln t$, we have the following inequality*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{4\left(\ln\frac{b}{a}\right)^\alpha}\left(\mathbf{J}_{a+}^\alpha F(b) + \mathbf{J}_{b-}^\alpha F(a)\right)\right| \leq \frac{I_{\ln}^{\alpha,s}(a,b)}{4\left(\ln\frac{b}{a}\right)^\alpha}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]
$$

where

$$I_{\ln}^{\alpha,s}(a,b) = L_{\ln}^{\alpha,s}(b,b) + L_{\ln}^{\alpha,s}(a,b) - L_{\ln}^{\alpha,s}(b,a) - L_{\ln}^{\alpha,s}(a,a).$$

References

[1] A.G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev. Colombiana Math., 28 (1994), 7-12.

[2] M. K. Bakula and J. Pečarić, *Note on some Hadamard-type inequalities,* Journal of Inequalities in Pure and Applied Mathematics, vol. 5, no. 3, article 74, 2004.

[3] S. Belarbi and Z. Dahmani, *On some new fractional integral inequalities*, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.

[4] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen,* Pupl. Inst. Math. 23(1978), 13-20.

[5] Z. Dahmani, *New inequalities in fractional integrals*, International Journal of Nonlinear Scinece, 9(4) (2010), 493-497.

[6] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(1) (2010), 51-58.

[7] Z. Dahmani, L. Tabharit, S. Taf, *Some fractional integral inequalities*, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.

[8] Z. Dahmani, L. Tabharit, S. Taf, *New generalizations of Gruss inequality usin Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

[9] J. Deng and J. Wang, *Fractional Hermite-Hadamard inequalities for* (α, m)*-logarithmically convex functions.* J. Inequal.Appl. 2013, Article ID 364 (2013)

[10] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[11] S. S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. lett., 11(5) (1998), 91-95.

[12] R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.

[13] H. Hudzik and L. Maligranda, *Some remarks on* $s-$convex functions, Aequationes Math. 48 (1994), 100-111.

[14] M. Jleli and B. Samet, *On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function*, J. Nonlinear Sci. Appl. 9 (2016), 1252-1260.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[16] S. Miller and B. Ross, *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.

[17] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[18] I. Podlubni, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[19] M.Z. Sarikaya and H. Ogunmez, *On new inequalities via Riemann-Liouville fractional integration*, Abstract and Applied Analysis, Volume 2012 (2012), Article ID 428983, 10 pages.

[20] M.Z. Sarikaya, E. Set, H. Yaldiz and N., Basak, *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, DOI:10.1016/j.mcm.2011.12.048, 57 (2013) 2403–2407.

[21] M.Z.Sarikaya, H. Filiz and M.E. Kiris, *On some generalized integral inequalities for Riemann-Liouville fractional integrals*, Filomat 29:6 (2015), 1307–1314, DOI 10.2298/FIL1506307S.

[22] M. Z. Sarikaya and H. Budak, *Generalized Hermite-Hadamard type integral inequalities for functions whose 3rd derivatives are s-convex,* Tbilisi Mathematical Journal, 7(2), 41-49, 2014.

[23] M.Z.Sarikaya and M.E. Kiris, *Some new inequalities of Hermite-Hadamard type for* $s-$*convex functions*, Miskolc Mathematical Notes 16(1), 2015, 491–501.

[24] M.Z.Sarikaya and H. Budak, *Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals*, Facta Universitatis Ser:. Math. Inform. 29(4), 2014, 371–384.

[25] E. Set, M. Z. Sarikaya, M. E. Ozdemir and H. Yildirim, *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, JAMSI, 2014, 10(2), 69-83.

[26] M. Tunc, *On new inequalities for h-convex functions via Riemann-Liouville fractional integration*, Filomat 27:4 (2013), 559–565.

[27] Y. Zhang and J. Wang, *On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals,* J. Inequal. Appl. 2013, Article ID 220 (2013)

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