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NEW REFINEMENTS AND APPLICATIONS OF OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WHOSE nth DERIVATIVES ARE OF BOUNDED VARIATION

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Abstract. The main aim of this paper is to establish some Ostrowski type integral inequalities using a newly developed special type of kernel for mappings whose nth derivatives are of bounded variation. We deduce some previous results as a special case. Some new efficient quadrature rules are also introduced.

1. Introduction

In 1938, Ostrowski [18] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many authors have worked on Ostrowski type inequalities for function of bounded variation, see for example ([1]-[14], [16]). Moreover, Dragomir proved some Ostrowski type inequalities for functions whose nth derivatives are of bounded variation in [15].

The following definitions will be frequently used to prove our results.

Definition 1. *Let $P: a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

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Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b (f) := \sup \left\{ \sum \Delta f(P) : P \in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [12], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f) \tag{1.2}$$

holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [8], authors gave the following Ostrowski type inequality:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a,b]$. Then we have the inequality*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2}\left[f(x) + f(a+b-x) \right] \right. \\ &\left. + \frac{1}{2}\left(x - \frac{3a+b}{4} \right) \left[f'(x) - f'(a+b-x) \right] \right| \\ \leq\ & \frac{1}{16} \left[\frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4\left| x - \frac{3a+b}{4} \right| \right] \bigvee_a^b (f') \end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2} \right]$.

In [5], Budak and Sarikaya obtained following Ostrowski type inequality in weighted form for the mappings whose first derivatives are of bounded variation:

Theorem 4. *Let $w:[a,b]\to\mathbb{R}$ be nonnegative and continous and let $f:[a,b]\to\mathbb{R}$ be differentiable mapping on $[a,b]$. If f' is of bounded variation on $[a,b]$, then we have the weighted inequality*

$$\begin{aligned} &\left| \left(\int_a^b (u-x)\, w(u) du \right) f'(x) + \left(\int_a^b w(u) du \right) f(x) - \int_a^b w(t) f(t) dt \right| \\ \leq\ & \left(\int_a^x (u-x)\, w(u) du \right) \bigvee_a^x (f') + \left(\int_x^b (u-x)\, w(u) du \right) \bigvee_x^b (f') \end{aligned}$$

for any $x \in [a,b]$.

Recently, Qayyum et. al [19]-[20], proved some Ostrowski inequality using multiple step kernel. In this paper, we obtain some Ostrowski type integral inequalities for functions whose *nth* derivatives are of bounded variation. The results presented here would provide extensions of those given in [9]- [10] and [12].

2. Refinements of Ostrowski type integral inequalities

Before we start our main results, we state and prove following lemmas:

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$, then the following identity holds*

$$\begin{aligned}
&\int_a^b P_n^1(x,t)df^{(n)}(t) \\
=\;& (-1)^n\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x) \\
&+(-1)^{n+1}\int_a^b f(t)dt.
\end{aligned} \tag{2.1}$$

where

$$P_n^1(x,t)=\begin{cases}\frac{(t-a)^{n+1}}{(n+1)!}, & a\le t\le x\\ \frac{(t-b)^{n+1}}{(n+1)!}, & x<t\le b\end{cases}$$

for all $x\in[a,b]$.

Proof. The proof of (2.1) is established using mathematical induction.

Take $n=1$,

$$\begin{aligned}
&\int_a^b P_1^1(x,t)df'(t) \\
=\;& \frac{1}{2}\left[\int_a^x(t-a)^2df'(t)+\int_x^b(t-b)^2df'(t)\right] \\
=\;& -(b-a)f(x)-(b-a)\left(x-\frac{a+b}{2}\right)+\int_a^b f(t)dt.
\end{aligned}$$

The identity (2.1) is provided for $n=1$.

Assume that (2.1) is true for n. We will show that (2.1) is true for $n+1$.

$$
\begin{aligned}
&\int_a^b P_{n+1}^1(x,t)df^{(n+1)}(t) \\
= \;& \frac{1}{(n+2)!}\left[\int_a^x (t-a)^{n+2}df^{(n+1)}(t)+\int_x^b (t-b)^{n+2}df^{(n+1)}(t)\right] \\
= \;& \frac{1}{(n+1)!}\left[(x-a)^{n+2}-(x-b)^{n+2}\right]f^{(n+1)}(x) \\
& -\frac{1}{(n+2)!}\left[\int_a^x (t-a)^{n+1}df^{(n)}(t)+\int_x^b (t-b)^{n+1}df^{(n)}(t)\right] \\
= \;& \frac{(-1)^{n+1}}{(n+2)!}\left[(-1)^{n+1}(x-a)^{n+2}+(b-x)^{n+2}\right]f^{(n+1)}(x) \\
& -(-1)^n\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)-(-1)^{n+1}\int_a^b f(t)dt \\
= \;& (-1)^{n+1}\sum_{k=0}^{n}\frac{1}{(k+1)!}\left[(-1)^k(x-a)^{k+1}+(b-x)^{k+1}\right]f^{(k)}(x)+(-1)^{n+2}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof. □

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$, then the following identity holds*

$$
\begin{aligned}
&\int_a^b P_n^2(x,t)df^{(n)}(t) \qquad (2.2)\\
= \;& \sum_{k=0}^{n}\frac{(-1)^{n+k}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
& +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
& +(-1)^{n+1}\int_a^b f(t)dt,
\end{aligned}
$$

where the mapping $P_n^2(x,t)$ *is defined by*

$$P_n^2(x,t)=\begin{cases} \frac{1}{(n+1)!}\left(t-a\right)^{n+1}, & t\in\left(a,\frac{a+x}{2}\right] \\ \frac{1}{(n+1)!}\left(t-\frac{3a+b}{4}\right)^{n+1}, & t\in\left(\frac{a+x}{2},x\right] \\ \frac{1}{(n+1)!}\left(t-\frac{a+b}{2}\right)^{n+1}, & t\in\left(x,a+b-x\right] \\ \frac{1}{(n+1)!}\left(t-\frac{a+3b}{4}\right)^{n+1}, & t\in\left(a+b-x,\frac{a+2b-x}{2}\right] \\ \frac{1}{(n+1)!}\left(t-b\right)^{n+1}, & t\in\left(\frac{a+2b-x}{2},b\right] \end{cases}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. We prove the Lemma using mathematical induction. Take $n=1$,

$$\begin{aligned} & \int_a^b P_1^2(x,t)df'(t) \\ = & \int_a^b f(t)dt-\frac{b-a}{4}\left[f\left(x\right)+f\left(a+b-x\right)+f\left(\frac{a+x}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right. \\ & +\left(x-\frac{5a+3b}{8}\right)\left\{f'\left(a+b-x\right)-f'\left(x\right)\right\} \\ & \left.+\frac{1}{2}\left(x-\frac{3a+b}{4}\right)\left\{f'\left(\frac{a+2b-x}{2}\right)-f'\left(\frac{a+x}{2}\right)\right\}\right]. \end{aligned}$$

The identity (2.2) is provided for $n=1$.

Assume that (2.1) is true for n. We will show that (2.1) is true for $n+1$.

$$
\begin{aligned}
& \int_a^b P_n^2(x,t)df^{(n)}(t) \\
= \; & \frac{(-1)^{2n+2}}{(k+1)!}\left[\frac{1}{2^{n+2}}\left\{(x-a)^{n+2}-\left(x-\frac{a+b}{2}\right)^{n+2}\right\}f^{(n+1)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{n+2}-\left(x-\frac{a+b}{2}\right)^{n+2}\right\}f^{(n+1)}(x) \\
& +(-1)^{n+2}\left\{\left(x-\frac{a+b}{2}\right)^{n+2}-\left(x-\frac{3a+b}{4}\right)^{n+2}\right\}f^{(n+1)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{n+2}\left\{\left(x-\frac{a+b}{2}\right)^{n+2}-(x-a)^{n+2}\right\}f^{(n+1)}\left(\frac{a+2b-x}{2}\right)\right] \\
& -\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
& +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
& -(-1)^{n+1}\int_a^b f(t)dt \\
= \; & \sum_{k=0}^{n+1}\frac{(-1)^{n+k+1}}{(k+1)!}\left[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\right. \\
& +\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x) \\
& +(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x) \\
& \left.+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\right] \\
& +(-1)^{n+2}\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof. □

Now using above identities, we state and prove the following theorems.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$. Then, for all $x \in [a,b]$, we have the inequality*

$$\left|(-1)^n \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(-1)^k (x-a)^{k+1} + (b-x)^{k+1}\right] f^{(k)}(x) + (-1)^{n+1} \int_a^b f(t)dt\right| \tag{2.3}$$
$$\leq \frac{1}{(n+1)!} \max\left\{(x-a)^{n+1}, (b-x)^{n+1}\right\} \bigvee_a^b (f^{(n)}).$$

where $\bigvee_a^b (f^{(n)})$ denotes the total variation of $f^{(n)}$ on $[a,b]$.

Proof. It is well known that if $g, f : [a,b] \to \mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exists and

$$\left|\int_a^b g(t)df(t)\right| \leq \sup_{t\in[a,b]} |g(t)| \bigvee_a^b (f). \tag{2.4}$$

In Lemma 1, by using (2.4), we get

$$\begin{aligned}
&\left|\int_a^b P_n^1(x,t)df^{(n)}(t)\right| \\
&\leq \frac{1}{(n+1)!}\left[\left|\int_a^x (t-a)^{n+1} df^{(n)}(t)\right| + \left|\int_x^b (t-b)^{n+1} df^{(n)}(t)\right|\right] \\
&\leq \frac{1}{(n+1)!}\left[\sup_{t\in[a,x]} |t-a|^{n+1} \bigvee_a^x (f^{(n)}) + \sup_{t\in[a,b]} |t-b|^{n+1} \bigvee_x^b (f^{(n)})\right] \\
&= \frac{1}{(n+1)!}\left[(x-a)^{n+1} \bigvee_a^x (f^{(n)}) + (b-x)^{n+1} \bigvee_x^b (f^{(n)})\right] \\
&\leq \frac{1}{(n+1)!} \max\left\{(x-a)^{n+1}, (b-x)^{n+1}\right\} \bigvee_a^b (f^{(n)}).
\end{aligned}$$

This completes the proof. □

Remark 1. *If we choose $n = 0$ in Theorem 5, the inequality (2.3) reduces the inequality (1.2).*

Corollary 1. *Under assumption of Theorem 5 with $n = 1$, we obtain the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f(x) - \left(\frac{a+b}{2} - x\right) f'(x)\right| \tag{2.5}$$
$$\leq \frac{1}{4}\left[\frac{1}{(b-a)}\left[\frac{1}{2}(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)^2\right] + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f').$$

Remark 2. *If we choose $x=\frac{a+b}{2}$ in (2.5), then we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq \frac{b-a}{8}\bigvee_a^b(f'),$$

which was given by Liu in [17].

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is a continuous function of bounded variation on $[a,b]$, then the following identity holds*

$$\begin{aligned}
&\Bigg|\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(k+1)!}\Bigg[\frac{1}{2^{k+1}}\left\{(x-a)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}\left(\frac{a+x}{2}\right)\\
&+\left\{\left(x-\frac{3a+b}{4}\right)^{k+1}-\left(x-\frac{a+b}{2}\right)^{k+1}\right\}f^{(k)}(x)\\
&+(-1)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-\left(x-\frac{3a+b}{4}\right)^{k+1}\right\}f^{(k)}(a+b-x)\\
&+\left(\frac{-1}{2}\right)^{k+1}\left\{\left(x-\frac{a+b}{2}\right)^{k+1}-(x-a)^{k+1}\right\}f^{(k)}\left(\frac{a+2b-x}{2}\right)\Bigg]\\
&+(-1)^{n+1}\int_a^b f(t)dt\Bigg|\\
\leq\ &\frac{1}{(n+1)!}\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\left(\frac{a+b}{2}-x\right)^{n+1},\frac{(x-a)^{n+1}}{2^{n+1}}\right\}\bigvee_a^b\left(f^{(n)}\right)
\end{aligned}$$

for all $x\in\left[a,\frac{a+b}{2}\right]$

Proof. In Lemma 2, by using (2.4), we get

$$\begin{aligned}
&\left|\int_a^b P_n^2(x,t)df^{(n)}(t)\right|\\
\leq\ &\frac{1}{(n+1)!}\Bigg[\left|\int_a^{\frac{a+x}{2}}(t-a)^{n+1}df^{(n)}(t)\right|+\left|\int_{\frac{a+x}{2}}^{x}\left(t-\frac{3a+b}{4}\right)^{n+1}df^{(n)}(t)\right|\\
&+\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^{n+1}df^{(n)}(t)\right|+\left|\int_{a+b-x}^{\frac{a+2b-x}{2}}\left(t-\frac{a+3b}{4}\right)^{n+1}df^{(n)}(t)\right|\\
&+\left|\int_{\frac{a+2b-x}{2}}^{b}(t-b)^{n+1}df^{(n)}(t)\right|\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(n+1)!}\left[\sup_{t\in\left[a,\frac{a+x}{2}\right]}|t-a|^{n+1}\bigvee_{a}^{\frac{a+x}{2}}(f^{(n)})+\sup_{t\in\left[\frac{a+x}{2},x\right]}\left|t-\frac{3a+b}{4}\right|^{n+1}\bigvee_{\frac{a+x}{2}}^{x}(f^{(n)})\right.\\
&\quad+\sup_{t\in[x,a+b-x]}\left|t-\frac{a+b}{2}\right|^{n+1}\bigvee_{x}^{a+b-x}(f^{(n)})+\sup_{t\in\left[a+b-x,\frac{a+2b-x}{2}\right]}\left|t-\frac{a+3b}{4}\right|^{n+1}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f^{(n)})\\
&\quad\left.+\sup_{t\in\left[\frac{a+2b-x}{2},b\right]}|t-b|^{n+1}\bigvee_{\frac{a+2b-x}{2}}^{b}(f^{(n)})\right]\\
&= \frac{1}{(n+1)!}\left[\frac{(x-a)^{n+1}}{2^{n+1}}\bigvee_{a}^{\frac{a+x}{2}}(f^{(n)})+\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}\right\}\bigvee_{\frac{a+x}{2}}^{x}(f^{(n)})\right.\\
&\quad+\left(\frac{a+b}{2}-x\right)^{n+1}\bigvee_{x}^{a+b-x}(f^{(n)})+\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\frac{1}{2^{n+1}}\left(\frac{a+b}{2}-x\right)^{n+1}\right\}\bigvee_{a+b-x}^{\frac{a+2b-x}{2}}(f^{(n)})\\
&\quad\left.+\frac{(x-a)^{n+1}}{2^{n+1}}\bigvee_{\frac{a+2b-x}{2}}^{b}(f^{(n)})\right]\\
&\leq \frac{1}{(n+1)!}\max\left\{\left|x-\frac{3a+b}{4}\right|^{n+1},\left(\frac{a+b}{2}-x\right)^{n+1},\frac{(x-a)^{n+1}}{2^{n+1}}\right\}\bigvee_{a}^{b}\left(f^{(n)}\right).
\end{aligned}
$$

This completes the proof. □

Remark 3. *If we choose $n=0$ in Theorem 6, we get the result proved by Budak and Sarikaya in* [9].

Remark 4. *If we choose $n=1$ in Theorem 6, we get the result proved by Budak et al. in* [10].

3. Derivation of Numerical Quadrature Rules

We propose some new quadrature rules involving nth-derivatives of the function f. In fact, the following new quadrature rules can be obtained while investigating error bounds using theorem 6.

$$
\begin{aligned}
Q_{n,1}(f) &:= \int_a^b f(t)dt\\
&\approx \sum_{k=0}^{n}\frac{1}{(k+1)!}\frac{(b-a)^{k+1}}{2^{k+1}}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right],
\end{aligned}
$$

$$Q_{n,2}(f) := \int_a^b f(t)dt$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \left[\frac{\left(1+(-1)^k\right)}{4^{k+1}2^{k+1}} (b-a)^{k+1} \left\{ f^{(k)}\left(\frac{7a+b}{8}\right) + f^{(k)}\left(\frac{a+7b}{8}\right) \right\} \right.$$
$$\left. + \frac{1}{4^{k+1}} (b-a)^{k+1} \left\{ (-1)^k f^{(k)}\left(\frac{3a+b}{4}\right) + f^{(k)}\left(\frac{a+3b}{4}\right) \right\} \right],$$

$$Q_{n,3}(f) := \int_a^b f(t)dt$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \left[\left\{ f^{(k)}\left(\frac{3a+b}{4}\right) + \left(1+(-1)^k\right) f^{(k)}\left(\frac{a+b}{2}\right) + (-1)^k f^{(k)}\left(\frac{a+3b}{4}\right) \right\} \right.$$
$$\left. \times \frac{1}{4^{k+1}} (b-a)^{k+1} \right]$$

Performance of the efficient quadrature rules

Sr. No.	**Method**	$n:\ Q_{n,1}(f)$	$n:\ Q_{n,2}(f)$	$n:\ Q_{n,3}(f)$	**Exact Value**
1.	$\int_0^1 f_1(x)dx$	**2:** 2.83333	**2:** 2.83333	**2:** 2.83333	2.83333
	Error:	0	0	0	
2.	$\int_0^1 f_2(x)dx$	**6:** 0.301168	**4:** 0.301169	**4:** 0.301168	0.301169
	Error:	5.5921×10^{-7}	7.20674×10^{-7}	1.07696×10^{-6}	
3.	$\int_0^1 f_3(x)dx$	**6:** 0.909332	**4:** 0.909329	**4:** 0.909333	0.909331
	Error:	1.22345×10^{-6}	1.66306×10^{-6}	2.48778×10^{-6}	
4.	$\int_0^1 f_4(x)dx$	**4:** 0.793022	**3:** 0.793023	**4:** 0.793031	0.793031
	Error:	8.63182×10^{-6}	8.08331×10^{-6}	1.03192×10^{-7}	
5.	$\int_0^1 f_5(x)dx$	**10:** 1.46266	**6:** 1.46265	**6:** 1.46265	1.46265
	Error:	5.8789×10^{-6}	1.54452×10^{-6}	2.29707×10^{-6}	
6.	$\int_0^1 f_6(x)dx$	**10:** 1.31384	**6:** 1.31383	**6:** 1.31383	1.31383
	Error	7.37624×10^{-6}	3.15394×10^{-7}	1.47843×10^{-7}	
7.	$\int_0^1 f_7(x)dx$	**5:**1.34147	**4:** 1.34147	**4:** 1.34147	1.34147
	Error:	2.46065×10^{-6}	1.26192×10^{-7}	1.88891×10^{-7}	
8.	$\int_0^1 f_8(x)dx$	**8:** 0.62977	**4:** 0.629773	**4:** 0.629762	0.629769
	Error:	1.18074×10^{-6}	4.86274×10^{-6}	6.3567×10^{-6}	

Table: 1

$$
\begin{aligned}
f_1(x) &= x^2+x+2, && f_2(x)=x\sin x,\\
f_3(x) &= e^x\sin x\ , && f_4(x)=x^2+\sin x,\\
f_5(x) &= e^{x^2}, && f_6(x)=e^x\cos\left(e^x-2x\right),\\
f_7(x) &= x+\cos x, && f_8(x)=\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right].
\end{aligned}
\tag{3.1}
$$

We conclude that all three quadrature rules show exact value of the integral of f_1 for $n=2$. For any polynomial of degree k, $n=k+1$ will give exact value of the integral f_1. Acceptable error estimates can be obtained for smaller values of n to save computational time.

In general $Q_{n,2}(f)$ gave better results as compared to other two quadrature rules for much smaller values of n. Therefore we can conclude that overall $Q_{n,2}(f)$ is computationally more efficient both in terms of error approximation, simplicity, and time. As a rough estimate we integrated $\log\left(x^2+2\right)\sin\left[\log\left(x^2+2\right)\right]$ using the built in algorithms of Mathematica 8.0 which took 26.30 seconds to give its approximate answer. To obtain similar approximation for the integral of f_8, $Q_{n,2}(f)$ took less than a second.

Based on this analysis, we can conjecture that $Q_{n,2}(f)$ is the most efficient quadrature rule. It should be noted that if desired the value of n can be adjusted to improve the error bounds or decrease computational time.

References

[1] M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] M.W. Alomari and S.S. Dragomir, *Mercer–Trapezoid rule for the Riemann–Stieltjes integral with applications*, Journal of Advances in Mathematics, 2 (2)(2013), 67–85.

[4] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.

[5] H. Budak and M.Z. Sarıkaya, *New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation,* RGMIA Research Report Collection, 18(2015), Article 43, 8 pp.

[6] H. Budak and M.Z. Sarikaya, *A new generalization of Ostrowski type inequality for mappings of bounded variation*, RGMIA Research Report Collection, 18(2015), Article 47, 9 pp.

[7] H. Budak and M.Z. Sarikaya, *On generalization of weighted Ostrowski type inequalities for functions of bounded variation,* RGMIA Research Report Collection, 18(2015), Article

[8] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan Journal of Pure and Applied Analysis, in press. 51, 11 pp.

[9] H. Budak and M.Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, RGMIA Research Report Collection, 19(2016), Article 24, 10 pp.

[10] H. Budak, M.Z. Sarikaya and A. Qayyum, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, RGMIA Research Report Collection, 19(2016), Article 25, 11 pp.

[11] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495-508.

[12] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4 (2001), no. 1, 59–66.

[13] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5 (2014) No. 1, 89-97 pp.

[14] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Research Report Collection, 16 (2013), Art. 93.

[15] S. S. Dragomir, *Approximating real functions which possess nth derivatives of bounded variation and applications*, Computers and Mathematics with Applications 56 (2008) 2268–2278.

[16] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).

[17] Z. Liu, *Some Ostrowski type inequalities*, Mathematical and Computer Modelling 48 (2008) 949–960.

[18] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[19] A. Qayyum, M. Shoaib and I. Faye, *Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications*, Journal of Nonlinear Science and Applications, 9 (2016), 537–552.

[20] A. Qayyum, M. Shoaib and I. Faye, *On new refinements and applications of efficient quadrature rules using n-times differentiable mappings*, RGMIA Research Report Collection, 19(2016), Article 9, 22 pp.

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