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INEQUALITIES FOR (m, M)-Ψ-CONVEX FUNCTIONS WITH APPLICATIONS TO OPERATOR NONCOMMUTATIVE PERSPECTIVES

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Abstract. In this paper we obtain some inequalities for (m, M)-Ψ-convex functions and apply them for operator noncommutative perspectives related to convex functions. Particular cases for weighted operator geometric mean and relative operator entropy are also given.

1. Introduction

Assume that the function $\Psi : J \subseteq \mathbb{R} \to \mathbb{R}$ (J is an interval) is convex on J and $m \in \mathbb{R}$. We shall say that the function $\Phi : J \to \mathbb{R}$ is *m-Ψ-lower convex* if $\Phi - m\Psi$ is a convex function on J. We may introduce the class of functions [2]

$$\mathcal{L}(J, m, \Psi) := \{\Phi : J \to \mathbb{R} | \Phi - m\Psi \text{ is convex on } J\}. \tag{1.1}$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of *M-Ψ-upper convex* functions by [2]

$$\mathcal{U}(J, M, \Psi) := \{\Phi : J \to \mathbb{R} | M\Psi - \Phi \text{ is convex on } J\}. \tag{1.2}$$

The intersection of these two classes will be called the class of *(m, M)-Ψ-convex functions* and will be denoted by [2]

$$\mathcal{B}(J, m, M, \Psi) := \mathcal{L}(J, m, \Psi) \cap \mathcal{U}(J, M, \Psi). \tag{1.3}$$

If $\Phi \in \mathcal{B}(J, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in J$).

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [14], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : J \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : J \to \mathbb{R}$, the function $f : J \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on J. In [14], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g-convex dominated functions can be obtained as a particular case from (m, M)-Ψ-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds [2].

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Lemma 1. *Let $\Psi, \Phi : J \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{J}$, the interior of J and Ψ is a convex function on J.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{J}, m, \Psi\right)$ iff*

$$m\left[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right] \le \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \tag{1.4}$$

for all $t, s \in \mathring{J}$.

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{J}, M, \Psi\right)$ iff*

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) \le M\left[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right] \tag{1.5}$$

for all $t, s \in \mathring{J}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{J}, m, M, \Psi\right)$ iff both (1.4) and (1.5) hold.*

Another elementary fact for twice differentiable functions also holds [2].

Lemma 2. *Let $\Psi, \Phi : J \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{J}$ and Ψ is convex on J.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{J}, m, \Psi\right)$ iff*

$$m\Psi''(t) \le \Phi''(t) \quad \text{for all } t \in \mathring{J}. \tag{1.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{J}, M, \Psi\right)$ iff*

$$\Phi''(t) \le M\Psi''(t) \quad \text{for all } t \in \mathring{J}. \tag{1.7}$$

(iii) *For M, $m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{J}, m, M, \Psi\right)$ iff both (1.6) and (1.7) hold.*

For various inequalities concerning these classes of function, see the survey paper [5].

Let Φ be a continuous function defined on the interval J of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{J}$. Then by using the continuous functional calculus, we can define the noncommutative *perspective* $\mathcal{P}_\Phi(B, A)$ by setting

$$\mathcal{P}_\Phi(B, A) := A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_\Phi(B, A) = A\Phi\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{J}$.

It is known that (see [18] and [17] or [19]), if Φ is an *operator convex function* defined in the positive half-line, then the mapping

$$(B, A) \to \mathcal{P}_\Phi(B, A)$$

defined in pairs of positive definite operators, is convex.

In the recent paper [10] we established the following reverse inequality for the perspective $\mathcal{P}_\Phi(B, A)$.

Let $\Phi : [a, b] \to \mathbb{R}$ be a *convex function* on the real interval $[a, b]$, A a positive invertible operator and B a selfadjoint operator such that

$$aA \leq B \leq bA, \tag{1.8}$$

then we have

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\left[\Phi(a)(bA - B) + \Phi(b)(B - aA)\right] - \mathcal{P}_{\Phi}(B, A) \\ &\leq \frac{\Phi'_{-}(b) - \Phi'_{+}(a)}{b-a}\left(bA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - aA^{1/2}\right) \\ &\leq \frac{1}{4}(b-a)\left[\Phi'_{-}(b) - \Phi'_{+}(a)\right]A. \end{aligned} \tag{1.9}$$

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq \Phi''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.10}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.8) is valid with $[a, b] \subset \mathring{J}$, then we have the following result as well [11]

$$\begin{aligned} &\frac{1}{2}d\left(bA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - aA^{1/2}\right) \\ &\leq \frac{1}{b-a}\left[\Phi(a)(bA - B) + \Phi(b)(B - aA)\right] - \mathcal{P}_{\Phi}(B, A) \\ &\leq \frac{1}{2}D\left(bA^{1/2} - BA^{-1/2}\right)\left(A^{-1/2}B - aA^{1/2}\right). \end{aligned} \tag{1.11}$$

If $d > 0$, then the first inequality in (1.11) is better than the same inequality in (1.9).

Motivated by the above results, in this paper we obtain some inequalities for (m, M)-Ψ-convex functions and apply them for operator noncommutative perspectives related to convex functions. Particular cases for weighted operator geometric mean and relative operator entropy are also given.

2. Scalar Inequalities for (m, M)-Ψ-Convex Functions

We have the following simple fact that has several particular cases of interest for integrals and special means:

Proposition 1. *Assume that the function $\Psi : J \subseteq \mathbb{R} \to \mathbb{R}$ is convex on J and M, $m \in \mathbb{R}$ with $M \geq m$. Then the function $\Phi : J \to \mathbb{R}$ belongs to $\mathcal{B}(J, m, M, \Psi)$ if and only iff it satisfies the double inequality*

$$\begin{aligned} &m\left[(1-\nu)\Psi(a) + \nu\Psi(b) - \Psi((1-\nu)a + \nu b)\right] \\ &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\ &\leq M\left[(1-\nu)\Psi(a) + \nu\Psi(b) - \Psi((1-\nu)a + \nu b)\right] \end{aligned} \tag{2.1}$$

for any $a, b \in J$ and any $\nu \in [0, 1]$.

Proof. We have that $\Phi \in \mathcal{B}(J, m, M, \Psi)$ iff $\Phi - m\Psi$ and $M\Psi - \Phi$ and by the definition of convexity, this is equivalent to (2.1). $\square$

Corollary 1. *Assume that the function $\Psi : J \subseteq \mathbb{R} \to \mathbb{R}$ is convex on J and M, $m \in \mathbb{R}$ with $M \geq m$. If the function $\Phi : J \to \mathbb{R}$ belongs to $\mathcal{B}(J, m, M, \Psi)$, then for any $a, b \in J$ with $a < b$ we have the inequalities*

$$m\left[\frac{(b-t)\Psi(a)+(t-a)\Psi(b)}{b-a}-\Psi(t)\right] \leq \frac{(b-t)\Phi(a)+(t-a)\Phi(b)}{b-a}-\Phi(t) \leq M\left[\frac{(b-t)\Psi(a)+(t-a)\Psi(b)}{b-a}-\Psi(t)\right] \tag{2.2}$$

and

$$m\left[\frac{(t-a)\Psi(a)+(b-t)\Psi(b)}{b-a}-\Psi(a+b-t)\right] \leq \frac{(t-a)\Phi(a)+(b-t)\Phi(b)}{b-a}-\Phi(a+b-t) \leq M\left[\frac{(b-t)\Psi(a)+(t-a)\Psi(b)}{b-a}-\Psi(a+b-t)\right] \tag{2.3}$$

for any $t \in [a, b]$.

In particular, we have

$$m\left[\frac{\Psi(a)+\Psi(b)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] \leq \frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right) \leq M\left[\frac{\Psi(a)+\Psi(b)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] \tag{2.4}$$

for any $a, b \in J$.

Proof. The inequality (2.2) follows by (2.1) on taking $\nu = \frac{t-a}{b-a} \in [0, 1]$ while (2.3) follows by (2.1) on taking $\nu = \frac{b-t}{b-a} \in [0, 1]$. □

Remark 1. *By adding the inequalities (2.2) and (2.3) and dividing by 2, we get*

$$m\left[\frac{\Psi(a)+\Psi(b)}{2}-\frac{\Psi(t)+\Psi(a+b-t)}{2}\right] \leq \frac{\Phi(a)+\Phi(b)}{2}-\frac{\Phi(t)+\Phi(a+b-t)}{2} \leq M\left[\frac{\Psi(a)+\Psi(b)}{2}-\frac{\Psi(t)+\Psi(a+b-t)}{2}\right] \tag{2.5}$$

for any $t \in [a, b]$.

If we replace in (2.4) a by t and b by $a+b-t$ we get

$$m\left[\frac{\Psi(t)+\Psi(a+b-t)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] \leq \frac{\Phi(t)+\Phi(a+b-t)}{2}-\Phi\left(\frac{a+b}{2}\right) \leq M\left[\frac{\Psi(t)+\Psi(a+b-t)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] \tag{2.6}$$

for any $t \in [a, b]$.

We have the following Hermite-Hadamard type result:

Theorem 1. *Assume that the function $\Psi : J \subseteq \mathbb{R} \to \mathbb{R}$ is convex on J and M, $m \in \mathbb{R}$ with $M \geq m$. If the function $\Phi : J \to \mathbb{R}$ belongs to $\mathcal{B}(J, m, M, \Psi)$, then for any $a, b \in J$, $b \neq a$ we have the inequalities*

$$\begin{aligned} m\left[\frac{\Psi(a)+\Psi(b)}{2}-\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right] &\\ \leq \frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b \Phi(t)\,dt &\\ \leq M\left[\frac{\Psi(a)+\Psi(b)}{2}-\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right] & \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} m\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt-\Psi\left(\frac{a+b}{2}\right)\right] &\\ \leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right) &\\ \leq M\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt-\Psi\left(\frac{a+b}{2}\right)\right]. & \end{aligned} \tag{2.8}$$

Proof. If we integrate the inequality (2.1) over $\nu \in [0, 1]$ and use the change of variable $t = (1-\nu)a + \nu b$, $\nu \in [0, 1]$ we get the desired result (2.7).

From the inequality (2.1) we have

$$\begin{aligned} m\left[\frac{\Psi(s)+\Psi(t)}{2}-\Psi\left(\frac{s+t}{2}\right)\right] &\leq \left[\frac{\Phi(s)+\Phi(t)}{2}-\Phi\left(\frac{s+t}{2}\right)\right] \\ &\leq M\left[\frac{\Psi(s)+\Psi(t)}{2}-\Psi\left(\frac{s+t}{2}\right)\right] \end{aligned} \tag{2.9}$$

for any $s, t \in J$.

If we take in (2.9) $s = (1-\nu)a + \nu b$ and $t = (1-\nu)b + \nu a$ with $a, b \in J$ and $\nu \in [0, 1]$, then we get

$$\begin{aligned} m\left[\frac{\Psi((1-\nu)a+\nu b)+\Psi((1-\nu)b+\nu a)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] &\\ \leq \left[\frac{\Phi((1-\nu)a+\nu b)+\Phi((1-\nu)b+\nu a)}{2}-\Phi\left(\frac{a+b}{2}\right)\right] &\\ \leq M\left[\frac{\Psi((1-\nu)a+\nu b)+\Psi((1-\nu)b+\nu a)}{2}-\Psi\left(\frac{a+b}{2}\right)\right] & \end{aligned} \tag{2.10}$$

for $a, b \in J$ and $\nu \in [0, 1]$.

If we integrate the inequality (2.10) over $\nu \in [0, 1]$ and use the fact that

$$\begin{aligned} \int_0^1 \Psi((1-\nu)a+\nu b)\,d\nu &= \int_0^1 \Psi((1-\nu)b+\nu a)\,d\nu \\ &= \frac{1}{b-a}\int_a^b f(t)\,dt, \end{aligned}$$

then we get the desired result (2.8). □

Theorem 2. *Let $\Psi, \Phi : J \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{J}$ and Ψ is a convex function on J. If the function $\Phi : J \to \mathbb{R}$ belongs to $\mathcal{B}(J, m, M, \Psi)$, then for any $a, b \in J$, $b \neq a$ we have the inequalities*

$$\begin{aligned}
(2.11)\qquad & m\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi(s) - \Psi'(s)\left(\frac{a+b}{2}-s\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi(s) - \Phi'(s)\left(\frac{a+b}{2}-s\right) \\
&\leq M\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi(s) - \Psi'(s)\left(\frac{a+b}{2}-s\right)\right]
\end{aligned}$$

for any $s \in \mathring{J}$ and

$$\begin{aligned}
(2.12)\qquad & m\left[\frac{1}{2}\left(\Psi(t) + \frac{\Psi(b)(b-t)+\Psi(a)(t-a)}{b-a}\right) - \frac{1}{b-a}\int_a^b \Psi(s)\,ds\right] \\
&\leq \frac{1}{2}\left(\Phi(t) + \frac{\Phi(b)(b-t)+\Phi(a)(t-a)}{b-a}\right) - \frac{1}{b-a}\int_a^b \Phi(s)\,ds \\
&\leq M\left[\frac{1}{2}\left(\Psi(t) + \frac{\Psi(b)(b-t)+\Psi(a)(t-a)}{b-a}\right) - \frac{1}{b-a}\int_a^b \Psi(s)\,ds\right]
\end{aligned}$$

for any $t \in J$.

Proof. Since $\Phi \in \mathcal{B}(J, m, M, \Psi)$, then by Lemma 1 we have

$$\begin{aligned}
(2.13)\qquad m[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)] &\leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\
&\leq M[\Psi(t) - \Psi(s) - \Psi'(s)(t-s)]
\end{aligned}$$

for all $s \in \mathring{J}$ and $t \in J$.

Let $a, b \in J$, $b \neq a$. The integral mean $\frac{1}{b-a}\int_a^b$ is a linear positive functional.

Now, by taking the integral mean over t in (2.13) we get

$$\begin{aligned}
& m\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi(s) - \Psi'(s)\left(\frac{1}{b-a}\int_a^b t\,dt - s\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi(s) - \Phi'(s)\left(\frac{1}{b-a}\int_a^b t\,dt - s\right) \\
&\leq M\left[\frac{1}{b-a}\int_a^b \Psi(t)\,dt - \Psi(s) - \Psi'(s)\left(\frac{1}{b-a}\int_a^b t\,dt - s\right)\right]
\end{aligned}$$

for any $s \in \mathring{J}$, that is equivalent to (2.11).

Now, if we take the integral mean in (2.13) over s we get

$$
\begin{aligned}
& m\left[\Psi(t)-\frac{1}{b-a}\int_a^b \Psi(s)\,ds-\frac{1}{b-a}\int_a^b \Psi'(s)(t-s)\,ds\right] \\
& \leq \Phi(t)-\frac{1}{b-a}\int_a^b \Phi(s)\,ds-\frac{1}{b-a}\int_a^b \Phi'(s)(t-s)\,ds \\
& \leq M\left[\Psi(t)-\frac{1}{b-a}\int_a^b \Psi(s)\,ds-\frac{1}{b-a}\int_a^b \Psi'(s)(t-s)\,ds\right]
\end{aligned} \tag{2.14}
$$

for any $t\in J$.

Observe that, integrating by parts we have

$$
\begin{aligned}
\int_a^b \Psi'(s)(t-s)\,ds &= \Psi(s)(t-s)|_a^b+\int_a^b \Psi(s)\,ds \\
&= \Psi(b)(t-b)-\Psi(a)(t-a)+\int_a^b \Psi(s)\,ds \\
&= -\Psi(b)(b-t)-\Psi(a)(t-a)+\int_a^b \Psi(s)\,ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \Psi(t)-\frac{1}{b-a}\int_a^b \Psi(s)\,ds-\frac{1}{b-a}\int_a^b \Psi'(s)(t-s)\,ds \\
& = \Psi(t)+\frac{\Psi(b)(b-t)+\Psi(a)(t-a)}{b-a}-\frac{2}{b-a}\int_a^b \Psi(s)\,ds
\end{aligned}
$$

and a similar equality for Φ.

By (2.14) we get then the desired result (2.12). □

Remark 2. *If we take $s=\frac{a+b}{2}$ in (2.11) then we recapture the inequality (2.8). The same choice for t in (2.12) produces the inequality*

$$
\begin{aligned}
& m\left[\frac{1}{2}\left(\Psi\left(\frac{a+b}{2}\right)+\frac{\Psi(a)+\Psi(b)}{2}\right)-\frac{1}{b-a}\int_a^b \Psi(s)\,ds\right] \\
& \leq \frac{1}{2}\left(\Phi\left(\frac{a+b}{2}\right)+\frac{\Phi(b)+\Phi(a)}{2}\right)-\frac{1}{b-a}\int_a^b \Phi(s)\,ds \\
& \leq M\left[\frac{1}{2}\left(\Psi\left(\frac{a+b}{2}\right)+\frac{\Psi(a)+\Psi(b)}{2}\right)-\frac{1}{b-a}\int_a^b \Psi(s)\,ds\right].
\end{aligned} \tag{2.15}
$$

Remark 3. *If the function $\Phi\in\mathcal{B}(J,m,M,\Psi)$, then $\Phi=m\Psi+g$ with Ψ and g are convex functions, implying that Φ is differentiable everywhere except a countably number of points in J and the inequality (2.13) holds for almost every $s\in\mathring{J}$ and every $t\in J$. Making use of a similar argument as above we conclude that the inequality (2.12) holds without the differentiability assumption.*

3. Inequalities for Perspectives

We have:

Theorem 3. *Let $\Phi : J \to \mathbb{R}$ be a convex function on the interval of real numbers J, M, $m \in \mathbb{R}$ with $M \geq m$ and the function $\Phi : J \to \mathbb{R}$ that belongs to $\mathcal{B}(J, m, M, \Psi)$. If A is a positive invertible operator and B is a selfadjoint operator such that*

$$aA \leq B \leq bA \tag{3.1}$$

with $[a, b] \subset \mathring{J}$ for some real numbers $a < b$, then we have the inequalities

$$\begin{aligned} & m\left[\frac{1}{b-a}\left[\Psi(a)(bA-B)+\Psi(b)(B-aA)\right]-\mathcal{P}_{\Psi}(B,A)\right] \\ & \leq \frac{1}{b-a}\left[\Phi(a)(bA-B)+\Phi(b)(B-aA)\right]-\mathcal{P}_{\Phi}(B,A) \\ & \leq M\left[\frac{1}{b-a}\left[\Psi(a)(bA-B)+\Psi(b)(B-aA)\right]-\mathcal{P}_{\Psi}(B,A)\right] \end{aligned} \tag{3.2}$$

is and

$$\begin{aligned} & m\left[\frac{1}{b-a}\Psi(a)(B-aA)+\Psi(b)(bA-B)-\mathcal{P}_{\Psi(a+b-\cdot)}(B,A)\right] \\ & \leq \frac{1}{b-a}\Phi(a)(B-aA)+\Phi(b)(bA-B)-\mathcal{P}_{\Phi(a+b-\cdot)}(B,A) \\ & \leq M\left[\frac{1}{b-a}\Psi(a)(B-aA)+\Psi(b)(bA-B)-\mathcal{P}_{\Psi(a+b-\cdot)}(B,A)\right], \end{aligned} \tag{3.3}$$

where

$$\mathcal{P}_{\Phi(a+b-\cdot)}(B,A) = A^{1/2}\Phi\left(A^{-1/2}\left[(a+b)A-B\right]A^{-1/2}\right)A^{1/2}$$

and a similar expression for Ψ.

Moreover, we have

$$\begin{aligned} & m\left[\frac{\Psi(a)+\Psi(b)}{2}A-\frac{\mathcal{P}_{\Psi}(B,A)+\mathcal{P}_{\Psi(a+b-\cdot)}(B,A)}{2}\right] \\ & \leq \frac{\Phi(a)+\Phi(b)}{2}A-\frac{\mathcal{P}_{\Phi}(B,A)+\mathcal{P}_{\Phi(a+b-\cdot)}(B,A)}{2} \\ & \leq M\left[\frac{\Psi(a)+\Psi(b)}{2}A-\frac{\mathcal{P}_{\Psi}(B,A)+\mathcal{P}_{\Psi(a+b-\cdot)}(B,A)}{2}\right]. \end{aligned} \tag{3.4}$$

Proof. Using the continuous functional calculus, for any selfadjoint operator X with $\mathrm{Sp}(X) \subseteq [a, b]$ we have from (2.2) that

$$\begin{aligned} & m\left[\frac{\Psi(a)(bI-X)+\Psi(b)(X-aI)}{b-a}-\Psi(X)\right] \\ & \leq \frac{\Phi(a)(bI-X)+\Phi(b)(X-aI)}{b-a}-\Phi(X) \\ & \leq M\left[\frac{\Psi(a)(bI-X)+\Psi(b)(X-aI)}{b-a}-\Psi(X)\right] \end{aligned} \tag{3.5}$$

in the operator order.

If (3.1) holds, then by multiplying both sides with $A^{-1/2}$ we get

$$aI \leq A^{-1/2}BA^{-1/2} \leq bI$$

and by writing the inequality (3.5) for $X = A^{-1/2}BA^{-1/2}$ we get

$$\begin{aligned}
&m\left[\frac{1}{b-a}\left[\Psi(a)\left(bI - A^{-1/2}BA^{-1/2}\right) + \Psi(b)\left(A^{-1/2}BA^{-1/2} - aI\right)\right]\right.\\
&\left.-\Psi\left(A^{-1/2}BA^{-1/2}\right)\right]\\
&\le \frac{1}{b-a}\left[\Phi(a)\left(bI - A^{-1/2}BA^{-1/2}\right) + \Phi(b)\left(A^{-1/2}BA^{-1/2} - aI\right)\right]\\
&-\Phi\left(A^{-1/2}BA^{-1/2}\right)\\
&\le M\left[\frac{1}{b-a}\left[\Psi(a)\left(bI - A^{-1/2}BA^{-1/2}\right) + \Psi(b)\left(A^{-1/2}BA^{-1/2} - aI\right)\right]\right.\\
&\left.-\Psi\left(A^{-1/2}BA^{-1/2}\right)\right]
\end{aligned}$$

that can be rewritten as

$$\begin{aligned}
&m\left[\frac{1}{b-a}\left[\Psi(a)A^{-1/2}(bA-B)A^{-1/2} + \Psi(b)A^{-1/2}(B-aA)A^{-1/2}\right]\right.\\
&\left.-\Psi\left(A^{-1/2}BA^{-1/2}\right)\right]\\
&\le \frac{1}{b-a}\left[\Phi(a)A^{-1/2}(bA-B)A^{-1/2} + \Phi(b)A^{-1/2}(B-aA)A^{-1/2}\right]\\
&-\Phi\left(A^{-1/2}BA^{-1/2}\right)\\
&\le M\left[\frac{1}{b-a}\Psi(a)A^{-1/2}(bA-B)A^{-1/2} + \Psi(b)A^{-1/2}(B-aA)A^{-1/2}\right.\\
&\left.-\Psi\left(A^{-1/2}BA^{-1/2}\right)\right].
\end{aligned}\tag{3.6}$$

If we multiply (3.6) both sides with $A^{1/2}$ we get the desired result (3.2).

The inequality (3.3) follows in a similar way from (3.3) and we omit the details.

If we add (3.2) and (3.3) and divide by 2 we get (3.4). □

Theorem 4. *Let $\Psi, \Phi : J \subseteq \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions on $\mathring{J}$ and Ψ is a convex function on J. If the function $\Phi : J \to \mathbb{R}$ belongs to $\mathcal{B}(J, m, M, \Psi)$, A is a positive invertible operator and B a selfadjoint operator such that the condition (3.1) is valid with $[a, b] \subset \mathring{J}$ for some real numbers $a < b$, then we have the inequalities*

$$\begin{aligned}
&m\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)A - \mathcal{P}_{\Psi}(B,A) - \mathcal{P}_{\Psi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}(B,A)\right]\\
&\le \left(\frac{1}{b-a}\int_a^b \Phi(t)\,dt\right)A - \mathcal{P}_{\Phi}(B,A) - \mathcal{P}_{\Phi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}(B,A)\\
&\le M\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)A - \mathcal{P}_{\Psi}(B,A) - \mathcal{P}_{\Psi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}(B,A)\right],
\end{aligned}\tag{3.7}$$

where

$$\mathcal{P}_{\Phi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}(B,A) = A^{1/2}\Phi'\left(A^{-1/2}BA^{-1/2}\right)A^{-1/2}\left(\frac{a+b}{2}A - B\right)$$

and a similar expression for Ψ.

Proof. Using the continuous functional calculus, for any selfadjoint operator X with $\operatorname{Sp}(X) \subseteq [a,b]$ we have from (2.11) that

$$m\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)I - \Psi(X) - \Psi'(X)\left(\frac{a+b}{2}I - X\right)\right] \tag{3.8}$$
$$\leq \left(\frac{1}{b-a}\int_a^b \Phi(t)\,dt\right)I - \Phi(X) - \Phi'(X)\left(\frac{a+b}{2}I - X\right)$$
$$\leq M\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)I - \Psi(X) - \Psi'(X)\left(\frac{a+b}{2}I - X\right)\right].$$

If we write the inequality (3.8) for $X = A^{-1/2}BA^{-1/2}$ we get

$$m\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)I - \Psi\left(A^{-1/2}BA^{-1/2}\right)\right. \tag{3.9}$$
$$\left.-\Psi'\left(A^{-1/2}BA^{-1/2}\right)\left(\frac{a+b}{2}I - A^{-1/2}BA^{-1/2}\right)\right]$$
$$\leq \left(\frac{1}{b-a}\int_a^b \Phi(t)\,dt\right)I - \Phi\left(A^{-1/2}BA^{-1/2}\right)$$
$$-\Phi'\left(A^{-1/2}BA^{-1/2}\right)\left(\frac{a+b}{2}I - A^{-1/2}BA^{-1/2}\right)$$
$$\leq M\left[\left(\frac{1}{b-a}\int_a^b \Psi(t)\,dt\right)I - \Psi\left(A^{-1/2}BA^{-1/2}\right)\right.$$
$$\left.-\Psi'\left(A^{-1/2}BA^{-1/2}\right)\left(\frac{a+b}{2}I - A^{-1/2}BA^{-1/2}\right)\right].$$

If we multiply (3.9) both sides with $A^{1/2}$ we get the desired result (3.7). □

We also have:

Theorem 5. *Let* $\Phi : J \to \mathbb{R}$ *be a convex function on the interval of real numbers* J, M, $m \in \mathbb{R}$ *with* $M \geq m$ *and the function* $\Phi : J \to \mathbb{R}$ *that belongs to* $\mathcal{B}(J, m, M, \Psi)$. *If* A *is a positive invertible operator and* B *is a selfadjoint operator such that the condition (3.1) is valid with* $[a,b] \subset \mathring{J}$ *for some real numbers* $a < b$, *then we have* *the inequalities*

$$
\begin{aligned}
(3.10)\qquad & m\left[\frac{1}{2}\left(\mathcal{P}_{\Psi}(B,A)+\frac{1}{b-a}\left[\Psi(b)(bA-B)+\Psi(a)(B-aA)\right]\right)\right.\\
&\left.-\left(\frac{1}{b-a}\int_a^b\Psi(s)\,ds\right)A\right]\\
&\le\frac{1}{2}\left(\mathcal{P}_{\Phi}(B,A)+\frac{1}{b-a}\left[\Phi(b)(bA-B)+\Phi(a)(B-aA)\right]\right)\\
&-\left(\frac{1}{b-a}\int_a^b\Phi(s)\,ds\right)A\\
&\le M\left[\frac{1}{2}\left(\mathcal{P}_{\Psi}(B,A)+\frac{1}{b-a}\left[\Psi(b)(bA-B)+\Psi(a)(B-aA)\right]\right)\right.\\
&\left.-\left(\frac{1}{b-a}\int_a^b\Psi(s)\,ds\right)A\right].
\end{aligned}
$$

The proof follows in a similar way as above by employing the scalar inequality (2.12) and we omit the details.

4. Applications for Power Function

Let $p\in(-\infty,0)\cup(1,\infty)$ and $\Phi:J\subset(0,\infty)\to\mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some γ, Γ we have

$$
\gamma x^{p-2}\le\Phi''(x)\le\Gamma x^{p-2}\text{ for any }x\in\mathring{J}. \tag{4.1}
$$

We observe that the functions $\Phi-\frac{\gamma}{p(p-1)}\ell^p$ and $\frac{\Gamma}{p(p-1)}\ell^p-\Phi$ where ℓ is the identity function, i.e. $\ell(t)=t$, are convex functions on J. Since $\Psi:=\ell^p$ is also a convex function, it follows that $\Phi\in\mathcal{B}(J,m,M,\Psi)$ with $m=\frac{\gamma}{p(p-1)}$ and $M=\frac{\Gamma}{p(p-1)}$.

If we use the inequality (2.1), then we have the inequality

$$
\begin{aligned}
(4.2)\qquad & \frac{\gamma}{p(p-1)}\left[(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\right]\\
&\le(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\\
&\le\frac{\Gamma}{p(p-1)}\left[(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\right]
\end{aligned}
$$

for any $a,b\in J$ and $\nu\in[0,1]$.

If we take $p=2$ in (4.2), then we get

$$
\begin{aligned}
(4.3)\qquad \frac{\gamma}{2}(1-\nu)\nu(b-a)^2 &\le(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\\
&\le\frac{\Gamma}{2}(1-\nu)\nu(b-a)^2
\end{aligned}
$$

for any $a,b\in J$ and $\nu\in[0,1]$, provided Φ is twice differentiable and

$$
\gamma\le\Phi''(x)\le\Gamma\text{ for any }x\in\mathring{J}. \tag{4.4}
$$

We observe that inequality (4.3) is a particular case of inequality (2.7) from [3], for $n=2$, $p_1=\nu$, $p_2=1-\nu$ and when the inner product space H reduces to $\mathbb{R}$.

It has been obtained in this form recently in [1] by an interesting two variable functions analysis argument and in [7] by a convexity argument similar to the one from above.

If Φ is twice differentiable and

$$\gamma x^{-3} \leq \Phi''(x) \leq \Gamma x^{-3} \text{for any } x \in \mathring{J}, \tag{4.5}$$

then by taking $p = -1$ in (4.2) we get

$$\begin{aligned} &\frac{\gamma}{2}\left[(1-\nu)a^{-1} + \nu b^{-1} - ((1-\nu)a + \nu b)^{-1}\right] \\ &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\ &\leq \frac{\Gamma}{2}\left[(1-\nu)a^{-1} + \nu b^{-1} - ((1-\nu)a + \nu b)^{-1}\right] \end{aligned} \tag{4.6}$$

for any $a, b \in J$ and $\nu \in [0,1]$.

Upon appropriate calculations, we have the following equivalent inequality

$$\begin{aligned} \frac{\gamma}{2}(1-\nu)\nu\frac{(b-a)^2}{ab[(1-\nu)a+\nu b]} &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\ &\leq \frac{\Gamma}{2}(1-\nu)\nu\frac{(b-a)^2}{ab[(1-\nu)a+\nu b]} \end{aligned} \tag{4.7}$$

for any $a, b \in J$ and $\nu \in [0,1]$.

Since

$$\frac{\min\{a,b\}}{ab} = \frac{1}{\max\{a,b\}} \leq \frac{1}{(1-\nu)a+\nu b} \leq \frac{1}{\min\{a,b\}} = \frac{\max\{a,b\}}{ab},$$

then by (4.7) we have the simpler, however coarser inequality

$$\begin{aligned} \frac{\gamma}{2}(1-\nu)\nu\min\{a,b\}\left(\frac{1}{a}-\frac{1}{b}\right)^2 &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\ &\leq \frac{\Gamma}{2}(1-\nu)\nu\max\{a,b\}\left(\frac{1}{a}-\frac{1}{b}\right)^2 \end{aligned} \tag{4.8}$$

for any $a, b \in J$ and $\nu \in [0,1]$.

Let $q \in (0,1)$ and $\Phi : J \subset (0,\infty) \to \mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some γ, Γ we have

$$\gamma x^{q-2} \leq \Phi''(x) \leq \Gamma x^{q-2} \text{ for any } x \in \mathring{J}. \tag{4.9}$$

Since $\Psi := -\ell^q$ is a convex function and

$$\Phi - \frac{\gamma}{q(1-q)}(-\ell^q) = \Phi - \frac{\gamma}{q(q-1)}\ell^q$$

and

$$\frac{\Gamma}{q(1-q)}(-\ell^q) - \Phi = \frac{\Gamma}{q(q-1)}\ell^q - \Phi,$$

it follows that $\Phi \in \mathcal{B}(J, m, M, \Psi)$ with $m = \frac{\gamma}{q(1-q)}$ and $M = \frac{\Gamma}{q(q-1)}$.

By using the inequality (2.1) we have

$$\begin{aligned}(4.10)\qquad &\frac{\gamma}{q(1-q)}\left[((1-\nu)a+\nu b)^q-(1-\nu)a^q-\nu b^q\right]\\&\leq(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\\&\leq\frac{\Gamma}{q(1-q)}\left[((1-\nu)a+\nu b)^q-(1-\nu)a^q-\nu b^q\right]\end{aligned}$$

for any $a, b \in J$ and $\nu \in [0,1]$.

If we take $q = \frac{1}{2}$, then we have

$$\begin{aligned}(4.11)\qquad &4\gamma\left[\sqrt{(1-\nu)a+\nu b}-(1-\nu)\sqrt{a}-\nu\sqrt{b}\right]\\&\leq(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)\\&\leq 4\Gamma\left[\sqrt{(1-\nu)a+\nu b}-(1-\nu)\sqrt{a}-\nu\sqrt{b}\right]\end{aligned}$$

for any $a, b \in J$ and $\nu \in [0,1]$.

For positive $x \neq y$ and $p \in \mathbb{R} \setminus \{-1, 0\}$, we define the *$p$-logarithmic mean* (*generalized logarithmic mean*) $L_p(x, y)$ by

$$L_p(x,y) := \left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

In fact the singularities at $p = -1,\ 0$ are removable and L_p can be defined for $p = -1,\ 0$ so as to make $L_p(x, y)$ a continuous function of p. In the limit as $p \to 0$ we obtain the *identric mean* $I(x, y)$, given by

$$(4.12)\qquad I(x,y) := \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},$$

and in the case $p \to -1$ the *logarithmic mean* $L(x, y)$, given by

$$L(x,y) := \frac{y-x}{\ln y-\ln x}.$$

In each case we define the mean as x when $y = x$, which occurs as the limiting value of $L_p(x, y)$ for $y \to x$.

We define the arithmetic mean as $A(x,y) := \frac{x+y}{2}$, the geometric mean as $G(x,y) := \sqrt{xy}$ and the harmonic mean as $H(x,y) = A^{-1}\left(x^{-1}, y^{-1}\right)$.

Let $p \in (-\infty, 0) \cup (1, \infty)$ and $\Phi : J \subset (0, \infty) \to \mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some γ, Γ we have the condition (4.1). Then by (2.7) and (2.8) we have

$$\begin{aligned}(4.13)\qquad \frac{\gamma}{p(p-1)}\left[A(a^p,b^p)-L_p^p(a,b)\right]&\leq\frac{\Phi(a)+\Phi(b)}{2}-\frac{1}{b-a}\int_a^b\Phi(t)\,dt\\&\leq\frac{\Gamma}{p(p-1)}\left[A(a^p,b^p)-L_p^p(a,b)\right]\end{aligned}$$

and

$$\begin{aligned}(4.14)\qquad \frac{\gamma}{p(p-1)}\left[L_p^p(a,b)-A^p(a,b)\right]&\leq\frac{1}{b-a}\int_a^b\Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right)\\&\leq\frac{\Gamma}{p(p-1)}\left[L_p^p(a,b)-A^p(a,b)\right].\end{aligned}$$

From (2.15) we have

$$\begin{aligned}
&\frac{\gamma}{p\left(p-1\right)}\left[\frac{1}{2}\left(A^{p}\left(a,b\right)+A\left(a^{p},b^{p}\right)\right)-L_{p}^{p}(a,b)\right] \\
&\leq\frac{1}{2}\left(\Phi\left(\frac{a+b}{2}\right)+\frac{\Phi\left(b\right)+\Phi\left(a\right)}{2}\right)-\frac{1}{b-a}\int_{a}^{b}\Phi\left(s\right)ds \\
&\leq\frac{\Gamma}{p\left(p-1\right)}\left[\frac{1}{2}\left(A^{p}\left(a,b\right)+A\left(a^{p},b^{p}\right)\right)-L_{p}^{p}(a,b)\right].
\end{aligned}\tag{4.15}$$

If we take $p=2$ in (4.13)-(4.15) then we get

$$\frac{1}{12}\gamma\left(b-a\right)^{2}\leq\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}\Phi\left(t\right)dt\leq\frac{1}{12}\Gamma\left(b-a\right)^{2},\tag{4.16}$$

$$\frac{1}{24}\gamma\left(b-a\right)^{2}\leq\frac{1}{b-a}\int_{a}^{b}\Phi\left(t\right)dt-\Phi\left(\frac{a+b}{2}\right)\leq\frac{1}{24}\Gamma\left(b-a\right)^{2}.\tag{4.17}$$

and

$$\begin{aligned}
&\frac{1}{48}\gamma\left(b-a\right)^{2}\leq\frac{1}{2}\left(\Phi\left(\frac{a+b}{2}\right)+\frac{\Phi\left(b\right)+\Phi\left(a\right)}{2}\right)-\frac{1}{b-a}\int_{a}^{b}\Phi\left(s\right)ds \\
&\leq\frac{1}{48}\Gamma\left(b-a\right)^{2},
\end{aligned}\tag{4.18}$$

provided Φ is twice differentiable and satisfies condition (4.4), see also [5].

If Φ is twice differentiable and the condition (4.5) is satisfied, then by taking $p=-1$ in (4.13)-(4.15) we get

$$\begin{aligned}
\frac{1}{2}\gamma\frac{L\left(a,b\right)-H\left(a,b\right)}{L\left(a,b\right)H\left(a,b\right)}&\leq\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}\Phi\left(t\right)dt \\
&\leq\frac{1}{2}\Gamma\frac{L\left(a,b\right)-H\left(a,b\right)}{L\left(a,b\right)H\left(a,b\right)},
\end{aligned}\tag{4.19}$$

$$\begin{aligned}
\frac{1}{2}\gamma\frac{A\left(a,b\right)-L\left(a,b\right)}{A\left(a,b\right)L\left(a,b\right)}&\leq\frac{1}{b-a}\int_{a}^{b}\Phi\left(t\right)dt-\Phi\left(\frac{a+b}{2}\right) \\
&\leq\frac{1}{2}\Gamma\frac{A\left(a,b\right)-L\left(a,b\right)}{A\left(a,b\right)L\left(a,b\right)}
\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}
&\frac{1}{4}\gamma\frac{L\left(a,b\right)\left[A\left(a,b\right)+H\left(a,b\right)\right]\right)-A\left(a,b\right)H\left(a,b\right)}{A\left(a,b\right)L\left(a,b\right)H\left(a,b\right)} \\
&\leq\frac{1}{2}\left(\Phi\left(\frac{a+b}{2}\right)+\frac{\Phi\left(b\right)+\Phi\left(a\right)}{2}\right)-\frac{1}{b-a}\int_{a}^{b}\Phi\left(s\right)ds \\
&\leq\frac{1}{4}\Gamma\frac{L\left(a,b\right)\left[A\left(a,b\right)+H\left(a,b\right)\right]\right)-A\left(a,b\right)H\left(a,b\right)}{A\left(a,b\right)L\left(a,b\right)H\left(a,b\right)}.
\end{aligned}\tag{4.21}$$

Assume that A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators [28]

$$A\nabla_{\nu}B:=\left(1-\nu\right)A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_{\nu}B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The definition $A\sharp_\nu B$ can be extended accordingly for any real number ν.

The following inequality is well as the operator *Young inequality* or operator *ν-weighted arithmetic-geometric mean inequality*:

$$A\sharp_\nu B \le A\nabla_\nu B \text{ for all } \nu \in [0,1]\,. \tag{4.22}$$

For recent results on operator Young inequality see [1], [15]-[24], [25] and [33]-[34].

Let $p \in (-\infty,0)\cup(1,\infty)$ and $\Phi : J \subset (0,\infty) \to \mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some γ, Γ we have the condition (4.1). If we use the inequalities (3.2), (3.7) and (3.10) we have the positive invertible operators A and B that satisfy the condition

$$aA \le B \le bB$$

for some $0 < a < b$, that

$$\begin{aligned}&\frac{\gamma}{p(p-1)}\left[\frac{1}{b-a}\left[a^p(bA-B)+b^p(B-aA)\right]-A\sharp_p B\right]\\&\le \frac{1}{b-a}\left[\Phi(a)(bA-B)+\Phi(b)(B-aA)\right]-\mathcal{P}_\Phi(B,A)\\&\le \frac{\Gamma}{p(p-1)}\left[\frac{1}{b-a}\left[a^p(bA-B)+b^p(B-aA)\right]-A\sharp_p B\right],\end{aligned} \tag{4.23}$$

$$\begin{aligned}&\frac{\gamma}{p(p-1)}\left[L_p^p(a,b)A-A\sharp_p B-p\left(A\sharp_{p-1}B\right)\left(\frac{a+b}{2}I-A^{-1}B\right)\right]\\&\le\left(\frac{1}{b-a}\int_a^b\Phi(t)\,dt\right)A-\mathcal{P}_\Phi(B,A)-\mathcal{P}_{\Phi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}(B,A)\\&\le \frac{\Gamma}{p(p-1)}\left[L_p^p(a,b)A-A\sharp_p B-p\left(A\sharp_{p-1}B\right)\left(\frac{a+b}{2}I-A^{-1}B\right)\right],\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}&\frac{\gamma}{p(p-1)}\left[\frac{1}{2}\left(A\sharp_p B+\frac{1}{b-a}\left[b^p(bA-B)+a^p(B-aA)\right]\right)\right.\\&\left.-L_p^p(a,b)A\right]\\&\le\frac{1}{2}\left(\mathcal{P}_\Phi(B,A)+\frac{1}{b-a}\left[\Phi(b)(bA-B)+\Phi(a)(B-aA)\right]\right)\\&-\left(\frac{1}{b-a}\int_a^b\Phi(s)\,ds\right)A\\&\le\frac{\Gamma}{p(p-1)}\left[\frac{1}{2}\left(A\sharp_p B+\frac{1}{b-a}\left[b^p(bA-B)+a^p(B-aA)\right]\right)\right.\\&\left.-L_p^p(a,b)A\right].\end{aligned} \tag{4.25}$$

Let $q\in(0,1)$ and $\Phi:J\subset(0,\infty)\to\mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some γ, Γ we have the condition (4.9). Then

$$\begin{aligned}
&\frac{\gamma}{q\left(1-q\right)}\left[A\sharp_{q}B-\frac{1}{b-a}\left[a^{q}\left(bA-B\right)+b^{q}\left(B-aA\right)\right]\right]\\
&\leq\frac{1}{b-a}\left[\Phi\left(a\right)\left(bA-B\right)+\Phi\left(b\right)\left(B-aA\right)\right]-\mathcal{P}_{\Phi}\left(B,A\right)\\
&\leq\frac{\Gamma}{q\left(1-q\right)}\left[A\sharp_{q}B-\frac{1}{b-a}\left[a^{q}\left(bA-B\right)+b^{q}\left(B-aA\right)\right]\right]
\end{aligned}\tag{4.26}$$

$$\begin{aligned}
&\frac{\gamma}{q\left(1-q\right)}\left[A\sharp_{q}B-q\left(A\sharp_{q-1}B\right)\left(\frac{a+b}{2}I-A^{-1}B\right)-L_{q}^{q}(a,b)A\right]\\
&\leq\left(\frac{1}{b-a}\int_{a}^{b}\Phi\left(t\right)dt\right)A-\mathcal{P}_{\Phi}\left(B,A\right)-\mathcal{P}_{\Phi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}\left(B,A\right)\\
&\leq\frac{\Gamma}{q\left(1-q\right)}\left[A\sharp_{q}B-q\left(A\sharp_{q-1}B\right)\left(\frac{a+b}{2}I-A^{-1}B\right)-L_{q}^{q}(a,b)A\right],
\end{aligned}\tag{4.27}$$

and

$$\begin{aligned}
&\frac{\gamma}{q\left(1-q\right)}\left[L_{q}^{q}(a,b)A\right.\\
&\left.-\frac{1}{2}\left(A\sharp_{q}B+\frac{1}{b-a}\left[b^{q}\left(bA-B\right)+a^{q}\left(B-aA\right)\right]\right)\right]\\
&\leq\frac{1}{2}\left(\mathcal{P}_{\Phi}\left(B,A\right)+\frac{1}{b-a}\left[\Phi\left(b\right)\left(bA-B\right)+\Phi\left(a\right)\left(B-aA\right)\right]\right)\\
&-\left(\frac{1}{b-a}\int_{a}^{b}\Phi\left(s\right)ds\right)A\\
&\leq\frac{\Gamma}{q\left(1-q\right)}\left[L_{q}^{q}(a,b)A\right.\\
&\left.-\frac{1}{2}\left(A\sharp_{q}B+\frac{1}{b-a}\left[b^{q}\left(bA-B\right)+a^{q}\left(B-aA\right)\right]\right)\right],
\end{aligned}\tag{4.28}$$

for positive invertible operators A and B that satisfy the condition (3.1).

5. Applications for Logarithm

Let $\Phi:J\subset(0,\infty)\to\mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some δ, Δ we have

$$\delta x^{-2}\leq\Phi''\left(x\right)\leq\Delta x^{-2}\text{ for any }x\in\mathring{J}.\tag{5.1}$$

We observe that the functions $\Phi-\delta\left(-\ln\right)$ and $\Delta\left(-\ln\right)-\Phi$ are convex functions on J. Since $\Psi:=-\ln$ is also a convex function, it follows that $\Phi\in\mathcal{B}\left(J,m,M,\Psi\right)$ with $m=\delta$ and $M=\Delta$.

We define the weighted arithmetic mean and geometric mean as follows

$$A_{\nu}\left(a,b\right):=\left(1-\nu\right)a+\nu b\text{ and }G_{\nu}\left(a,b\right):=a^{1-\nu}b^{\nu}$$

If we use the inequality (2.1) we have

$$\ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{\delta} \leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a+\nu b) \leq \ln\left(\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)^{\Delta} \tag{5.2}$$

for any $a, b \in J$ and any $\nu \in [0,1]$.

From (2.7) and (2.8) we have

$$\ln\left(\frac{I(a,b)}{G(a,b)}\right)^{\delta} \leq \frac{\Phi(a)+\Phi(b)}{2} - \frac{1}{b-a}\int_a^b \Phi(t)\,dt \leq \ln\left(\frac{I(a,b)}{G(a,b)}\right)^{\Delta} \tag{5.3}$$

and

$$\ln\left(\frac{A(a,b)}{I(a,b)}\right)^{\delta} \leq \frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) \leq \ln\left(\frac{A(a,b)}{I(a,b)}\right)^{M}. \tag{5.4}$$

From the inequality (2.15) we have

$$\begin{aligned}
&\ln\left(\frac{I(a,b)}{\sqrt{A(a,b)G(a,b)}}\right)^{\delta} \\
&\leq \frac{1}{2}\left(\Phi\left(\frac{a+b}{2}\right) + \frac{\Phi(b)+\Phi(a)}{2}\right) - \frac{1}{b-a}\int_a^b \Phi(s)\,ds \\
&\leq \ln\left(\frac{I(a,b)}{\sqrt{A(a,b)G(a,b)}}\right)^{\Delta}.
\end{aligned} \tag{5.5}$$

For similar results see [5].

Kamei and Fujii [20], [21] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$S(A|B) := A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}, \tag{5.6}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [32].

For some recent results on relative operator entropy see [12]-[13], [26]-[27] and [29]-[30].

Consider the logarithmic function ln. Then the relative operator entropy can be interpreted as the permanent of ln, namely

$$\mathcal{P}_{\ln}(B,A) = S(A|B).$$

If we use the inequalities (3.2), (3.7) and (3.10) we have the positive invertible operators A and B that satisfy the condition $aA \leq B \leq bB$, then we have

$$\begin{aligned}
&\delta\left[S(A|B) - \frac{1}{b-a}\left[\ln a(bA-B) + \ln b(B-aA)\right]\right] \\
&\leq \frac{1}{b-a}\left[\Phi(a)(bA-B) + \Phi(b)(B-aA)\right] - \mathcal{P}_{\Phi}(B,A) \\
&\leq \Delta\left[S(A|B) - \frac{1}{b-a}\left[\ln a(bA-B) + \ln b(B-aA)\right]\right],
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
(5.8)\qquad & \delta\left[S\left(A|B\right)+\frac{a+b}{2}AB^{-1}A-\ln I\left(a,b\right)A-A\right] \\
& \leq\left(\frac{1}{b-a}\int_a^b\Phi\left(t\right)dt\right)A-\mathcal{P}_{\Phi}\left(B,A\right)-\mathcal{P}_{\Phi'(\cdot)\left(\frac{a+b}{2}-\cdot\right)}\left(B,A\right) \\
& \leq\Delta\left[S\left(A|B\right)+\frac{a+b}{2}AB^{-1}A-\ln I\left(a,b\right)A-A\right]
\end{aligned}$$

and

$$\begin{aligned}
(5.9)\qquad & \delta\left[\ln I\left(a,b\right)A\right. \\
& \left.-\frac{1}{2}\left(S\left(A|B\right)+\frac{1}{b-a}\left[\ln b\left(bA-B\right)+\ln a\left(B-aA\right)\right]\right)\right] \\
& \leq\frac{1}{2}\left(\mathcal{P}_{\Phi}\left(B,A\right)+\frac{1}{b-a}\left[\Phi\left(b\right)\left(bA-B\right)+\Phi\left(a\right)\left(B-aA\right)\right]\right) \\
& -\left(\frac{1}{b-a}\int_a^b\Phi\left(s\right)ds\right)A \\
& \leq\Delta\left[\ln I\left(a,b\right)A\right. \\
& \left.-\frac{1}{2}\left(S\left(A|B\right)+\frac{1}{b-a}\left[\ln b\left(bA-B\right)+\ln a\left(B-aA\right)\right]\right)\right].
\end{aligned}$$

Let $\Phi:J\subset(0,\infty)\to\mathbb{R}$ be twice differentiable on $\mathring{J}$ and such that for some θ, Θ we have

$$(5.10)\qquad \theta x^{-1}\leq\Phi''\left(x\right)\leq\Theta x^{-1}\text{ for any }x\in\mathring{J}.$$

We observe that the functions $\Phi-\theta\ell\ln$ and $\Theta\ell\ln-\Phi$ are convex functions on J, where ℓ is the identity function $\ell\left(t\right)=t$. Since $\Psi:=\ell\ln$ is also a convex function, it follows that $\Phi\in\mathcal{B}\left(J,m,M,\Psi\right)$ with $m=\theta$ and $M=\Theta$.

If we consider the entropy function $\eta\left(t\right)=-t\ln t$, then it is well known that for any positive invertible operators A, B we have

$$(5.11)\qquad S\left(A|B\right)=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}.$$

The function $f\left(t\right)=t\ln t=-\eta\left(t\right),\ t>0$, is convex, then the perspective of this function is

$$\mathcal{P}_{(\cdot)\ln(\cdot)}\left(B,A\right)=-A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=-S\left(B|A\right),$$

where for the last equality we used (5.11) for A replacing B.

From the inequality (3.2) we have, for the convex function $\Psi\left(t\right)=t\ln t$, $t>0$ that

$$\begin{aligned}
(5.12)\qquad & \theta\left[\frac{1}{b-a}\left[a\ln a\left(bA-B\right)+b\ln b\left(B-aA\right)\right]+S\left(B|A\right)\right] \\
& \leq\frac{1}{b-a}\left[\Phi\left(a\right)\left(bA-B\right)+\Phi\left(b\right)\left(B-aA\right)\right]-\mathcal{P}_{\Phi}\left(B,A\right) \\
& \leq\Theta\left[\frac{1}{b-a}\left[a\ln a\left(bA-B\right)+b\ln b\left(B-aA\right)\right]+S\left(B|A\right)\right]
\end{aligned}$$

provided that $\Phi : J \subset (0, \infty) \to \mathbb{R}$ is twice differentiable on $\mathring{J}$ and such that the condition (5.10) holds while the operators A, B satisfy the condition (3.1) with $[a, b] \subset \mathring{J}$.

Similar inequalities can be stated by utilizing the results in (3.7) and (3.10), however the details are not presented here.

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