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SOME q-ANALOGUES OF HERMITE-HADAMARD INEQUALITY OF FUNCTIONS OF TWO VARIABLES ON FINITE RECTANGLES IN THE PLANE

M. A. LATIF[1], S. S. DRAGOMIR[2], AND E. MOMONIAT[3]

Abstract. Preliminaries of q-calculus for functions of two variables over finite rectangles in the plane are introduced. Some q-analogues of the famous Hermite-Hadamard inequality of functions of two variables defined on finite rectangles in plane are presented. A q_1q_2-Hölder inequality for functions of two variables over finite rectangles is also established to provide some quantum estimates of trapezoidal type inequality for functions of two variables whose q_1q_2-partial derivatives in absolute value with certain powers satisfy the criteria of convexity on co-ordinates.

1. Introduction

Quantum calculus or q-calculus is the study of calculus without limits. In the eighteenth century, Euler initiated the study q-calculus by introducing the q in Newton's work of infinite series. Many remarkable results such as Jacobi's triple product identity and the theory of q-hypergeometric functions were obtained in the nineteenth century. In early twentieth century, Jackson [7] has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in different areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics and in theory of relativity. This subject has received exceptional consideration by many researchers and hence it has appeared as an interdisciplinary subject between mathematics and physics. Interested readers are referred to [4, 5, 8] for some recent developments in the theory of quantum calculus and theory of inequalities in quantum calculus.

Theory of inequalities and theory of convex functions have been observed to be profoundly dependent on each other and consequently a vast literature on inequalities has been produced by a number of researchers by using convex functions, see [1, 2, 6]. The following result has been extensively studied during the past three decades which provides a necessary and sufficient condition for a function $f : I \subset \mathbb{R} \to \mathbb{R}$ to be convex on $[a, b]$, where $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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The inequalities in (1.1) are known as Hermite-Hadamard inequalities. Tariboon et al. [16, 18] introduced the concept of quantum derivatives and quantum integrals on finite intervals and developed various quantum analogues for Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Grüss, Grüss-Čebyšev and other integral inequalities using classical convexity. Most recently, Noor et al. [12, 13, 14] and Zhuang et al. [20] have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions. Motivated by the recent progress in the field quantum calculus, our aim is to further develop this theory for functions of two variables and to provide some quantum analogues of Hermite-Hadamard inequality of functions of two variables over finite rectangles. At the next step, we will also provide some quantum estimates for the right part of the q-analogue of Hermite-Hadamard inequality of functions of two variables by using convexity and quasi-convexity on co-ordinates of the absolute value of the q_1q_2-partial derivatives.

2. Preliminaries

In this section we recall some q-calculus essentials over finite intervals and introduce some results of quantum calculus over finite rectangles from the plane.

Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$, q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ is given in the following definition.

Definition 1. [16] *Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f at x is defined by the expression*

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, x \neq a. \tag{2.1} $$

Since $f : J \to \mathbb{R}$ is a continuous function, thus we have ${}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x)$. The function f is said to be q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$. If $a = 0$ in (2.1), then ${}_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is the well-known q-derivative of f defined by the expression

$$ D_q f(x) = \frac{f(qx) - f(x)}{(1-q)x}, x \neq 0. \tag{2.2} $$

For more details on q-derivative given above by (2.2), we refer the reader to [8].

Definition 2. [16] *Let $f : J \to \mathbb{R}$ be a continuous function. A second-order q-derivative on J denoted by ${}_aD_q^2 f$, provided ${}_aD_q f$ is q-differentiable on J, is defined as ${}_aD_q^2 f = {}_aD_q ({}_aD_q f) : J \to \mathbb{R}$. Similarly higher order q-derivatives on J is defined by ${}_aD_q^n f = {}_aD_q \left({}_aD_q^{n-1} f\right) : J \to \mathbb{R}$.*

The following result is very important to evaluate q-derivative of monomials.

Lemma 1. [16] *Let $\alpha \in \mathbb{R}$ and $0 < q < 1$, we have*

$$ {}_aD_q (x-a)^\alpha = \left(\frac{1-q^\alpha}{1-q}\right)(x-a)^{\alpha-1}. $$

One can find further properties of q-derivatives in [19].

Definition 3. [16] *Suppose that $f : J \to \mathbb{R}$ is a continuous function. Then the definite q-integral on J is defined by*

$$\int_a^x f(x)_a \, d_q x = (x-a)(1-q) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n) a) \tag{2.3}$$

for $x \in J$. If $c \in (a, x)$, then the definite q-integral on J is defined as

$$\begin{aligned} \int_c^x f(x)_a \, d_q x &= \int_a^x f(x)_a \, d_q x - \int_a^c f(x)_a \, d_q x \\ &= (x-a)(1-q) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n) a) \\ &+ (c-a)(1-q) \sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n) a). \end{aligned}$$

If $a = 0$ in (2.3), then we get the classical q-definite integral defined by (see [4]*)*

$$\int_0^x f(x)_0 \, d_q x = (1-q) x \sum_{n=0}^{\infty} q^n f(q^n x), x \in [0, \infty).$$

The following results hold about definite q-integrals.

Theorem 1. [19] *Let $f : J \to \mathbb{R}$ be a continuous function. Then*

(1) ${}_a D_q \int_a^x f(t)_a \, d_q t = f(x)$
(2) $\int_c^x {}_a D_q f(t)_a \, d_q t = f(x) - f(c)$, $c \in (a, x)$.

Theorem 2. [19] *Suppose that $f, g : J \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

(1) $\int_a^x [f(t) + g(t)]_a \, d_q t = \int_a^x f(t)_a \, d_q t + \int_a^x g(t)_a \, d_q t$;
(2) $\int_a^x \alpha f(t)_a \, d_q t = \alpha \int_a^x f(t)_a \, d_q t$;
(3) $\int_c^x f(t)_a \, D_q g(t)_a \, d_q t = f(t) g(t)|_c^x - \int_c^x g(qt + (1-q) a)_a \, D_q f(t)_a \, d_q t$, $c \in (a, x)$.

The following is a valuable result to evaluate definite q-integrals of monomials.

Lemma 2. [16] *For $\alpha \in \mathbb{R} \setminus \{-1\}$ and $0 < q < 1$, the following formula holds:*

$$\int_a^x (x-a)^{\alpha} \; {}_a d_q x = \left(\frac{1-q}{1-q^{\alpha+1}} \right) (x-a)^{\alpha+1}.$$

We will also use the following definite q-integrals to prove our results.

Lemma 3. [16] *Let $0 < q < 1$, the following hold*

$$\Delta_q := \int_0^1 t \, |1 - (1+q) t| \; {}_0 d_q t = \frac{q\left(1 + 4q + q^2\right)}{(1+q+q^2)(1+q)^3},$$

$$\Psi_q := \int_0^1 (1-t) \, |1 - (1+q) t| \; {}_0 d_q t = \frac{q\left(1 + 3q^2 + 2q^3\right)}{(1+q+q^2)(1+q)^3}$$

and

$$\Phi_q := \int_0^1 |1 - (1+q) t| \; {}_0 d_q t = \frac{2q}{(1+q)^2}.$$

In what follows we introduce q-partial derivatives and definite q-integrals for functions of two variables.

Definition 4. *Let $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function of two variables and $0<q_1<1$, $0<q_2<1$, the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives at $(x,y)\in[a,b]\times[c,d]$ can be defined as follows*

$$\frac{{}_a\partial_{q_1}f(x,y)}{{}_a\partial_{q_1}x}=\frac{f(q_1x+(1-q_1)a,y)-f(x,y)}{(1-q_1)(x-a)},x\neq a$$

$$\frac{{}_c\partial_{q_2}f(x,y)}{{}_c\partial_{q_2}y}=\frac{f(x,q_2y+(1-q_2)c)-f(x,y)}{(1-q_2)(y-c)},y\neq c,$$

and

$$\frac{{}_{a,c}\partial^2_{q_1,q_2}f(x,y)}{{}_a\partial_{q_1}x\,{}_c\partial_{q_2}y}=\frac{1}{(1-q_1)(1-q_2)(y-c)(x-a)}\left[f(q_1x+(1-q_1)a,q_2y+(1-q_2)c)\right.$$
$$\left.-f(q_1x+(1-q_1)a,y)-f(x,q_2y+(1-q_2)c)+f(x,y)\right],x\neq a,y\neq c.$$

The function $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is said to be partially q_1-, q_2- and q_1q_2-differentiable on $[a,b]\times[c,d]$ if $\frac{{}_a\partial_{q_1}f(x,y)}{{}_a\partial_{q_1}x}$, $\frac{{}_c\partial_{q_2}f(x,y)}{{}_c\partial_{q_2}y}$ and $\frac{{}_{a,c}\partial^2_{q_1,q_2}f(x,y)}{{}_a\partial_{q_1}x_c\,\partial_{q_2}y}$ exist for all $(x,y)\in[a,b]\times[c,d]$. We can similarly define higher order partial derivatives.

Definition 5. *Suppose that $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is a continuous function. Then the definite q_1q_2-integral on $[a,b]\times[c,d]$ is defined by*

$$\int_c^y\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_cd_{q_2}y=(x-a)(y-c)(1-q_1)(1-q_2)\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^mf(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c)\tag{2.4}$$

for $(x,y)\in[a,b]\times[c,d]$. It is clear from (2.4) that

$$\int_c^y\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y=\int_a^x\int_c^y f(x,y)\,{}_cd_{q_2}y\,{}_ad_{q_1}x.$$

If $(x_1,y_1)\in(a,x)\times(c,y)$, then

(2.5)
$$\int_{y_1}^y\int_{x_1}^x f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y=\int_{y_1}^y\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y-\int_{y_1}^y\int_a^{x_1} f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y$$
$$=\int_c^y\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y-\int_c^{y_1}\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y$$
$$-\int_c^y\int_a^{x_1} f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y+\int_c^{y_1}\int_a^{x_1} f(x,y)\,{}_ad_{q_1}x\,{}_ad_{q_2}y.$$

From (2.5), we also note that

$$\int_c^y\int_a^x f(x,y)\,{}_ad_{q_1}x\,{}_cd_{q_2}y=\int_c^y\left(\int_a^x f(x,y)\,{}_ad_{q_1}x\right)\,{}_cd_{q_2}y$$
$$=\int_a^x\left(\int_c^y f(x,y)\,{}_cd_{q_2}y\right)\,{}_ad_{q_1}x.$$

The following theorems hold for definite q_1q_2-double integrals.

Theorem 3. *Let $f:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Then*

(1) $\frac{{}_{a,c}\partial^2_{q_1,q_2}}{{}_a\partial_{q_1}x_c\,\partial_{q_2}y}\int_c^y\int_a^x f(t,s)\,{}_ad_{q_1}t\,{}_cd_{q_2}s=f(x,y)$

(2) $\int_c^y \int_a^x \frac{{}_{a,c}\partial^2_{q_1,q_2} f(t,s)}{{}_a\partial_{q_1} t_c\ \partial_{q_2} s}\ {}_a d_{q_1} t_c d_{q_2} s = f(x,y)$

(3) $\int_{y_1}^y \int_{x_1}^x \frac{{}_{a,c}\partial^2_{q_1,q_2} f(t,s)}{{}_a\partial_{q_1} t_c\ \partial_{q_2} s}\ {}_a d_{q_1} t_c d_{q_2} s = f(x,y) - f(x,y_1) - f(x_1,y) + f(x_1,y_1)$, $(x_1,y_1) \in (a,x)\times(c,y)$.

Proof. (1) By Definition 5 and the definition of partial q_1q_2-derivatives, we have

$$\frac{{}_{a,c}\partial^2_{q_1,q_2}}{{}_a\partial_{q_1} x\ {}_c\partial_{q_2} y}\Bigg[(x-a)(y-c)(1-q_1)(1-q_2)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right)\Bigg]$$
$$= \frac{1}{(1-q_1)(1-q_2)(y-c)(x-a)}\Bigg[q_1q_2(1-q_1)(1-q_2)(y-c)(x-a)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^{n+1} x + \left(1-q_1^{n+1}\right)a, q_2^{m+1} y + \left(1-q_2^{m+1}\right)c\right)$$
$$-q_1(x-a)(y-c)(1-q_1)(1-q_2)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^{n+1} x + \left(1-q_1^{n+1}\right)a, q_2^{m} y + \left(1-q_2^{m}\right)c\right)$$
$$-q_2(x-a)(y-c)(1-q_1)(1-q_2)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^{n} x + \left(1-q_1^{n}\right)a, q_2^{m+1} y + \left(1-q_2^{m+1}\right)c\right)$$
$$+(x-a)(y-c)(1-q_1)(1-q_2)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right)\Bigg]$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right)$$
$$- \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right)$$
$$- \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right)$$
$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} q_1^n q_2^m f\left(q_1^n x + (1-q_1^n)a, q_2^m y + (1-q_2^m)c\right) = f(x,y)$$

(2) By the definition of partial q_1q_2-derivatives and Definition 5, we have

$$\int_c^y \int_a^x \frac{{}_{a,c}\partial^2_{q_1,q_2} f(t,s)}{{}_a\partial_{q_1} t_c\ \partial_{q_2} s}\ {}_c d_{q_2} s_a d_{q_1} t$$
$$= \int_c^y \int_a^x \frac{1}{(1-q_1)(1-q_2)(s-c)(t-a)}\left[f\left(q_1 t + (1-q_1)a, q_2 s + (1-q_2)c\right)\right.$$
$$\left. - f\left(q_1 t + (1-q_1)a, s\right) - f\left(t, q_2 s + (1-q_2)c\right) + f(t,s)\right]_c d_{q_2} s_a d_{q_1} t$$

$$
\begin{aligned}
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[f\left(q_1^{n+1}x+\left(1-q_1^{n+1}\right)a,q_2^{m+1}y+\left(1-q_2^{m+1}\right)c\right)\right.\\
&\quad -f\left(q_1^{n+1}x+\left(1-q_1^{n+1}\right)a,q_2^{m}y+\left(1-q_2^{m}\right)c\right)\\
&\quad -f\left(q_1^{n}x+\left(1-q_1^{n}\right)a,q_2^{m+1}y+\left(1-q_2^{m+1}\right)c\right)\\
&\quad \left.+f\left(\left(q_1^{n}x+\left(1-q_1^{n}\right)a\right),\left(q_2^{m}y+\left(1-q_2^{m}\right)c\right)\right)\right]\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f\left(q_1^{n}x+\left(1-q_1^{n}\right)a,q_2^{m}y+\left(1-q_2^{m}\right)c\right)\\
&\quad -\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}f\left(q_1^{n}x+\left(1-q_1^{n}\right)a,q_2^{m}y+\left(1-q_2^{m}\right)c\right)\\
&\quad -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}f\left(q_1^{n}x+\left(1-q_1^{n}\right)a,q_2^{m}y+\left(1-q_2^{m}\right)c\right)\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f\left(\left(q_1^{n}x+\left(1-q_1^{n}\right)a\right),\left(q_2^{m}y+\left(1-q_2^{m}\right)c\right)\right)=f\left(x,y\right).
\end{aligned}
$$

(3) By using (2.5) and applying the result (2), we obtain

$$
\begin{aligned}
&\int_{y_1}^{y}\int_{x_1}^{x}\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(t,s\right)}{{}_{a}\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s\\
&=\int_{c}^{y}\int_{a}^{x}\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(t,s\right)}{{}_{a}\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s-\int_{c}^{y_1}\int_{a}^{x}\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(t,s\right)}{{}_{a}\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s\\
&-\int_{c}^{y}\int_{a}^{x_1}\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(t,s\right)}{{}_{a}\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s+\int_{c}^{y_1}\int_{a}^{x_1}\frac{{}_{a,c}\partial^2_{q_1,q_2}f\left(t,s\right)}{{}_{a}\partial_{q_1}x_c\ \partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s\\
&=f\left(x,y\right)-f\left(x,y_1\right)-f\left(x_1,y\right)+f\left(x_1,y_1\right).
\end{aligned}
$$

□

Theorem 4. *Suppose that $f,g:[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ are continuous functions, $\alpha\in\mathbb{R}$. Then, for $(x,y)\in[a,b]\times[c,d]$,*

1. $\int_c^y\int_a^x\left[f\left(t,s\right)+g\left(t,s\right)\right]_a d_{q_1}t_c d_{q_2}s=\int_c^y\int_a^x f\left(t,s\right)_a d_q t+\int_c^y\int_a^x g\left(t,s\right)_a d_{q_1}t_c d_{q_2}s.$
2. $\int_c^y\int_a^x \alpha f\left(t,s\right)_a d_{q_1}t_c d_{q_2}s=\alpha\int_c^y\int_a^x f\left(t,s\right)_a d_{q_1}t_c d_{q_2}s.$
3. *The following integration by parts formula for interated q_1q_2-double integrals holds:*

$$
\begin{aligned}
&\int_{y_1}^{y}\int_{x_1}^{x}f\left(t,s\right)\frac{{}_{a,c}\partial^2_{q_1,q_2}g\left(t,s\right)}{{}_{a}\partial_{q_1}x\ {}_{c}\partial_{q_2}y}\ {}_{a}d_{q_1}t_c d_{q_2}s\\
&=f\left(x,y\right)g\left(x,y\right)-f\left(x,y_1\right)g\left(x,y_1\right)-f\left(x_1,y\right)g\left(x_1,y\right)+f\left(x_1,y_1\right)g\left(x_1,y_1\right)\\
&\quad -\int_{y_1}^{y}g\left(x,q_2s+\left(1-q_2\right)c\right)\frac{{}_{c}\partial_{q_2}f\left(x,s\right)}{{}_{c}\partial_{q_2}s}\ {}_{c}d_{q_2}s\\
&\quad +\int_{y_1}^{y}g\left(x_1,q_2s+\left(1-q_2\right)c\right)\frac{{}_{c}\partial_{q_2}f\left(x_1,s\right)}{{}_{c}\partial_{q_2}s}\ {}_{c}d_{q_2}s\\
&\quad -\int_{x_1}^{x}g\left(q_1t+\left(1-q_1\right)a,y\right)\frac{{}_{a}\partial_{q_1}f\left(t,y\right)}{{}_{a}\partial_{q_1}t}\ {}_{a}d_{q_1}t
\end{aligned}
$$

$$+\int_{a}^{x} g\left(q_{1}t+\left(1-q_{1}\right)a, y_{1}\right) \frac{{}_{a}\partial_{q_{1}} f\left(t, y_{1}\right)}{{}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t$$

$$+\int_{y_{1}}^{y} \int_{x_{1}}^{x} g\left(q_{1}t+\left(1-q_{1}\right)a, q_{2}s+\left(1-q_{2}\right)c\right)$$

$$\times \frac{{}_{c,a}\partial_{q_{2},q_{1}}^{2} f\left(t, s\right)}{{}_{c}\partial_{q_{2}} s\, {}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t\, {}_{c}d_{q_{2}} s, \left(x_{1}, y_{1}\right) \in (a, x) \times (c, y).$$

Proof. The proof of (1) and (2) follows by definition of definite q_1q_2-double integrals. (3) By applying (3) of Theorem 2, we have

$$\int_{y_{1}}^{y}\left(\int_{x_{1}}^{x} f\left(t, s\right) \frac{{}_{a,c}\partial_{q_{1},q_{2}}^{2} g\left(t, s\right)}{{}_{a}\partial_{q_{1}} t\, {}_{c}\ \partial_{q_{2}} s} \ {}_{a}d_{q_{1}} t\right) \ {}_{c}d_{q_{2}} s$$

$$=\int_{y_{1}}^{y}\left[f\left(x, s\right) \frac{{}_{c}\partial_{q_{2}} g\left(x, s\right)}{{}_{c}\partial_{q_{2}} s}-f\left(x_{1}, s\right) \frac{{}_{c}\partial_{q_{2}} g\left(a, s\right)}{{}_{c}\partial_{q_{2}} s}\right.$$

$$\left.-\int_{x_{1}}^{x} \frac{{}_{c}\partial_{q_{2}} g\left(q_{1}t+\left(1-q_{1}\right)a, s\right)}{{}_{c}\partial_{q_{2}} s} \frac{{}_{a}\partial_{q_{1}} f\left(t, s\right)}{{}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t\right] \ {}_{c}d_{q_{2}} s$$

$$=\int_{y_{1}}^{y} f\left(x, s\right) \frac{{}_{c}\partial_{q_{2}} g\left(x, s\right)}{{}_{c}\partial_{q_{2}} s} \ {}_{c}d_{q_{2}} s-\int_{y_{1}}^{y} f\left(x_{1}, s\right) \frac{{}_{c}\partial_{q_{2}} g\left(x_{1}, s\right)}{{}_{c}\partial_{q_{2}} s} \ {}_{c}d_{q_{2}} s$$

$$-\int_{x_{1}}^{x}\left(\int_{y_{1}}^{y} \frac{{}_{c}\partial_{q_{2}} g\left(q_{1}t+\left(1-q_{1}\right)a, s\right)}{{}_{c}\partial_{q_{2}} s} \frac{{}_{a}\partial_{q_{1}} f\left(t, s\right)}{{}_{a}\partial_{q_{1}} t} \ {}_{c}d_{q_{2}} s\right) \ {}_{a}d_{q_{1}} t$$

$$=f\left(x, y\right) g\left(x, y\right)-f\left(x, y_{1}\right) g\left(x, y_{1}\right)$$

$$-\int_{y_{1}}^{y} g\left(x, q_{2}s+\left(1-q_{2}\right)c\right) \frac{{}_{c}\partial_{q_{2}} f\left(x, s\right)}{{}_{c}\partial_{q_{2}} s} \ {}_{c}d_{q_{2}} s$$

$$-f\left(x_{1}, y\right) g\left(x_{1}, y\right)+f\left(x_{1}, y_{1}\right) g\left(x_{1}, y_{1}\right)$$

$$+\int_{y_{1}}^{y} g\left(x_{1}, q_{2}s+\left(1-q_{2}\right)c\right) \frac{{}_{c}\partial_{q_{2}} f\left(x_{1}, s\right)}{{}_{c}\partial_{q_{2}} s} \ {}_{c}d_{q_{2}} s$$

$$-\int_{x_{1}}^{x} g\left(q_{1}t+\left(1-q_{1}\right)a, y\right) \frac{{}_{a}\partial_{q_{1}} f\left(t, y\right)}{{}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t$$

$$+\int_{x_{1}}^{x} g\left(q_{1}t+\left(1-q_{1}\right)a, y_{1}\right) \frac{{}_{a}\partial_{q_{1}} f\left(t, y_{1}\right)}{{}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t$$

$$+\int_{y_{1}}^{y} \int_{x_{1}}^{x} g\left(q_{1}t+\left(1-q_{1}\right)a, q_{2}s+\left(1-q_{2}\right)c\right) \frac{{}_{c,a}\partial_{q_{2},q_{1}}^{2} f\left(t, s\right)}{{}_{c}\partial_{q_{2}} s\, {}_{a}\partial_{q_{1}} t} \ {}_{a}d_{q_{1}} t\, {}_{c}d_{q_{2}} s$$

which is the expected result. □

3. Main Results

In this section, we first prove Hermite-Hadamard type inequality for functions of two variable which are convex on the co-ordinates on $[a,b]\times[c,d]$.

We recall the following definition before proving our main results.

Definition 6. [3] *A mapping* $f:[a,b]\times[c,d]\to\mathbb{R}$ *is said to be convex on* $[a,b]\times[c,d]$ *if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ *and* $\lambda\in[0,1]$.

Definition 7. [3] *A function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is said to be convex on co-ordinates on* $[a, b] \times [c, d]$ *if the partial mappings* $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ *and* $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ *are convex where defined for all* $x \in [a, b], y \in [c, d]$.

A different approach of stating convexity of f on co-ordinates on $[a, b] \times [c, d]$ is given in the definition below.

Definition 8. [9] *A function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is said to be convex on co-ordinates on* $[a, b] \times [c, d]$ *if the inequality*

$$\begin{aligned}&f(tx + (1 - t)y, sz + (1 - s)w)\\&\leq tsf(x, z) + t(1 - s)f(x, w) + s(1 - t)f(y, z) + (1 - t)(1 - s)f(y, w)\end{aligned}$$

holds for all $(t, s) \in [0, 1] \times [0, 1]$ *and* $(x, z), (y, w) \in [a, b] \times [c, d]$.

Definition 9. [15] *A function* $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be quasi-convex on* $[a, b] \times [c, d]$ *if the inequality*

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \max\{f(x, y), f(z, w)\}$$

holds for all $(x, y), (z, w) \in [a, b] \times [c, d]$ *and* $\lambda \in [0, 1]$.

Definition 10. [15] *A function* $f : [a, b] \times [c, d] \to \mathbb{R}$ *is said to be quasi-convex on the co-ordinates on* $[a, b] \times [c, d]$ *if the partial mappings* $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ *and* $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ *are quasi-convex where defined for all* $x \in [a, b], y \in [c, d]$.

Another way of expressing the concept of co-ordinated quasi-convex functions is stated in the definition below.

Definition 11. [15] *A function* $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ *is said to be quasi-convex on co-ordinates on* $[a, b] \times [c, d]$ *if*

$$f(tx + (1 - t)z, sy + (1 - s)w) \leq \max\{f(x, y), f(x, w), f(z, y), f(z, w)\}$$

holds for all $(x, y), (z, w) \in [a, b] \times [c, d]$ *and* $(s, t) \in [0, 1] \times [0, 1]$.

Theorem 5. *Let* $f : [a, b] \times [c, d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be convex on co-ordinates on* $[a, b] \times [c, d]$, *the following inequality holds*

$$\begin{aligned}f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2(b-a)}\int_a^b f\left(x, \frac{c+d}{2}\right) {}_a d_{q_1}x + \frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2}, y\right) {}_c d_{q_2}y\\&\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x, y)_c\, d_{q_2}y_a d_{q_1}x\\&\leq \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a, y)_c\, d_{q_2}y + \frac{1}{2(1+q_1)(d-c)}\int_c^d f(b, y)_c\, d_{q_2}y\\&+ \frac{1}{2(1+q_2)(b-a)}\int_a^b f(x, d)_a\, d_{q_1}x + \frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x, c)_a\, d_{q_1}x\\&\leq \frac{q_1q_2 f(a, c) + q_1 f(a, d) + q_2 f(b, c) + f(b, d)}{(1+q_1)(1+q_2)}.\end{aligned}$$

Proof. Since $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is convex on co-ordinates on $[a,b] \times [c,d]$, we have

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = f\left(\frac{ta+(1-t)\,b+tb+(1-t)\,a}{2}, \frac{c+d}{2}\right)$$
$$\leq \frac{1}{2} f\left(ta+(1-t)\,b, \frac{c+d}{2}\right) + \frac{1}{2} f\left(ta+(1-t)\,b, \frac{c+d}{2}\right)$$

The q_1-integration with respect to t over $[0,1]$, q_2-integration with respect to y over $[c,d]$ on both sides of the above inequality and by the change of variables, gives

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \;{}_a d_{q_1} x. \tag{3.1}$$

Now

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = f\left(\frac{a+b}{2}, \frac{cs+(1-s)\,d+sd+(1-s)\,c}{2}\right)$$
$$\leq \frac{1}{2} f\left(\frac{a+b}{2}, cs+(1-s)\,d\right) + \frac{1}{2} f\left(\frac{a+b}{2}, cd+(1-s)\,c\right)$$

The q_1-integration with respect to x over $[a,b]$, q_2-integration with respect to s over $[0,1]$ on both sides of the above inequality and by the change of variables, gives

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \;{}_c d_{q_2} y. \tag{3.2}$$

Adding (3.1) and (3.2) and dividing both sides by 2, we get

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2\,(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) \;{}_a d_{q_1} x + \frac{1}{2\,(d-c)} \int_c^d f\left(\frac{a+b}{2}, y\right) \;{}_c d_{q_2} y. \tag{3.3}$$

Consider now

$$\frac{1}{2\,(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right)$$
$$= \frac{1}{2\,(b-a)} \int_a^b f\left(x, \frac{cs+(1-s)\,d+sd+(1-s)\,c}{2}\right) \;{}_a d_{q_1} x$$
$$\leq \frac{1}{4\,(b-a)} \int_a^b f\left(x, cs+(1-s)\,d\right) \;{}_a d_{q_1} x + \frac{1}{4\,(b-a)} \int_a^b f\left(x, sd+(1-s)\,c\right) \;{}_a d_{q_1} x$$

The q_2-integration with respect to s over $[0,1]$, yields

$$\frac{1}{2\,(b-a)} \int_a^b f\left(x, \frac{c+d}{2}\right) \tag{3.4}$$
$$\leq \frac{1}{4\,(b-a)} \int_a^b \int_c^d f\,(x,y)_c \, d_{q_2} y_a d_{q_1} x + \frac{1}{4\,(b-a)} \int_a^b \int_c^d f\,(x,y) \;{}_c d_{q_2} y_a d_{q_1} x$$
$$= \frac{1}{2\,(b-a)\,(d-c)} \int_a^b \int_c^d f\,(x,y)_c \, d_{q_2} y_a d_{q_1} x.$$

Similarly

$$\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y \leq \frac{1}{2(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_c\, d_{q_2}y_a d_{q_1}x. \tag{3.5}$$

Addition of (3.4) and (3.5), gives

$$\begin{aligned}\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)+\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)\ {}_c d_{q_2}y \\ \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_c\, d_{q_2}y_a d_{q_1}x.\end{aligned} \tag{3.6}$$

We also observe that

$$\begin{aligned}\int_a^b\int_c^d f(x,y)_c\, d_{q_2}y_a d_{q_1}x &= (b-a)\int_0^1\int_c^d f(tb+(1-t)a,y)_c\, d_{q_2}y_0 d_{q_1}t \\ \leq (b-a)\int_0^1\int_c^d (1-t)f(a,y)_c\, d_{q_2}y_0 d_{q_1}t &+ (b-a)\int_0^1\int_c^d tf(b,y)_c\, d_{q_2}y_0 d_{q_1}t \\ &= \frac{q_1(b-a)}{1+q_1}\int_c^d f(a,y)_c\, d_{q_2}y+\frac{(b-a)}{1+q_1}\int_c^d f(b,y)_c\, d_{q_2}y.\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\int_a^b\int_c^d f(x,y)_c\, d_{q_2}y_a d_{q_1}x &= (d-c)\int_a^b\int_0^1 f(x,sd+(1-s)c)_c\, d_{q_2}s_0 d_{q_1}x \\ \leq (d-c)\int_a^b\int_0^1 sf(x,d)_c\, d_{q_2}s_0 d_{q_1}x &+ (d-c)\int_a^b\int_c^d (1-s)f(x,c)_c\, d_{q_2}s_0 d_{q_1}x \\ &= \frac{(d-c)}{1+q_2}\int_a^b f(x,d)_a\, d_{q_1}x+\frac{q_2(d-c)}{1+q_2}\int_a^b f(x,c)_a\, d_{q_1}x.\end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8) and multiplying the resulting inequality by $\frac{1}{2(b-a)(d-c)}$, we get

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)_c\, d_{q_2}y_a d_{q_1}x \\ &\leq \frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)_c\, d_{q_2}y+\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)_c\, d_{q_2}y \\ &+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)_a\, d_{q_1}x+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)_a\, d_{q_1}x.\end{aligned} \tag{3.9}$$

Lastly, we have

$$\begin{aligned}(3.10)\quad &\frac{q_1}{2(1+q_1)(d-c)}\int_c^d f(a,y)_c\,d_{q_2}y+\frac{1}{2(1+q_1)(d-c)}\int_c^d f(b,y)_c\,d_{q_2}y\\ &+\frac{1}{2(1+q_2)(b-a)}\int_a^b f(x,d)_a\,d_{q_1}x+\frac{q_2}{2(1+q_2)(b-a)}\int_a^b f(x,c)_a\,d_{q_1}x\\ \le&\frac{q_1}{2(1+q_1)}\int_0^1 f(a,sd+(1-s)c)_0\,d_{q_2}s+\frac{1}{2(1+q_1)}\int_0^1 f(b,sd+(1-s)c)_0\,d_{q_2}s\\ &+\frac{1}{2(1+q_2)}\int_0^1 f(tb+(1-t)a,d)_0\,d_{q_1}t+\frac{q_2}{2(1+q_2)}\int_0^1 f(tb+(1-t)a,c)_0\,d_{q_1}t\\ \le&\frac{q_1f(a,d)}{2(1+q_1)}\int_0^1 s_0d_{q_2}s+\frac{q_1f(a,c)}{2(1+q_1)}\int_0^1(1-s)\ {}_0d_{q_2}s\\ &+\frac{f(b,d)}{2(1+q_1)}\int_0^1 s_0d_{q_2}s+\frac{f(b,c)}{2(1+q_1)}\int_0^1(1-s)\ {}_0d_{q_2}s\\ &+\frac{f(b,d)}{2(1+q_2)}\int_0^1 t_0d_{q_1}t+\frac{f(a,d)}{2(1+q_2)}\int_0^1(1-t)_0\,d_{q_1}t\\ &+\frac{q_2f(b,c)}{2(1+q_2)}\int_0^1 t_0d_{q_1}t+\frac{q_2f(a,c)}{2(1+q_2)}\int_0^1(1-t)_0\,d_{q_1}t\\ =&\frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)}.\end{aligned}$$

□

Remark 1. *When $q_1\to1^-$ and $q_2\to1^-$, Theorem 5 becomes Theorem 1 from* [3, page 778].

We need the following results to prove our next results.

Theorem 6 (Hölder inequality for double sums)**.** *Suppose $(a_{nm})_{n,m\in\mathbb{N}}$, $(b_{nm})_{n,m\in\mathbb{N}}$ with $a_{nm},b_{nm}\in\mathbb{R}$ or $\mathbb{C}$ and $\frac{1}{p}+\frac{1}{p'}=1$, $p,\,p'>1$, the following Hölder inequality for double sums holds*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}b_{nm}|\le\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}\right)^{\frac{1}{p'}},$$

where all the sums are assumed to be finite.

Proof. Let

$$x=\frac{|a_{nm}|}{\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^p\right)^{\frac{1}{p}}}\ \text{and}\ y=\frac{|b_{nm}|}{\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}\right)^{\frac{1}{p'}}}.$$

Applying the Young's inequality

$$xy\le\frac{x^p}{p}+\frac{y^{p'}}{p'},\ \frac{1}{p}+\frac{1}{p'}=1,$$

we get

$$\frac{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}||b_{nm}|}{\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^{p}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}\right)^{\frac{1}{p'}}}$$
$$\leq \frac{1}{p}\frac{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^{p}}{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{nm}|^{p}}+\frac{1}{p'}\frac{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}}{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|b_{nm}|^{p'}}$$
$$=\frac{1}{p}+\frac{1}{p'}=1$$

and thus the proof follows. □

Theorem 7 (q_1q_2-Hölder inequality for functions of two variables)**.** *Let f and g be functions defined on $[a,b]\times[c,d]$ and $0<q_1,q_2<1$. If $\frac{1}{r_1}+\frac{1}{r_2}$ with $r_1>1$, the following q_1q_2-Hölder inequality holds*

$$\int_a^x\int_c^y |f(x,y)\,g(x,y)|\ {}_cd_{q_2}y\ {}_ad_{q_1}x \tag{3.11}$$
$$\leq\left(\int_a^b\int_c^d |f(x,y)|^{r_1}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_1}}\left(\int_a^b\int_c^d |g(x,y)|^{r_2}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_2}}.$$

Proof. By the definition of q_1q_2-integral and applying Theorem 6, we have

$$\int_a^b\int_c^d |f(x,y)\,g(x,y)|\ {}_cd_{q_2}y\ {}_ad_{q_1}x=(1-q_1)(1-q_2)(x-a)(y-c)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\left|f\left(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|$$
$$\times\left|g\left(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[(1-q_1)(1-q_2)(x-a)(y-c)\right]^{\frac{1}{r_1}}\left(q_1^nq_2^m\right)^{\frac{1}{r_1}}$$
$$\times\left|f\left(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|$$
$$\times\left[(1-q_1)(1-q_2)(x-a)(y-c)\right]^{\frac{1}{r_2}}\left(q_1^nq_2^m\right)^{\frac{1}{r_2}}$$
$$\times\left|g\left(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|$$
$$\leq\Bigg((1-q_1)(1-q_2)(x-a)(y-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m$$
$$\times\left|f\left(q_1^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|^{r_1}\Bigg)^{\frac{1}{r_1}}$$
$$\times\Bigg((1-q_1)(1-q_2)(x-a)(y-c)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m$$
$$\times\left|g\left(q_{11}^nx+(1-q_1^n)a,q_2^my+(1-q_2^m)c\right)\right|^{r_2}\Bigg)^{\frac{1}{r_2}}$$
$$=\left(\int_a^b\int_c^d |f(x,y)|^{r_1}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_1}}\left(\int_a^b\int_c^d |g(x,y)|^{r_2}\ {}_cd_{q_2}y\ {}_ad_{q_1}x\right)^{\frac{1}{r_2}}.$$

□

Lemma 4. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° for $0<q_1,q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f(t,s)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$, then the following equality holds*

$$\tag{3.12} \begin{aligned} \Upsilon_{q_1,q_2}(a,b,c,d)(f) &:= \frac{q_1q_2f(a,c)+q_1f(a,d)+q_2f(b,c)+f(b,d)}{(1+q_1)(1+q_2)} \\ &- \frac{q_2}{(1+q_2)(b-a)}\int_a^b f(x,c)\ {}_ad_{q_1}x - \frac{1}{(1+q_2)(b-a)}\int_a^b f(x,d)\ {}_ad_{q_1}x \\ &- \frac{q_1}{(1+q_1)(d-c)}\int_c^d f(a,y)\ {}_cd_{q_2}y - \frac{1}{(1+q_1)(d-c)}\int_c^d f(b,y)\ {}_cd_{q_2}y \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x \\ &= \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\int_0^1\int_0^1 (1-(1+q_1)t)(1-(1+q_2)s) \\ &\quad\times \frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\ {}_0d_{q_1}t\ {}_0d_{q_2}s. \end{aligned}$$

Proof. By the definition of partial q_1q_2-derivatives and definite q_1q_2-integrals, we have

$$\tag{3.13} \begin{aligned} &\int_0^1\int_0^1 (1-(1+q_1)t)(1-(1+q_2)s) \\ &\quad\times \frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t{}_c\partial_{q_1}s}\ {}_0d_{q_1}t\ {}_0d_{q_2}s \\ &= \frac{1}{(1-q_1)(1-q_2)(b-a)(d-c)}\int_0^1\int_0^1 \frac{(1-(1+q_1)t)(1-(1+q_2)s)}{st} \\ &\times[f(tq_1b+(1-tq_1)a,sq_2d+(1-sq_2)c)-f(tq_1b+(1-tq_1)a,sd+(1-s)c) \\ &-f(tb+(1-t)a,q_2sd+(1-q_2s)c)+f(tb+(1-t)a,sd+(1-s)c)]\ {}_0d_{q_1}t\ {}_0d_{q_2}s \\ &= \frac{1}{(b-a)(d-c)}\sum_{n=0}^\infty\sum_{m=0}^\infty (1-(1+q_1)q_1^n)(1-(1+q_2)q_2^m) \\ &\quad\times\left[f\left(q_1^{n+1}b+\left(1-q_1^{n+1}\right)a,q_2^{m+1}d+\left(1-q_2^{m+1}\right)c\right)\right. \\ &\quad-f\left(q_1^{n+1}b+\left(1-q_1^{n+1}\right)a,q_2^md+(1-q_2^m)c\right) \\ &\quad-f\left(q_1^nb+(1-q_1^n)a,q_2^{m+1}d+\left(1-q_2^{m+1}\right)c\right) \\ &\quad\left.+f(q_1^nb+(1-q_1^n)a,q_2^md+(1-q_2^m)c)\right] \\ &= \frac{1}{(b-a)(d-c)}\sum_{n=1}^\infty\sum_{m=1}^\infty f(q_1^nb+(1-q_1^n)a,q_2^md+(1-q_2^m)c) \\ &- \frac{1}{(b-a)(d-c)}\sum_{n=1}^\infty\sum_{m=0}^\infty f(q_1^nb+(1-q_1^n)a,(1-q_2^m)c+q_2^md) \\ &\quad- \frac{1}{(b-a)(d-c)}\sum_{n=0}^\infty\sum_{m=1}^\infty f(q_1^nb+(1-q_1^n)a,q_2^md+(1-q_2^m)c) \end{aligned}$$

$$+\frac{1}{(d-c)(b-a)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$

$$+\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right)$$
$$+\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$-\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$-\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$+\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right)$$
$$+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$-\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$+\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$-\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right)$$
$$-\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)$$
$$+\frac{(1+q_1)(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right).$$

We observe that

$$\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right) \tag{3.14}$$
$$=\frac{f(a,c)}{(b-a)(d-c)}+\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right),$$

$$-\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right) \tag{3.15}$$

$$=-\frac{f(a,d)}{(b-a)(d-c)}-\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right),$$

$$-\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right) \tag{3.16}$$

$$=-\frac{f(b,c)}{(b-a)(d-c)}-\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right),$$

$$\frac{1}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right) \tag{3.17}$$

$$=\frac{f(b,d)}{(b-a)(d-c)}+\frac{1}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right),$$

$$\begin{aligned}&-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right)\\&=\frac{(1+q_1)f(b,c)}{q_1(b-a)(d-c)}-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,c\right)\\&\quad-\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right),\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right)\\&=-\frac{(1+q_1)f(b,d)}{q_1(b-a)(d-c)}+\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,d\right)\\&\quad+\frac{(1+q_1)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,(1-q_2^m)c+q_2^m d\right),\end{aligned} \tag{3.19}$$

$$\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right) \tag{3.20}$$

$$\begin{aligned}&=\frac{(1+q_1)}{(b-a)(d-c)}\left[-\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,d\right)+\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,c\right)\right]\\&\quad+\frac{(1+q_1)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a,q_2^m d+(1-q_2^m)c\right),\end{aligned}$$

$$
\begin{aligned}
(3.21)\quad & -\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right)\\
&=\frac{(1+q_2)f(a,d)}{q_2(b-a)(d-c)}-\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(a, q_2^m d+(1-q_2^m)c\right)\\
&\quad-\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.22)\quad & \frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a, (1-q_2^m)c+q_2^m d\right)\\
&=-\frac{(1+q_2)}{(b-a)(d-c)}\left[-\sum_{m=0}^{\infty}q_2^m f\left(b,(1-q_2^m)c+q_2^m d\right)+\sum_{m=0}^{\infty}q_2^m f\left(a,(1-q_2^m)c+q_2^m d\right)\right]\\
&\quad+\frac{(1+q_2)}{(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a, (1-q_2^m)c+q_2^m d\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.23)\quad & \frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right)\\
&=-\frac{(1+q_2)f(b,d)}{q_2(b-a)(d-c)}+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b, q_2^m d+(1-q_2^m)c\right)\\
&\quad+\frac{(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.24)\quad & \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right)\\
&=-\frac{(1+q_1)(1+q_2)f(b,d)}{q_1q_2(b-a)(d-c)}-\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a, d\right)\\
&\quad-\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b, q_2^m d+(1-q_2^m)c\right)\\
&\quad+\frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right)
\end{aligned}
$$

$$
\begin{aligned}
(3.25)\quad & -\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, (1-q_2^m)c+q_2^m d\right)\\
&=\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b,(1-q_2^m)c+q_2^m d\right)\\
&\quad-\frac{(1+q_1)(1+q_2)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, (1-q_2^m)c+q_2^m d\right)
\end{aligned}
$$

and

$$\begin{aligned}(3.26)\quad & -\frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right) \\ &= \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a, d\right) \\ &- \frac{(1+q_1)(1+q_2)}{q_2(b-a)(d-c)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right).\end{aligned}$$

Using (3.14)-(3.26) in (3.13) and simplifying, we get

$$\begin{aligned}(3.27)\quad & \int_0^1\int_0^1 (1-(1+q_1)t)(1-(1+q_2)s) \\ & \times \frac{{}_{a,c}\partial^2_{q_1,q_2} f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}x\,{}_c\partial_{q_2}y}\ {}_0d_{q_1}t\ {}_0d_{q_2}s \\ &= \frac{q_1q_2 f(a,c)+q_1 f(a,d)+q_2 f(b,c)+f(b,d)}{q_1q_2(b-a)(d-c)} \\ &- \frac{(1+q_1)(1-q_1)}{q_1(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a, c\right) \\ &- \frac{(1+q_1)(1-q_1)}{q_1q_2(b-a)(d-c)}\sum_{n=0}^{\infty}q_1^n f\left(q_1^n b+(1-q_1^n)a, d\right) \\ &- \frac{(1+q_2)(1-q_2)}{q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(a, q_2^m d+(1-q_2^m)c\right) \\ &- \frac{(1+q_2)(1-q_2)}{q_1q_2(b-a)(d-c)}\sum_{m=0}^{\infty}q_2^m f\left(b,(1-q_2^m)c+q_2^m d\right) \\ &+ \frac{(1+q_1)(1+q_2)(1-q_1)(1-q_2)}{q_1q_2(b-a)(d-c)} \\ &\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^n q_2^m f\left(q_1^n b+(1-q_1^n)a, q_2^m d+(1-q_2^m)c\right) \\ &= \frac{q_1q_2 f(a,c)+q_1 f(a,d)+q_2 f(b,c)+f(b,d)}{q_1q_2(b-a)(d-c)} \\ &- \frac{(1+q_1)}{q_1(b-a)^2(d-c)}\int_a^b f(x,c)\ {}_ad_{q_1}x - \frac{(1+q_1)}{q_1q_2(b-a)^2(d-c)}\int_a^b f(x,d)\ {}_ad_{q_1}x \\ &- \frac{(1+q_2)}{q_2(b-a)(d-c)^2}\int_c^d f(a,y)\ {}_cd_{q_2}y - \frac{(1+q_2)}{q_1q_2(b-a)(d-c)^2}\int_c^d f(b,y)\ {}_cd_{q_2}y \\ &+ \frac{(1+q_1)(1+q_2)}{q_1q_2(b-a)^2(d-c)^2}\int_a^b\int_c^d f(x,y)\ {}_cd_{q_2}y\ {}_ad_{q_1}x.\end{aligned}$$

Multiplying both sides of (3.27) by $\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}$, we get the desired equality. □

Remark 2. *As* $q_1 \to 1^-$ *and* $q_2 \to 1^-$, $\Upsilon_{q_1,q_2}(a,b,c,d)(f) \to \Upsilon(a,b,c,d)(f)$, *where*

$$\begin{aligned}\Upsilon(a,b,c,d)(f) := &\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(b-a)(d-c)}\\&-\frac{1}{2(b-a)}\int_a^b f(x,c)\,dx-\frac{1}{2(b-a)}\int_a^b f(x,d)\,dx\\&-\frac{1}{2(d-c)}\int_c^d f(a,y)\,dy-\frac{1}{2(d-c)}\int_c^d f(b,y)\,dy\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\end{aligned}$$

and hence the result of Lemma 1 becomes Lemma 1 proved in [17, 139].

Now can present some integral inequalities for functions whose partial q_1q_2-derivatives satisfy the assumptions of convexity on co-ordinates on $[a,b]\times[c,d]$.

Theorem 8. *Let* $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* q_1q_2*-differentiable function on* Λ° *with* $0<q_1<1$ *and* $0<q_2<1$. *If partial* q_1q_2*-derivative* $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ *is continuous and integrable on* $[a,b]\times[c,d]\subseteq\Lambda^\circ$ *and* $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ *is convex on co-ordinates on* $[a,b]\times[c,d]$ *for* $r\geq 1$, *then the following inequality holds*

$$\begin{aligned}(3.28)\quad |\Upsilon_{q_1,q_2}(a,b,c,d)(f)| \leq &\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}(\Phi_{q_1}\Phi_{q_2})^{1-\frac{1}{r}}\\&\times\left\{\Psi_{q_1}\Psi_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+\Delta_{q_1}\Psi_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right.\\&\left.+\Delta_{q_2}\Psi_{q_1}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+\Delta_{q_1}\Delta_{q_2}\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right\}^{\frac{1}{r}}.\end{aligned}$$

Proof. Taking the absolute value on both sides of the equality of Lemma 1, using the q_1q_2-Hölder inequality for functions of two variables and convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ on co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}(3.29)\quad &|\Upsilon_{q_1,q_2}(a,b,c,d)(f)|\\&\leq\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|\right.\\&\quad\times\left[\left((1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+(1-t)s\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right.\right.\\&\quad\left.\left.\left.+(1-s)t\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r+st\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right]\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r}}.\end{aligned}$$

From Lemma 3, we observe that

$$\begin{aligned}&\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)| \ {}_0d_{q_1}t \ {}_0d_{q_2}s\\&=\left(\int_0^1 |(1-(1+q_1)t)| \ {}_0d_{q_1}t\right)\left(\int_0^1 |(1-(1+q_2)s)| \ {}_0d_{q_2}s\right)=\Phi_{q_1}\Phi_{q_2},\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1 (1-t)(1-s)|(1-(1+q_1)t)(1-(1+q_2)s)| \ {}_0d_{q_1}t \ {}_0d_{q_2}s\\&=\left(\int_0^1 (1-t)|1-(1+q_1)t| \ {}_0d_{q_1}t\right)\left(\int_0^1 (1-s)|1-(1+q_2)s| \ {}_0d_{q_2}s\right)\\&=\Psi_{q_1}\Psi_{q_2},\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1 (1-t)s|(1-(1+q_1)t)(1-(1+q_2)s)| \ {}_0d_{q_1}t \ {}_0d_{q_2}s\\&=\left(\int_0^1 (1-t)|1-(1+q_1)t| \ {}_0d_{q_1}t\right)\left(\int_0^1 s|1-(1+q_2)s| \ {}_0d_{q_2}s\right)=\Delta_{q_1}\Psi_{q_2},\end{aligned}$$

$$\begin{aligned}&\int_0^1\int_0^1 t(1-s)|(1-(1+q_1)t)(1-(1+q_2)s)| \ {}_0d_{q_1}t \ {}_0d_{q_2}s\\&=\left(\int_0^1 t|1-(1+q_1)t| \ {}_0d_{q_1}t\right)\left(\int_0^1 (1-s)|1-(1+q_2)s| \ {}_0d_{q_2}s\right)=\Delta_{q_2}\Psi_{q_1}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 ts|(1-(1+q_1)t)(1-(1+q_2)s)| \ {}_0d_{q_1}t \ {}_0d_{q_2}s\\&=\left(\int_0^1 t|1-(1+q_1)t| \ {}_0d_{q_1}t\right)\left(\int_0^1 (1-s)|1-(1+q_2)s| \ {}_0d_{q_2}s\right)=\Delta_{q_1}\Delta_{q_2}.\end{aligned}$$

By using the values of the above q_1q_2-integrals, we get the required inequality. □

Remark 3. *When $q_1\to 1^-$ and $q_2\to 1^-$ in Theorem 8, we get the result proved in Theorem 4 in* [17, page 146].

Theorem 9. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $r_1>1$, then the following inequality holds*

$$\begin{aligned}(3.30)\quad |\Upsilon_{q_1,q_2}(a,b,c,d)(f)|&\le\frac{q_1q_2(b-a)(d-c)}{[(1+q_1)(1+q_2)]^{1+\frac{1}{r_1}}}(A_{q_1}(r_2)A_{q_2}(r_2))^{\frac{1}{r_2}}\\&\times\left\{q_1q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+q_1\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right.\\&\left.+q_2\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right\}^{\frac{1}{r_1}},\end{aligned}$$

where $\frac{1}{r_1}+\frac{1}{r_2}=1$.

Proof. Taking the absolute value on both sides of the equality of Lemma 1, using the q_1q_2-Hölder inequality for functions of two variables and convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ on co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}(3.31)\quad &\left|\Upsilon_{q_1,q_2}(a,b,c,d)(f)\right| \\ &\leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\int_0^1\int_0^1 \left|(1-(1+q_1)t)(1-(1+q_2)s)\right|^{r_2}\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_2}} \\ &\times\left(\int_0^1\int_0^1\left[(1-t)(1-s)\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+(1-t)s\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right.\right. \\ &\left.\left.+(1-s)t\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}+st\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}\right]\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_1}}.\end{aligned}$$

By using Lemma 2 and the properties of q-integrals, we have the following observations

$$\begin{aligned}(3.32)\quad &\int_0^1\left|1-(1+q_1)t\right|^{r_2}\ {}_0d_{q_1}t \\ &=\int_0^{\frac{1}{1+q_1}}(1-(1+q_1)t)^{r_2}\ {}_0d_{q_1}t+\int_{\frac{1}{1+q_1}}^1((1+q_1)t-1)^{r_2}\ {}_0d_{q_1}t.\end{aligned}$$

Consider the first q_1-integral from (3.32) and making use of the substitution $1-(1+q_1)t=s$, we obtain

$$\begin{aligned}(3.33)\quad \int_0^{\frac{1}{1+q_1}}(1-(1+q_1)t)^{r_2}\ {}_0d_{q_1}t &= -\frac{1}{1+q_1}\int_1^0 s^{r_2}\ {}_0d_{q_1}s \\ &=\frac{1}{1+q_1}\int_0^1 s^{r_2}\ {}_0d_{q_1}s=\frac{1-q_1}{(1+q_1)\left(1-q_1^{r_2+1}\right)}.\end{aligned}$$

Consider the second q_2-integral from (3.32) and making use of the substitution $1-(1+q_1)t=s$, we get

$$\begin{aligned}(3.34)\quad \int_{\frac{1}{1+q_1}}^1((1+q_1)t-1)^{r_2}\ {}_0d_{q_1}t &= \frac{1}{1+q_1}\int_0^{q_1}s^{r_2}\ {}_0d_{q_1}s \\ &=\frac{(1-q_1)q_1^{r_2+1}}{(1+q_1)\left(1-q_1^{r_2+1}\right)}.\end{aligned}$$

Substitution of (3.33) and (3.34) in (3.32) gives

$$(3.35)\quad \int_0^1\left|1-(1+q_1)t\right|^{r_2}\ {}_0d_{q_1}t=\frac{(1-q_1)\left(1+q_1^{r_2+1}\right)}{(1+q_1)\left(1-q_1^{r_2+1}\right)}=A_{q_1}(r_2).$$

Similarly, one can have

$$(3.36)\quad \int_0^1\left|1-(1+q_2)s\right|^{r_2}\ {}_0d_{q_2}s=\frac{(1-q_2)\left(1+q_2^{r_2+1}\right)}{(1+q_2)\left(1-q_2^{r_2+1}\right)}=A_{q_2}(r_2).$$

Finally, we also have

$$\int_0^1 (1-t)\ {}_0d_{q_1}t = \frac{q_1}{1+q_1}, \int_0^1 (1-s)\ {}_0d_{q_2}s = \frac{q_2}{1+q_2},$$
$$\int_0^1 t\ {}_0d_{q_1}t = \frac{1}{1+q_1} \text{ and } \int_0^1 s\ {}_0d_{q_2}s = \frac{1}{1+q_2}.$$

□

Remark 4. *When $q_1 \to 1^-$ and $q_2 \to 1^-$ in Theorem 9, we get the following inequality proved in* [17, page 144]

$$|\Upsilon(a,b,c,d)(f)| \leq \frac{(b-a)(d-c)}{4}\left(\frac{1}{r_2+1}\right)^{\frac{2}{r_2}} \times \left\{\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^{r_1}+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^{r_1}}{4}\right\}^{\frac{1}{r_1}}. \tag{3.37}$$

Indeed, the inequality (3.37) follows by applying L'Hospital rule to the limits

$$\lim_{q_1\to 1^-}\frac{1-q_1}{1-q_1^{r_2+1}} \text{ and } \lim_{q_2\to 1^-}\frac{1-q_2}{1-q_2^{r_2+1}}.$$

The next two results are for quasi-convex functions on co-ordinates on $[a,b]\times[c,d]$.

Theorem 10. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ is quasi-convex on co-ordinates on $[a,b]\times[c,d]$ for $r\geq 1$, then the following inequality holds*

$$|\Upsilon_{q_1,q_2}(a,b,c,d)(f)| \leq \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\frac{4q_1q_2}{(1+q_1)^2(1+q_2)^2}\right) \times\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\right\}. \tag{3.38}$$

Proof. Lemma 1, an application of the q_1q_2-Hölder inequality and quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ on $[a,b]\times[c,d]$, yield

$$\begin{aligned}
&|\Upsilon_{q_1,q_2}(a,b,c,d)(f)| \qquad (3.39)\\
&\le \frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{1-\frac{1}{r}}\\
&\quad\times\left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\right.\\
&\quad\left.\times\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r}}\\
&=\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\int_0^1\int_0^1 |(1-(1+q_1)t)(1-(1+q_2)s)|\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)\\
&\times\left(\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r\right\}\right)^{\frac{1}{r}}.
\end{aligned}$$

Now by using the properties of supremum and Lemma 3, we get the required result from (3.39). □

Theorem 11. *Let $f:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}$ is continuous and integrable on $[a,b]\times[c,d]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_1}$ is convex on co-ordinates on $[a,b]\times[c,d]$ for $r_1>1$, then the following inequality holds*

$$\begin{aligned}
&|\Upsilon_{q_1,q_2}(a,b,c,d)(f)|\le\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(A_{q_1}(r_2)A_{q_2}(r_2)\right)^{\frac{1}{r_2}} \qquad (3.40)\\
&\times\sup\left\{\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,c)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(a,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|,\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f(b,d)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|\right\}.
\end{aligned}$$

Proof. With the similar reasoning as in proving (3.38), we notice that

$$\begin{aligned}
&|\Upsilon_{q_1,q_2}(a,b,c,d)(f)| \qquad (3.41)\\
&\le\frac{q_1q_2(b-a)(d-c)}{(1+q_1)(1+q_2)}\left(\int_0^1\int_0^1|(1-(1+q_1)t)(1-(1+q_2)s)|^{r_1}\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_1}}\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f((1-t)a+tb,(1-s)c+sd)}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^{r_2}\ {}_0d_{q_1}t\ {}_0d_{q_2}s\right)^{\frac{1}{r_2}}.
\end{aligned}$$

By using the properties of the supremum, the quasi-convexity of $\left|\frac{{}_{a,c}\partial^2_{q_1,q_2}f}{{}_a\partial_{q_1}t\ {}_c\partial_{q_2}s}\right|^r$ on $[a,b]\times[c,d]$, (3.35) and (3.36), we get (3.40). □

Remark 5. *As $q_1\to1^-$ and $q_2\to1^-$ in Theorem 10 and Theorem 11, we get the corresponding results of classical calculus of functions of two variables.*

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[1]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail address: m_amer_latif@hotmail.com

[2]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

[2]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail address: sever.dragomir@vu.edu.au

[3]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail address: ebrahim.momoniat@gmail.com