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SOME REVERSES OF THE CAUCHY-SCHWARZ AND TRIANGLE INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract. In this paper, we give some reverses of the Cauchy-Schwarz inequality and triangle inequality in 2-inner product spaces and also, by using the main results, applications for determinantal integral inequalities.

1. **Introduction**

The Cauchy-Schwarz inequality plays an important role in the theory of inner product spaces (see, for instance, [20, 21]), which is one of the classical inequalities. It is well known that, in a semi-inner product space $(\mathscr{X}, \langle\cdot,\cdot\rangle)$, the Cauchy-Schwarz inequality has the form

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle$$

for all $x, y \in \mathscr{X}$. In recent years, many authors have studied some related topics such as the reverse of the Cauchy-Schwarz inequality, the triangle and Bessel inequality as well as Grüss inequality (see [7, 10, 11, 16]). The probably first reverse of the Cauchy-Schwarz inequality for positive real numbers was obtained by Pólya and Szegö in 1925 (see [18, p. 57 and 213–214] and [19, p. 71–72 and 253–255]). Since then, there exist a lot of generalizations of the reverse of the Cauchy-Schwarz inequality. For example, in 2007, Dragomir [6, Chapter 2] contributed much to the reverses of the Cauchy-Schwarz inequality and also similar results for integrals, isotonic functionals as well as generalizations in the setting of inner product spaces are well-studied and understood (see the book [5]). Some other interesting inequalities for the reverse of the Cauchy-Schwarz inequality can be found in [8, 9, 12].

In this paper, we continue and complement this research by proving some new reverses of the Cauchy-Schwarz inequality in framework of 2-inner product spaces.

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Furthermore, as applications, some reverse results for the generalized triangle inequality, i.e., upper bounds for the quantity

$$(0 \leq) \sum_{i=1}^{n} \|x_i, z\| - \left\| \sum_{i=1}^{n} x_i, z \right\|$$

under various assumptions for the vectors $z, x_i \in X,\ i \in \{1, \cdots, n\}$, are established and, also, some applications for the generalized triangle inequality are also given.

2. **Preliminaries**

The concept of 2-normed spaces was introduced by Gähler [14] in 1963. After that, in 1973 and 1977, Diminnie, Gähler and White introduced the concept of 2-inner product spaces ([3, 4]). For more details on 2-inner product spaces, see [2, 13, 15, 17]. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let $\mathscr{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a $\mathbb{K}$-valued function defined on $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ satisfying the following conditions:

(2I-1) $\langle x, x|z \rangle \geq 0$ and $\langle x, x|z \rangle = 0$, if and only if x and z are linearly dependent;
(2I-2) $\langle x, x|z \rangle = \langle z, z|x \rangle$;
(2I-3) $\langle y, x|z \rangle = \overline{\langle x, y|z \rangle}$;
(2I-4) $\langle \alpha x, y|z \rangle = \alpha \langle x, y|z \rangle$, for any $\alpha \in \mathbb{K}$;
(2I-5) $\langle x + x', y|z \rangle = \langle x, y|z \rangle + \langle x', y|z \rangle$.

$\langle \cdot, \cdot | \cdot \rangle$ is called a *2-inner product* on $\mathscr{X}$ and $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner products $\langle \cdot, \cdot | \cdot \rangle$ can be immediately obtained as follows:

(1) If $\mathbb{K} = \mathbb{R}$, then (2I-3) reduces to $\langle y, x|z \rangle = \langle x, y|z \rangle$;
(2) $\langle 0, y|z \rangle = \langle x, 0|z \rangle = \langle x, y|0 \rangle = 0$;
(3) $\langle x, y|\alpha z \rangle = |\alpha|^2 \langle x, y|z \rangle$, for all $x, y, z \in \mathscr{X}$ and $\alpha \in \mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x, y|z \rangle|^2 \leq \langle x, x|z \rangle \langle y, y|z \rangle .$$

In any given 2-inner product space $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$, we can define a function $\|\cdot, \cdot\|$ on $\mathscr{X} \times \mathscr{X}$ by

$$\|x, z\| = \sqrt{\langle x, x|z \rangle} \tag{2.1}$$

for all $x, z \in \mathscr{X}$. It is easy to see that this function satisfies the following conditions:

(2N-1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent;
(2N-2) $\|x, z\| = \|z, x\|$;
(2N-3) $\|\alpha x, z\| = |\alpha| \, \|x, z\|$ for any scalar $\alpha \in \mathbb{K}$;
(2N-4) $\|x + x', z\| \leq \|x, z\| + \|x', z\|$.

Any function $\|\cdot, \cdot\|$ defined on $\mathscr{X} \times \mathscr{X}$ and satisfying the above conditions is called a *2-norm* on $\mathscr{X}$ and $(\mathscr{X}, \|\cdot, \cdot\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ is given, we consider it as a linear 2-norm space $(\mathscr{X}, \|\cdot,\cdot\|)$ with the 2-norm defined by (2.1).

Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field K. Then there exists $(e_i)_{1\le i\le n}$ in a 2-inner product space X such that $(e_i)_{1\le i\le n}$ are linearly dependent vectors and, for any $z \in \mathscr{X}$,

$$\langle e_i, e_j|z\rangle = \delta_{ij}$$

for all $i, j \in \{1, \cdots, n\}$, where δ_{ij} is the Kronecker delta (we say that the family $(e_i)_{1\le i\le n}$ is *z-orthonormal*).

3. Some reverses of the Cauchy-Schwarz inequality

First, we have the following:

Theorem 3.1. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If, for all $x, y, z \in \mathscr{X}$ and $r_1, r_2 > 0$,*

$$r_1 \le |\|x, z\| - \|y, z\|| \le \|x - y, z\| \le r_2, \tag{3.1}$$

then

$$\begin{aligned} \|x, z\|\,\|y, z\| - |\langle x, y|z\rangle| &\le \|x, z\|\,\|y, z\| - |\mathrm{Re}\,\langle x, y|z\rangle| \\ &\le \|x, z\|\,\|y, z\| - \mathrm{Re}\,\langle x, y|z\rangle \\ &\le \frac{1}{2}\left(r_2^2 - r_1^2\right). \end{aligned} \tag{3.2}$$

Proof. Taking the square in the second inequality in (3.1), we have

$$\|x, z\|^2 - 2\,\mathrm{Re}\,\langle x, y, z\rangle + \|y, z\|^2 \le r_2^2,$$

which is equivalent to

$$2\left(\|x, z\|\,\|y, z\| - \mathrm{Re}\,\langle x, y|z\rangle\right)\left(\|x\| - \|y\|\right) \le r_2^2. \tag{3.3}$$

Using the first inequality in (3.1), we have

$$r_1^2 \le \left(\|x, z\| - \|y, z\|\right)^2. \tag{3.4}$$

Therefore, from (3.3) and (3.4), we have (3.2). This completes the proof. □

Corollary 3.2. *With all the assumptions of Theorem 3.1, the following holds:*

$$\|x, z\| + \|y, z\| - \|x + y, z\| \le \sqrt{r_2^2 - r_1^2}. \tag{3.5}$$

Proof. It follows from (3.2) that

$$\begin{aligned} \left(\|x, z\| + \|y, z\|\right)^2 - \|x + y, z\|^2 &= 2\left(\|x, z\|\,\|y, z\| - \mathrm{Re}\,\langle x, y|z\rangle\right) \\ &\le r_2^2 - r_1^2 \end{aligned}$$

gives

$$\left(\|x, z\| + \|y, z\|\right)^2 \le \|x + y, z\|^2 + r_2^2 - r_1^2. \tag{3.6}$$

Taking the square root in (3.6) and taking into account that

$$\sqrt{\alpha + \beta} \le \sqrt{\alpha} + \sqrt{\beta}$$

for all $\alpha, \beta \geq 0$, we have the desired inequality (3.5). This completes the proof. $\square$

Theorem 3.3. *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$,*

$$\left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \leq r \tag{3.7}$$

and

$$\|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \leq \frac{1}{2} r^2 \|x, z\| \, \|y, z\| \tag{3.8}$$

are equivalent

Proof. It is obvious by taking the square in (3.8) and performing the required calculations. $\square$

Remark 3.4. Since

$$\begin{aligned} \| \|y, z\| \, x - \|x, z\| \, y, z\| &= \| \|y, z\| \, (x - y) + (\|y, z\| - \|x, z\|) \, y, z\| \\ &\leq \|y, z\| \, \|x - y, z\| + \| \|y, z\| - \|x, z\| \| \, \|y, z\| \\ &\leq 2 \|y, z\| \, \|x - y, z\| , \end{aligned}$$

the sufficient condition for (3.7) to hold is

$$\|x - y, z\| \leq \frac{r}{2} \|x, z\| .$$

Theorem 3.5. *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Then, for any $x, y, z \in \mathscr{X}$ and $p \geq 1$,*

$$\begin{aligned} 0 &\leq \|x, z\| \, \|y, z\| - |\langle x, y | z \rangle| \\ &\leq \|x, z\| \, \|y, z\| - |\langle x, y | z \rangle| \\ &\leq \frac{1}{2} \begin{cases} \left((\|x, z\| + \|y, z\|)^{2p} - \|x + y, z\|^{2p} \right)^{\frac{1}{p}}, \\ \left(\|x - y, z\|^{2p} - | \|x, z\| - \|y, z\| |^{2p} \right)^{\frac{1}{p}}. \end{cases} \end{aligned} \tag{3.9}$$

Proof. Firstly, observe that

$$2 \left(\|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \right) = (\|x, z\| + \|y, z\|)^2 - \|x + y, z\|^2 .$$

Denoting

$$D := \|x, z\| \, \|y, z\| - \operatorname{Re} \langle x, y | z \rangle ,$$

we have

$$2D + \|x + y, z\|^2 = (\|x, z\| + \|y, z\|)^2 . \tag{3.10}$$

Taking in (3.10) the power $p \geq 1$ and using the elementary inequality

$$(a + b)^p \geq a^p + b^p, \; a, b \geq 0,$$

we have

$$(\|x, z\| + \|y, z\|)^{2p} = \left(2D + \|x + y, z\|^2 \right)^p \geq 2^p D^p + \|x + y, z\|^{2p} .$$

This implies that

$$D^p \leq \frac{1}{2^p} \left((\|x, z\| + \|y, z\|)^{2p} - \|x + y, z\|^{2p} \right) , \tag{3.11}$$

which is clearly equivalent to the first branch of the third inequality in (3.9).

With the above notation, we also have

$$2D + (\|x, z\| - \|y, z\|)^2 = \|x - y, z\|^2. \tag{3.12}$$

Taking the power $p \geq 1$ in (3.12) and using the inequality (3.11), we have

$$\|x - y, z\|^{2p} \geq 2^p D^p + |\|x, z\| - \|y, z\||^{2p}$$

and so we have the last part of (3.9). This completes the proof. □

We stated the following result that provides an invariant property for the constant in the Cauchy-Schwarz inequality:

Theorem 3.6. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{C}$,*

$$\|x, z\|^2 \|y, z\|^2 - |\langle x, y|z\rangle|^2 = \|x - \lambda, z\|^2 \|y, z\|^2 - |\langle x - \lambda y, y|z\rangle|^2.$$

Proof. By properties of 2-inner product, it follows that, for any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&\|x - \lambda, z\|^2 \|y, z\|^2 - |\langle x - \lambda y, y|z\rangle|^2 \\
&= \left(\|x, z\|^2 - 2\operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) + |\lambda|^2 \|y, z\|^2\right) \|y, z\|^2 \\
&\quad - \left|\langle x, y|z\rangle - \lambda \|y, z\|^2\right|^2 \\
&= \|x, z\|^2 \|y, z\|^2 - 2\|y, z\|^2 \operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) + |\lambda|^2 \|y, z\|^4 \\
&\quad - |\langle y, x|z\rangle|^2 + 2\|y, z\|^2 \operatorname{Re}\left(\overline{\lambda}\langle x, y|z\rangle\right) - |\lambda|^2 \|y, z\|^4 \\
&= \|y, z\|^2 \|x, z\|^2 - |\langle y, x|z\rangle|^2.
\end{aligned}$$

This completes the proof. □

Corollary 3.7. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda \in \mathbb{C}$,*

$$\|x, z\|^2 \|y, z\|^2 - |\langle x, y|z\rangle|^2 \leq \|x - \lambda, z\|^2 \|y, z\|^2. \tag{3.13}$$

The equalities holds in (3.13) *if and only if* $\langle x, y|z\rangle = \lambda \|y, z\|^2$.

For two parameters, we can get the following:

Theorem 3.8. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$,*

$$\begin{aligned}
&\left(\|x, z\|^2 \|y, z\|^2 - |\langle x, y|z\rangle|^2\right) |\mu - \lambda|^2 \\
&= \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2 - |\langle x - \lambda y, x - \mu y|z\rangle|^2.
\end{aligned} \tag{3.14}$$

Proof. Denote $w := x - \lambda y$. Using some properties of a 2-inner product, we have

$$\begin{aligned}
\left|\langle x-\lambda y, x-\mu y|z\rangle\right|^2 &= \left|\langle w, x-\mu y|z\rangle\right|^2 \\
&= \left|\langle w, x-\lambda y+(\lambda-\mu)\, y|z\rangle\right|^2 \\
&= \left|\langle w, w+(\lambda-\mu)\, y|z\rangle\right|^2 \\
&= \left|\|w,z\|^2+\overline{(\lambda-\mu)}\,\langle w,y|z\rangle\right|^2 \\
&= \|w,z\|^4+2\|w,z\|^2 \operatorname{Re}(\lambda-\mu)\,\overline{\langle w,y|z\rangle} \\
&\quad +|\lambda-\mu|^2|\langle w,y|z\rangle|^2 \\
&= \|w,z\|^4+2\|w,z\|^2 \operatorname{Re}\left((\lambda-\mu)\,\overline{\langle w,y|z\rangle}\right) \\
&\quad +|\lambda-\mu|^2\|w,z\|^2\|y,z\|^2 \\
&\quad -|\lambda-\mu|^2\left(\|w,z\|^2\|y,z\|^2-|\langle w,y|z\rangle|^2\right). \qquad (3.15)
\end{aligned}$$

Observe also that

$$\begin{aligned}
&\|w,z\|^4+2\|w,z\|^2 \operatorname{Re}\left((\lambda-\mu)\,\overline{\langle w,y|z\rangle}\right)+|\lambda-\mu|^2\|w,z\|^2\|y,z\|^2 \\
&=\|w,z\|^2\left(\|w,z\|^2+2\operatorname{Re}\left(\overline{(\lambda-\mu)\,\overline{\langle w,y|z\rangle}}\right)+|\lambda-\mu|^2\|y,z\|^2\right) \\
&=\|w,z\|^2\left(\|w+(\lambda-\mu)\,y,z\|^2\right) \\
&=\|x-\lambda y\|^2\|x-\mu y\|^2. \qquad (3.16)
\end{aligned}$$

Therefore, from (3.15) and (3.16), we have the desired result (3.14). This completes the proof. □

Corollary 3.9. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. For any $x, y, z \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$,*

$$\|x,z\|^2\|y,z\|^2-|\langle x,y|z\rangle|^2 \le \frac{1}{|\mu-\lambda|^2}\|x-\lambda y,z\|^2\|x-\mu y,z\|^2.$$

As an application of Theorem 3.8, we have the following:

Proposition 3.10. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Then, for all $x, y, z, e \in \mathscr{X}$ with $\|e,z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{C}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$,*

$$\begin{aligned}
&|\langle x,y|z\rangle-\langle x,e|z\rangle\langle e,y|z\rangle| \\
&\le \frac{1}{|\lambda-\mu|\,|\gamma-\eta|}\left(\|x-\lambda e,z\|\,\|x-\mu e,z\|\,\|y-\gamma e,z\|\,\|y-\eta e,z\|\right. \\
&\quad \left.-|\langle x-\lambda e,x-\mu e|z\rangle|\,|\langle y-\gamma e,y-\eta e|z\rangle|\right) \\
&\le \frac{1}{|\lambda-\mu|\,|\gamma-\eta|}\,\|x-\lambda e,z\|\,\|x-\mu e,z\|\,\|x-\gamma e,z\|\,\|x-\eta e,z\|. \qquad (3.17)
\end{aligned}$$

Proof. Applying the Cauchy-Schwarz inequality for the vectors $x - \langle x,e|z\rangle e$ and $y - \langle y,e|z\rangle e$ and taking into account that

$$\langle x-\langle x,e|z\rangle e, y-\langle y,e|z\rangle e|z\rangle = \langle x,y|z\rangle-\langle x,e|z\rangle\langle e,y|z\rangle,$$

$$\|x - \langle x, e|z\rangle\, e, z\|^2 = \|x, z\|^2 - |\langle x, e|z\rangle|^2$$

and

$$\|y - \langle y, e|z\rangle\, e, z\|^2 = \|y, z\|^2 - |\langle y, e|z\rangle|^2,$$

we have

$$\begin{aligned} &|\langle x, y|z\rangle\, \langle x, e|z\rangle\, \langle e, y|z\rangle| \\ &\le \left(\|x, z\|^2 - |\langle x, e|z\rangle|^2\right)^{\frac{1}{2}} \left(\|y, z\|^2 - |\langle y, e|z\rangle|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.18}$$

for any $x, y, z, e \in \mathscr{X}$ with $\|e, z\| = 1$. From (3.14), it follows that

$$\begin{aligned} &\left(\|x, z\|^2 - |\langle x, e|z\rangle|^2\right)^{\frac{1}{2}} \\ &= \frac{1}{|\mu - \lambda|} \left(\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 - |\langle x - \lambda e, x - \mu e|z\rangle|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &\left(\|y, z\|^2 - |\langle y, e|z\rangle|^2\right)^{\frac{1}{2}} \\ &= \frac{1}{|\gamma - \eta|} \left(\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 - |\langle y - \gamma e, y - \eta e|z\rangle|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.20}$$

for any $x, y, z, e \in \mathscr{X}$ with $\|e, z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{C}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$.

Now, if we multiply (3.19) with (3.20), then we have

$$\begin{aligned} &\left(\|x, z\|^2 - |\langle x, e|z\rangle|^2\right)^{\frac{1}{2}} \left(\|y, z\|^2 - |\langle y, e|z\rangle|^2\right)^{\frac{1}{2}} \\ &\le \frac{1}{|\mu - \lambda|\, |\gamma - \eta|} \\ &\quad \times \left(\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e|z\rangle|^2\right)^{\frac{1}{2}} \\ &\quad \times \left(\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e|z\rangle|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.21}$$

Further, if we use the elementary inequality

$$\left(a^2 - b^2\right)^{\frac{1}{2}} \left(c^2 - d^2\right)^{\frac{1}{2}} \le as - bd$$

for all $a \ge b \ge 0$ and $c \ge d \ge 0$, then we also have

$$\begin{aligned} &\left(\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e|z\rangle|^2\right)^{\frac{1}{2}} \\ &\quad \times \left(\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e|z\rangle|^2\right)^{\frac{1}{2}} \\ &\le \|x - \lambda e, z\|\, \|x - \mu e, z\|\, \|y - \gamma e, z\|\, \|y - \eta e, z\| \\ &\quad - |\langle x - \lambda e, x - \mu e|z\rangle|\, |\langle y - \gamma e, y - \eta e|z\rangle|\,. \end{aligned} \tag{3.22}$$

Finally, using (3.18), (3.21) and (3.22), we have the desired inequality (3.17). This completes the proof. □

4. **Some reverses of the triangle inequality**

In this section, we give some reverses of the triangle inequality.

Theorem 4.1. *Let $(\mathscr{X}, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Let $z, x_i, \in \mathscr{X},\ i \in \{1, \cdots, n\}$, and $r_{ij} > 0$ for each $1 \le i \le j \le n$ be such that*

$$0 \le \|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle \le r_{ij}. \tag{4.1}$$

Then the following quadratic reverse of the triangle inequality holds:

$$\left(\sum_{i=1}^{n} \|x_i, z\|\right)^2 \le \left\|\sum_{i=1}^{n} x_i, z\right\|^2 + 2 \sum_{1\le i\le j\le n} r_{ij}. \tag{4.2}$$

The case of the equality holds in (4.2) if and only if it holds in (4.1) for each i, j with $1 \le i \le j \le n$.

Proof. Observe that

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \|x_i, z\|\right)^2 - \left\|\sum_{i=1}^{n} x_i, z\right\|^2 \\
&= \sum_{i,j=1}^{n} \|x_i, z\| \, \|x_j, z\| - \left\langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j | z \right\rangle \\
&= \sum_{i,j=1}^{n} \|x_i, z\| \, \|x_j, z\| - \sum_{i,j=1}^{n} \operatorname{Re}\langle x_i, x_j|z\rangle \\
&= \sum_{i,j=1}^{n} (\|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle) \\
&= \sum_{1\le i\le j\le n} (\|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle) \\
&\quad + \sum_{1\le j\le i\le n} (\|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle) \\
&= 2 \sum_{1\le i\le j\le n} (\|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle).
\end{aligned} \tag{4.3}$$

Using the condition (4.1), we have

$$\sum_{1\le i\le j\le n} (\|x_i, z\| \, \|x_j, z\| - \operatorname{Re}\langle x_i, x_j|z\rangle) \le \sum_{1\le i\le j\le n} r_{ij}$$

and, by (4.3), we have the desired inequality (4.2). The case of the equality is obvious by the identity (4.3) and we omit the details. This completes the proof. ☐

Remark 4.2. From (4.2), one may deduce the coarser inequality that might be useful in some applications:

$$0 \leq \sum_{i=1}^{n} \|x_i, z\| \left\| \sum_{i=1}^{n} x_i, z \right\| \leq \sqrt{2} \left(\sum_{1 \leq i \leq j \leq n} r_{ij} \right)^{\frac{1}{2}}.$$

Theorem 4.3. *Let $(\mathscr{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. Let $z, x_i, \in \mathscr{X}$, $i \in \{1, \cdots, n\}$, and $r > 0$ be such that*

$$\|x_i - x_j, z\| \leq r \tag{4.4}$$

for each $1 \leq i \leq j \leq n$. Then

$$\left(\sum_{i=1}^{n} \|x_i, z\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i, z \right\|^2 + \frac{n(n-1)}{2} r^2. \tag{4.5}$$

The case of the equality holds in (4.5) if and only if

$$\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle = \frac{1}{2} r^2 \tag{4.6}$$

for each $1 \leq i \leq j \leq n$.

Proof. The inequality (4.4) is obviously equivalent to

$$\|x_i, z\|^2 + \|x_j, z\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j | z \rangle + r^2$$

for each $1 \leq i \leq j \leq n$. Since

$$2 \|x_i, z\| \|x_j, z\| \leq \|x_i, z\|^2 + \|x_j, z\|^2$$

for each $1 \leq i \leq j \leq n$, we have

$$\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \leq \frac{1}{2} r^2$$

for each $1 \leq i \leq j \leq n$. Applying Theorem 4.1 for $r_{ij} := \frac{1}{2} r^2$ and taking into account that

$$\sum_{1 \leq i \leq j \leq n} r_{ij} = \frac{n(n-1)}{4} r^2,$$

we deduce the desired inequality (4.5). The case of the equality is also obvious by the Theorem 4.1 and we omit the details. This completes the proof. $\square$

5. Applications for determinantal integral inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_\rho^2(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2-ρ-integrable on Ω, i.e.,

$$\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty,$$

where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω. We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by the formula

$$\begin{aligned}&\langle f,g|h\rangle_\rho\\ &:= \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix}\begin{vmatrix} g(s) & g(t)\\ h(s) & h(t)\end{vmatrix} d\mu(s)\,d\mu(t),\end{aligned}$$

where, by

$$\begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{vmatrix} g(s) & g(t)\\ h(s) & h(t)\end{vmatrix},$$

generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f,h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)\begin{vmatrix} f(s) & f(t)\\ h(s) & h(t)\end{vmatrix} d\mu(s)\,d\mu(t)\right)^{\frac{1}{2}}. \tag{5.1}$$

A simple calculation with integrals reveals that

$$\langle f,g|h\rangle_\rho = \begin{vmatrix} \int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu\\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix} \tag{5.2}$$

and

$$\|f,h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho fh d\mu\\ \int_\Omega \rho fh d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix}^{\frac{1}{2}}, \tag{5.3}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s)\, d\mu(s)$, we have written $\int_\Omega \rho fg d\mu$. Using the representations (5.2), (5.3) and the Theorem 4.1, one may state interesting determinantal integral inequality, as follows:

Proposition 5.1. *Let $f_1,\cdots,f_n,g,h \in L^2_\rho(\Omega)$, where $\rho : \Omega \to [0,\infty)$ is a measurable function on Ω and $r_{ij} > 0$ such that*

$$\begin{vmatrix} \int_\Omega \rho(f_i-f_j)^2 d\mu & \int_\Omega \rho(f_i-f_j)h d\mu\\ \int_\Omega \rho(f_i-f_j)h d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix}^{\frac{1}{2}} \le r$$

for each $1 \le i \le j \le n$. Then

$$\left(\sum_{i=1}^n \begin{vmatrix} \int_\Omega \rho {f_i}^2 d\mu & \int_\Omega \rho f_i h d\mu\\ \int_\Omega \rho f_i h d\mu & \int_\Omega \rho h^2 d\mu\end{vmatrix}^{\frac{1}{2}}\right)^2 \le \begin{vmatrix} \sum_{i=1}^n\int_\Omega \rho {f_i}^2 d\mu & \sum_{i=1}^n\int_\Omega \rho f_i h d\mu\\ \sum_{i=1}^n\int_\Omega \rho f_i h d\mu & \sum_{i=1}^n\int_\Omega \rho h^2 d\mu\end{vmatrix} + 2\sum_{1\le i\le j\le n} r_{ij}.$$

Proof. The proof follows by Theorem 4.1, applied for the 2-inner product $\langle\cdot,\cdot|\cdot\rangle_\rho$ and we omit the details. $\square$

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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