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GENERALIZATION OF SOME INEQUALITIES FOR THE RATIO OF GAMMA FUNCTIONS

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Abstract. We present some monotonic functions and some generalized inequalities involving the ratios of analogues of the Gamma function.

1. Introduction

The classical Euler's Gamma function $\Gamma(t)$ is commonly defined as

$$\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}\,dx,\qquad t>0. \tag{1}$$

The p-digamma function $\psi_p(t)$, q-digamma function $\psi_q(t)$ and k-digamma function $\psi_k(t)$ are respectively defined as follows.

$$\psi_p(t)=\frac{d}{dt}\ln(\Gamma_p(t))=\frac{\Gamma_p'(t)}{\Gamma_p(t)},\qquad t>0. \tag{2}$$

where $\Gamma_p(t)$ is the p-analogue of the Gamma function defined by (see [2], [3])

$$\Gamma_p(t)=\frac{p!p^t}{t(t+1)\dots(t+p)}=\frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})},\quad p\in N,\quad t>0, \tag{3}$$

$$\psi_q(t)=\frac{d}{dt}\ln(\Gamma_q(t))=\frac{\Gamma_q'(t)}{\Gamma_q(t)},\qquad t>0 \tag{4}$$

where $\Gamma_q(t)$ is the q-analogue of the Gamma function defined by (see [4])

$$\Gamma_q(t)=(1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}},\quad q\in(0,1),\quad t>0, \tag{5}$$

and

$$\psi_k(t)=\frac{d}{dt}\ln(\Gamma_k(t))=\frac{\Gamma_k'(t)}{\Gamma_k(t)},\quad t>0 \tag{6}$$

where $\Gamma_k(t)$ is the k-analogue of the Gamma function defined by (see [1], [5])

$$\Gamma_k(t)=\int_0^\infty e^{-\frac{x^k}{k}}x^{t-1}\,dx,\quad k>0,\quad t>0. \tag{7}$$

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In a recent paper [6], Nantomah and Iddrisu proved that the following double inequalities hold:

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)p^t\Gamma_p(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)p^{t-1}\Gamma_p(\alpha+1)} \tag{8}$$

for $k>0$, $p\in N$ and

$$\frac{\alpha e^{t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)}{(\alpha+t)(1-q)^{-t}\Gamma_q(\alpha)} < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(\alpha+1)e^{(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+1)}{(\alpha+t)(1-q)^{1-t}\Gamma_q(\alpha+1)} \tag{9}$$

for $k>0$, $q\in(0,1)$. Where $t\in(0,1)$ and α a positive real number.

Our objective in this paper is to establish some generalizations of the inequalities (8) and (9).

2. Preliminary Results

The following auxiliary results are crucial to the main results of the paper.

Lemma 2.1. *The functions $\psi_p(t)$, $\psi_q(t)$ and $\psi_k(t)$ as defined above have the following series representations.*

$$\psi_p(t)=\ln p-\sum_{n=0}^{p}\frac{1}{n+t}, \quad p\in N, \quad t>0 \tag{10}$$

$$\psi_q(t)=-\ln(1-q)+\ln q\sum_{n=0}^{\infty}\frac{q^{t+n}}{1-q^{t+n}}, \quad q\in(0,1), \quad t>0 \tag{11}$$

$$\psi_k(t)=\frac{\ln k-\gamma}{k}-\frac{1}{t}+\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}, \quad k>0, \quad t>0. \tag{12}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [3], [5] and [6] and the references therein.

Lemma 2.2. *Let $a>0$, $b>0$ and $t>0$. Then,*

$$-a(\frac{\ln k-\gamma}{k})+b\ln p+\frac{a}{t}+a\psi_k(t)-b\psi_p(t)>0.$$

Proof. Using the series representations in equations (10) and (12) we have,

$$\begin{aligned}&-a(\frac{\ln k-\gamma}{k})+b\ln p+\frac{a}{t}+a\psi_k(t)-b\psi_p(t)\\ &\qquad\qquad\qquad\qquad=a\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}+b\sum_{n=0}^{p}\frac{1}{(n+t)}>0.\end{aligned}$$

Lemma 2.3. *Let $a>0$, $b>0$ and $\alpha+\beta t>0$. Then*

$$-a(\frac{\ln k-\gamma}{k})+b\ln p+\frac{a}{\alpha+\beta t}+a\psi_k(\alpha+\beta t)-b\psi_p(\alpha+\beta t)>0$$

Proof. This follows directly from Lemma 2.2.

Lemma 2.4. *Let $a > 0$, $b > 0$ and $t > 0$. Then,*

$$-a(\frac{\ln k - \gamma}{k}) + b\ln(1-q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t) > 0$$

Proof. Using the series representations in equations (11) and (12) we have,

$$\begin{aligned}
&-a(\frac{\ln k - \gamma}{k}) + b\ln(1-q) + \frac{a}{t} + a\psi_k(t) - b\psi_q(t) \\
&\qquad\qquad = a\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)} - b\ln q\sum_{n=0}^{\infty}\frac{q^{t+n}}{1-q^{t+n}} > 0
\end{aligned}$$

Lemma 2.5. *Let $a > 0$, $b > 0$ and $\alpha + \beta t > 0$. Then,*

$$-a(\frac{\ln k - \gamma}{k}) + b\ln(1-q) + \frac{a}{\alpha+\beta t} + a\psi_k(\alpha+\beta t) - b\psi_p(\alpha+\beta t) > 0$$

Proof. This follows directly from Lemma 2.4.

3. Main Results

We now state and prove the results of this paper.

Theorem 3.1. *Define a function Λ by*

$$\Lambda(t) = \frac{(\alpha+\beta t)^a e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{p^{-b\beta t}\Gamma_p(\alpha+\beta t)^b}, \quad t\in(0,\infty),\ k>0,\ p\in N. \tag{13}$$

where a, b, α, β are positive real numbers. Then Λ is increasing on $t \in (0,\infty)$ and the inequality

$$\frac{\alpha^a e^{a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)^a}{(\alpha+\beta t)^a p^{b\beta t}\Gamma_p(\alpha)^b} < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b} < \frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b} \tag{14}$$

holds for every $t \in (0,1)$.

Proof. Let $g(t) = \ln\Lambda(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned}
g(t) &= \ln\frac{(\alpha+\beta t)^a e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{p^{-b\beta t}\Gamma_p(\alpha+\beta t)^b} \\
&= -a\beta t(\frac{\ln k-\gamma}{k}) + b\beta t\ln p + a\ln(\alpha+\beta t) + a\ln\Gamma_k(\alpha+\beta t) - b\ln\Gamma_p(\alpha+\beta t)
\end{aligned}$$

Then,

$$\begin{aligned}
g'(t) &= -a\beta(\frac{\ln k-\gamma}{k}) + b\beta\ln p + \frac{a\beta}{\alpha+\beta t} + a\beta\psi_k(\alpha+\beta t) - b\beta\psi_p(\alpha+\beta t) \\
&= \beta\left[-a(\frac{\ln k-\gamma}{k}) + b\ln p + \frac{a}{\alpha+\beta t} + a\psi_k(\alpha+\beta t) - b\psi_p(\alpha+\beta t)\right] > 0
\end{aligned}$$

as a result of Lemma 2.3. This proves that g is increasing on $t \in (0, \infty)$. Hence Λ is increasing on $t \in (0, \infty)$. Thus, for every $t \in (0, 1)$ we have

$$\Lambda(0) < \Lambda(t) < \Lambda(1),$$

yielding

$$\frac{\alpha^a e^{a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)^a}{(\alpha+\beta t)^a p^{b\beta t}\Gamma_p(\alpha)^b} < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b} < \frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b}.$$

Corollary 3.2. *If $t \in [1, \infty)$, then the following inequality holds.*

$$\frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a p^{b\beta(t-1)}\Gamma_p(\alpha+\beta)^b} \leq \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_p(\alpha+\beta t)^b}$$

Proof. If $t \in [1, \infty)$, then we have $\Lambda(1) \leq \Lambda(t)$ yielding the result.

Theorem 3.3. *Define a function Υ by*

$$\Upsilon(t) = \frac{(\alpha+\beta t)^a e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{(1-q)^{b\beta t}\Gamma_q(\alpha+\beta t)^b}, \quad t \in (0,\infty),\ k>0,\ q \in (0,1). \tag{15}$$

where a, b, α, β are positive real numbers. Then Υ is increasing on $t \in (0, \infty)$ and the inequality

$$\frac{\alpha^a e^{a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)^a}{(\alpha+\beta t)^a (1-q)^{-b\beta t}\Gamma_q(\alpha)^b} < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} < \frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a (1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b} \tag{16}$$

holds for every $t \in (0, 1)$.

Proof. Let $h(t) = \ln \Upsilon(t)$ for every $t \in (0, \infty)$. Then,

$$\begin{aligned} h(t) &= \ln \frac{(\alpha+\beta t)^a e^{-a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta t)^a}{(1-q)^{b\beta t}\Gamma_q(\alpha+\beta t)^b} \\ &= -a\beta t(\frac{\ln k-\gamma}{k}) - b\beta t \ln(1-q) + a\ln(\alpha+\beta t) \\ &\quad + a\ln\Gamma_k(\alpha+\beta t) - b\ln\Gamma_q(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} h'(t) &= -a\beta(\frac{\ln k-\gamma}{k}) - b\beta\ln(1-q) + \frac{a\beta}{\alpha+\beta t} + a\beta\psi_k(\alpha+\beta t) - b\beta\psi_q(\alpha+\beta t) \\ &= \beta\left[-a(\frac{\ln k-\gamma}{k}) - b\ln(1-q) + \frac{a}{\alpha+\beta t} + a\psi_k(\alpha+\beta t) - b\psi_p(\alpha+\beta t)\right] > 0 \end{aligned}$$

as a result of Lemma 2.5. This proves that h is increasing on $t \in (0, \infty)$. Hence Υ is increasing on $t \in (0, \infty)$ and for every $t \in (0, 1)$ we have

$$\Upsilon(0) < \Upsilon(t) < \Upsilon(1)$$

yielding

$$\frac{\alpha^a e^{a\beta t(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha)^a}{(\alpha+\beta t)^a(1-q)^{-b\beta t}\Gamma_q(\alpha)^b} < \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b} < \frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a(1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b}.$$

Corollary 3.4. *If $t \in [1,\infty)$, then the following inequality holds.*

$$\frac{(\alpha+\beta)^a e^{a\beta(t-1)(\frac{\ln k-\gamma}{k})}\Gamma_k(\alpha+\beta)^a}{(\alpha+\beta t)^a(1-q)^{b\beta(1-t)}\Gamma_q(\alpha+\beta)^b} \leq \frac{\Gamma_k(\alpha+\beta t)^a}{\Gamma_q(\alpha+\beta t)^b}$$

Proof. If $t \in [1,\infty)$, then we have $\Upsilon(1) \leq \Upsilon(t)$ yielding the result.

4. Concluding Remarks

Remark 4.1. By putting $a = b = \beta = 1$ into inequalities (14) and (16), we thus obtain respectively, inequalities (8) and (9) as in [6].

Remark 4.2. This paper is a corrected version of the paper [7].

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