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NEW REFINEMENTS AND REVERSES OF HERMITE-HADAMARD INEQUALITY AND APPLICATIONS TO YOUNG'S OPERATOR INEQUALITY

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Abstract. In this paper we obtain some new refinements and reverses of Hermite-Hadamard inequality and provide some examples for basic convex/concave functions of interest such as the norm, the exponential and the logarithm. Applications to Young's operator inequality are given as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)<\frac{1}{b-a}\int_a^b f(x)dx<\frac{f(a)+f(b)}{2},\quad a,b\in\mathbb{R}. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [35]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [35]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,\ y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [10, p. 2], [11, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \le p < \infty$) is a convex function, then for any x, $y \in X$ we have the following norm inequality from (1.2)

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x + ty\|^p dt \le \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

In this paper we obtain some new refinements and reverses of Hermite-Hadamard inequality and provide some examples for basic convex/concave functions of interest such as the norm, the exponential and the logarithm. Applications to Young's operator inequality are given as well.

We need some preliminary results as follows.

2. Some Preliminary Facts

Jensen's inequality for convex function is one of the most known and extensively used inequality in various filed of Modern Mathematics. It is a source of many classical inequalities including the generalized triangle inequality, the arithmetic mean-geometric mean-harmonic mean inequality, the positivity of *relative entropy* in Information Theory, Schannon's inequality, Ky Fan's inequality, Levinson's inequality and other results. For classical and contemporary developments related to the Jensen inequality, see [3], [36], [41] and [6] where further references are provided.

To be more specific, we recall that, if X is a linear space and $C \subseteq X$ a convex subset in X, then for any convex function $f : C \to \mathbb{R}$ and any $z_i \in C, r_i \ge 0$ for $i \in \{1, ..., k\}, k \ge 2$ with $\sum_{i=1}^k r_i = R_k > 0$ one has the *weighted Jensen's inequality:*

$$\frac{1}{R_k}\sum_{i=1}^k r_i f(z_i) \ge f\left(\frac{1}{R_k}\sum_{i=1}^k r_i z_i\right). \tag{J}$$

If $f : C \to \mathbb{R}$ is strictly convex and $r_i > 0$ for $i \in \{1, ..., k\}$ then the equality case hods in (J) if and only if $z_1 = ... = z_n$.

By $\mathcal{P}_n$ we denote the set of all nonnegative n-tuples $(p_1, ..., p_n)$ with the property that $\sum_{i=1}^n p_i = 1$. Consider the *normalised Jensen functional*

$$\mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \ge 0,$$

where $f : C \to \mathbb{R}$ be a convex function on the convex set C and $\mathbf{x} = (x_1, ..., x_n) \in C^n$ and $\mathbf{p} \in \mathcal{P}_n$.

The following result holds [5]:

Lemma 1. *If* $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}_n$, $q_i > 0$ *for each* $i \in \{1, ..., n\}$ *then*

$$(0 \le) \min_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_n(f, \mathbf{x}, \mathbf{q}) \le \mathcal{J}_n(f, \mathbf{x}, \mathbf{p}) \le \max_{1\le i\le n}\left\{\frac{p_i}{q_i}\right\}\mathcal{J}_n(f, \mathbf{x}, \mathbf{q}). \tag{2.1}$$

In the case $n = 2$, if we put $p_1 = 1 - p$, $p_2 = p$, $q_1 = 1 - q$ and $q_2 = q$ with $p \in [0, 1]$ and $q \in (0, 1)$ then by (2.1) we get

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q)f(x) + qf(y) - f((1-q)x + qy)] \\ &\le [(1-p)f(x) + pf(y) - f((1-p)x + py)] \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[(1-q)f(x) + qf(y) - f((1-q)x + qy)] \end{aligned} \tag{2.2}$$

for any $x,\, y \in C$.

If we take $q = \frac{1}{2}$ in (2.2), then we get

$$\begin{aligned}
(2.3) \qquad & 2\min\{t, 1-t\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\
&\le [(1-t)f(x) + tf(y) - f((1-t)x+ty)] \\
&\le 2\max\{t, 1-t\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right]
\end{aligned}$$

for any $x,\, y \in C$ and $t \in [0,1]$.

We consider the weighted Arithmetic, Geometric and Harmonic means defined by

$$A_\nu(a,b) := (1-\nu)a + \nu b,\; G_\nu(a,b) := a^{1-\nu}b^\nu \text{ and } H_\nu(a,b) = A_\nu^{-1}\left(a^{-1}, b^{-1}\right)$$

where $a,\, b > 0$ and $\nu \in [0,1]$.

If we take the convex function $f : \mathbb{R} \to (0,\infty)$, $f(x) = \exp(\alpha x)$, with $\alpha \neq 0$, then we have from (2.2) that

$$\begin{aligned}
(2.4) \qquad & \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a,b))] \\
&\le A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a,b)) \\
&\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_q(a,b))]
\end{aligned}$$

for any $p \in [0,1]$ and $q \in (0,1)$ and any $x,\, y \in \mathbb{R}$.

For $q = \frac{1}{2}$ we have by (2.4) that

$$\begin{aligned}
(2.5) \qquad & 2\min\{p, 1-p\}[A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a,b))] \\
&\le A_p(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A_p(a,b)) \\
&\le 2\max\{p, 1-p\}[A(\exp(\alpha x), \exp(\alpha y)) - \exp(\alpha A(a,b))]
\end{aligned}$$

for any $p \in [0,1]$ and any $x,\, y \in \mathbb{R}$.

If we take $x = \ln a$ and $y = \ln b$ in (2.4), then we get

$$\begin{aligned}
(2.6) \qquad & \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[A_q(a^\alpha, b^\alpha) - G_q^\alpha(a,b)\right] \\
&\le A_p(a^\alpha, b^\alpha) - G_p^\alpha(a,b) \\
&\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[A_q(a^\alpha, b^\alpha) - G_q^\alpha(a,b)\right]
\end{aligned}$$

for any $a,\, b > 0$, for any $p \in [0,1]$, $q \in (0,1)$ and $\alpha \neq 0$.

For $q = \frac{1}{2}$ we have by (2.6) that

$$\begin{aligned}
(2.7) \qquad \min\{p, 1-p\}\left(b^{\frac{\alpha}{2}} - a^{\frac{\alpha}{2}}\right)^2 &\le A_p(a^\alpha, b^\alpha) - G_p^\alpha(a,b) \\
&\le \max\{p, 1-p\}\left(b^{\frac{\alpha}{2}} - a^{\frac{\alpha}{2}}\right)^2
\end{aligned}$$

for any $a,\, b > 0$, for any $p \in [0,1]$ and $\alpha \neq 0$.

For $\alpha = 1$ we get from (2.7) that

$$\min\{p, 1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 \le A_p(a,b) - G_p(a,b) \le \max\{p, 1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2 \tag{2.8}$$

for any a, $b > 0$ and for any $p \in [0,1]$, which are the inequalities obtained by Kittaneh and Manasrah in [29] and [30].

For $\alpha = 1$ in (2.6) we obtain

$$\begin{aligned}&\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(a,b) - G_q(a,b)] \\ &\le A_p(a,b) - G_p(a,b) \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[A_q(a,b) - G_q(a,b)],\end{aligned} \tag{2.9}$$

for any a, $b > 0$, for any $p \in [0,1]$, which is the inequality (2.1) from [1] in the particular case $\lambda = 1$ in a slightly more general form for the weights p, q.

If we take in (2.2) $f(x) = -\ln x$, then we get

$$\left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} \le \frac{A_p(x,y)}{G_p(x,y)} \le \left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} \tag{2.10}$$

for any x, $y > 0$ and for any $p \in [0,1]$, $q \in (0,1)$.

This inequality is a particular case for $n = 2$ of the inequality (4.2) from [5].

For $q = \frac{1}{2}$ we have by (2.10) (for $x = a$, $y = b$) that

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2\min\{p,1-p\}} \le \frac{A_p(a,b)}{G_p(a,b)} \le \left(\frac{A(a,b)}{G(a,b)}\right)^{2\max\{p,1-p\}} \tag{2.11}$$

for any a, $b > 0$ and for any $p \in [0,1]$.

The first inequality in (2.11) was obtained in an equivalent form in terms of Kantorovich constant by Zou et al. in [44] while the second by Liao et al. [31].

If we take in (2.2) $f(x) = -\ln x$ and $x = \exp a$, $y = \exp b$, with $a, b \in \mathbb{R}$, then we get

$$\begin{aligned}&\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[\ln(A_q(\exp a, \exp b)) - A_q(a,b)] \\ &\le \ln(A_p(\exp a, \exp b)) - A_p(a,b) \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}[\ln(A_q(\exp a, \exp b)) - A_q(a,b)]\end{aligned} \tag{2.12}$$

for any $p \in [0,1]$, $q \in (0,1)$.

This inequality can be written in an equivalent form as

$$\begin{aligned}\left(\frac{A_q(\exp a, \exp b)}{\exp A_q(a,b)}\right)^{\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} &\le \frac{A_p(\exp a, \exp b)}{\exp A_p(a,b)} \\ &\le \left(\frac{A_q(\exp a, \exp b)}{\exp A_q(a,b)}\right)^{\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}}\end{aligned} \tag{2.13}$$

for any a, $b \in \mathbb{R}$ and $p \in [0,1]$, $q \in (0,1)$.

3. Trapezoidal Type Integral Inequalities

We have:

Theorem 1. *Let X be a linear space and $C \subseteq X$ a convex subset in X, then for any convex function $f : C \to \mathbb{R}$ and $x, y \in C$ we have*

$$\begin{aligned}
&\frac{1}{2}\left[(1-q) f(x)+q f(y)-f((1-q) x+q y)\right] \\
&\leq \frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) \, dt \\
&\leq \frac{1}{2} \frac{q^2-q+1}{q(1-q)}\left[(1-q) f(x)+q f(y)-f((1-q) x+q y)\right],
\end{aligned} \tag{3.1}$$

or, equivalently

$$\begin{aligned}
&2 \frac{q(1-q)}{q^2-q+1}\left[\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) \, dt\right] \\
&\leq(1-q) f(x)+q f(y)-f((1-q) x+q y) \\
&\leq 2\left[\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t) x+t y) \, dt\right],
\end{aligned} \tag{3.2}$$

for any $q \in (0,1)$.

Proof. If we integrate over $p \in [0,1]$ the inequality (2.2), then we get

$$\begin{aligned}
&\left[(1-q) f(x)+q f(y)-f((1-q) x+q y)\right] \int_0^1 \min \left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} dp \\
&\leq \frac{f(x)+f(y)}{2}-\int_0^1 f((1-p) x+p y) \, dp \\
&\leq\left[(1-q) f(x)+q f(y)-f((1-q) x+q y)\right] \int_0^1 \max \left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} dp
\end{aligned} \tag{3.3}$$

for any $x, y \in C$ and $q \in (0,1)$.

Observe that

$$\frac{p}{q}-\frac{1-p}{1-q}=\frac{p-q}{q(1-q)}$$

showing that

$$\min \left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}=\begin{cases}\frac{p}{q} \text{ if } 0 \leq p \leq q \leq 1 \\ \\ \frac{1-p}{1-q} \text{ if } 0 \leq q \leq p \leq 1\end{cases}$$

and

$$\max \left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}=\begin{cases}\frac{1-p}{1-q} \text{ if } 0 \leq p \leq q \leq 1 \\ \\ \frac{p}{q} \text{ if } 0 \leq q \leq p \leq 1 .\end{cases}$$

Then

$$\begin{aligned}
\int_0^1 \min \left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} dp &= \int_0^q \frac{p}{q} dp+\int_q^1 \frac{1-p}{1-q} dp \\
&=\frac{q^2}{2 q}+\frac{1}{1-q}\left(1-q-\left(\frac{1-q^2}{2}\right)\right)=\frac{1}{2}
\end{aligned}$$

and

$$\int_0^1 \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} dp = \int_0^q \frac{1-p}{1-q} dp + \int_q^1 \frac{p}{q} dp$$
$$= \frac{1}{1-q}\left(q - \frac{q^2}{2}\right) + \frac{1-q^2}{2q}$$
$$= \frac{q^2-q+1}{2q(1-q)}$$

and by (3.3) we obtain the desired result (3.1). □

Remark 1. *If we take $q = \frac{1}{2}$ in (3.1), then we get*

$$\begin{aligned} &\frac{1}{2}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ &\le \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \\ &\le \frac{3}{2}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right], \end{aligned} \tag{3.4}$$

for any x, $y \in C$.

Remark 2. *If the function f is defined on the real interval I and $a, b \in I$, with $a \neq b$, then by (3.1) and (3.2) we have*

$$\begin{aligned} &\frac{1}{2}\left[(1-q)f(a) + qf(b) - f((1-q)a+qb)\right] \\ &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds \\ &\le \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[(1-q)f(a) + qf(b) - f((1-q)a+qb)\right], \end{aligned} \tag{3.5}$$

or, equivalently

$$\begin{aligned} &2\frac{q(1-q)}{q^2-q+1}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds\right] \\ &\le (1-q)f(a) + qf(b) - f((1-q)a+qb) \\ &\le 2\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(s)\,ds\right], \end{aligned} \tag{3.6}$$

for any $q \in (0,1)$.

Let $(X, \|\cdot\|)$ be a real or complex normed linear space. The function $f : X \to [0,\infty)$, $f(x) = \|x\|^r$ with $r \geq 1$ is a convex function on X. Then by (2.2) and (3.1) we have the norm inequalities

$$\begin{aligned} &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[(1-q)\|x\|^r + q\|y\|^r - \|(1-q)x+qy\|^r\right] \\ &\le (1-p)\|x\|^r + p\|y\|^r - \|(1-p)x+py\|^r \\ &\le \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\left[(1-q)\|x\|^r + q\|y\|^r - \|(1-q)x+qy\|^r\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}(3.8)\qquad &\frac{1}{2}\left[(1-q)\|x\|^r+q\|y\|^r-\|(1-q)x+qy\|^r\right]\\ &\leq \frac{\|x\|^r+\|y\|^r}{2}-\int_0^1\|(1-t)x+ty\|^r dt\\ &\leq \frac{1}{2}\frac{q^2-q+1}{q(1-q)}\left[(1-q)\|x\|^r+q\|y\|^r-\|(1-q)x+qy\|^r\right],\end{aligned}$$

for any $x, y\in X$, $p\in[0,1]$ and $q\in(0,1)$.

For positive $x\neq y$ and $p\in\mathbb{R}\setminus\{-1,0\}$, we define the *p-logarithmic mean* (*generalized logarithmic mean*) $L_p(x,y)$ by

$$L_p(x,y):=\left[\frac{y^{p+1}-x^{p+1}}{(p+1)(y-x)}\right]^{1/p}.$$

In fact the singularities at $p=-1,\ 0$ are removable and L_p can be defined for $p=-1,\ 0$ so as to make $L_p(x,y)$ a continuous function of p. In the limit as $p\to 0$ we obtain the *identric mean* $I(x,y)$, given by

$$I(x,y):=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}, \tag{3.9}$$

and in the case $p\to-1$ the *logarithmic mean* $L(x,y)$, given by

$$L(x,y):=\frac{y-x}{\ln y-\ln x}.$$

In each case we define the mean as x when $y=x$, which occurs as the limiting value of $L_p(x,y)$ for $y\to x$.

We define the arithmetic mean as $A(x,y):=\frac{x+y}{2}$, the geometric mean as $G(x,y):=\sqrt{xy}$ and the harmonic mean as $H(x,y)=A^{-1}\left(x^{-1},y^{-1}\right)$.

If we take the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(\alpha x)$, with $\alpha\neq 0$, then we have from (3.6) that

$$\begin{aligned}(3.10)\qquad &2\frac{q(1-q)}{q^2-q+1}\left[A(\exp(\alpha x),\exp(\alpha y))-E_\alpha(x,y)\right]\\ &\leq A_q(\exp(\alpha x),\exp(\alpha y))-\exp(A_q(x,y))\\ &\leq 2\left[A(\exp(\alpha x),\exp(\alpha y))-E_\alpha(x,y)\right],\end{aligned}$$

for any $x, y\in\mathbb{R}$, $x\neq y$ and $q\in[0,1]$ where the *exponential integral mean* $E_\alpha(x,y)$ is defined by

$$E_\alpha(x,y):=\frac{1}{y-x}\int_x^y\exp(\alpha s)\,ds=\frac{\exp(\alpha y)-\exp(\alpha x)}{\alpha(y-x)}. \tag{3.11}$$

Now, if we take in (3.10) $x=\ln a$ and $y=\ln b$ for $a, b>0$, then we get (for $q=p$) that

$$\begin{aligned}(3.12)\qquad 2\frac{p(1-p)}{p^2-p+1}\left[A(a^\alpha,b^\alpha)-L(a^\alpha,b^\alpha)\right]&\leq A_p(a^\alpha,b^\alpha)-G_p^\alpha(a,b)\\ &\leq 2\left[A(a^\alpha,b^\alpha)-L(a^\alpha,b^\alpha)\right],\end{aligned}$$

for any $\alpha\in\mathbb{R}$ with $\alpha\neq 0$ and $p\in[0,1]$.

For $\alpha = 1$ we have

$$2\frac{p(1-p)}{p^2-p+1}\left[A(a,b)-L(a,b)\right] \le A_p(a,b)-G_p(a,b) \le 2\left[A(a,b)-L(a,b)\right], \tag{3.13}$$

for any $a,\ b>0$ and $p\in[0,1]$.

If we take $\alpha=-1$ in (3.12), then we get

$$2\frac{p(1-p)}{p^2-p+1}\frac{G^2(a,b)-H(a,b)L(a,b)}{H(a,b)G^2(a,b)} \le \frac{G_p(a,b)-H_p(a,b)}{G_p(a,b)H_p(a,b)} \le 2\frac{G^2(a,b)-H(a,b)L(a,b)}{H(a,b)G^2(a,b)}, \tag{3.14}$$

for any $a,\ b>0$ and $p\in[0,1]$.

If we take in (3.12) $\alpha=2$, then we get

$$\begin{aligned} &2\frac{p(1-p)}{p^2-p+1}\left[A\left(a^2,b^2\right)-A(a,b)L(a,b)\right] \\ &\le A_p\left(a^2,b^2\right)-G_p^2(a,b) \\ &\le 2\left[A\left(a^2,b^2\right)-A(a,b)L(a,b)\right], \end{aligned} \tag{3.15}$$

for any $a,\ b>0$ and $p\in[0,1]$.

If we take in (3.6) $f(x)=-\ln x$, then we get

$$\begin{aligned} &2\frac{p(1-p)}{p^2-p+1}\left[\frac{1}{b-a}\int_a^b \ln s ds-\ln(G(a,b))\right] \\ &\le \ln A_p(a,b)-\ln(G_p(a,b)) \\ &\le 2\left[\frac{1}{b-a}\int_a^b \ln s ds-\ln(G(a,b))\right], \end{aligned} \tag{3.16}$$

and since

$$\frac{1}{b-a}\int_a^b \ln s ds=\ln I(a,b),$$

then by (3.16) we get

$$\left(\frac{I(a,b)}{G(a,b)}\right)^{2\frac{p(1-p)}{p^2-p+1}} \le \frac{A_p(a,b)}{G_p(a,b)} \le \left(\frac{I(a,b)}{G(a,b)}\right)^2, \tag{3.17}$$

for any $a,\ b>0$ and $p\in[0,1]$.

We observe that the inequality (2.8) provides the following upper bound

$$\max\{p,1-p\}\left(\sqrt{b}-\sqrt{a}\right)^2$$

for the quantity

$$A_p(a,b)-G_p(a,b)$$

while the inequality (3.13) provides the upper bound

$$2\left[A(a,b)-L(a,b)\right]$$

where $a,\ b>0$ and $p\in[0,1]$.

Let $a = 1$, $b = x > 0$ and $y = p \in [0, 1]$ in the above and consider the difference function

$$D_1(x, y) := \max\{y, 1-y\} \left(\sqrt{x} - 1\right)^2 - 2\left(\frac{x+1}{2} - \frac{x-1}{\ln x}\right).$$

The plot of the function $D_1(x, y)$ in the box $[0, 2] \times [0, 1]$, see Figure 1 below, shows that it takes both positive and negative values meaning that neither of the upper bounds is always best.

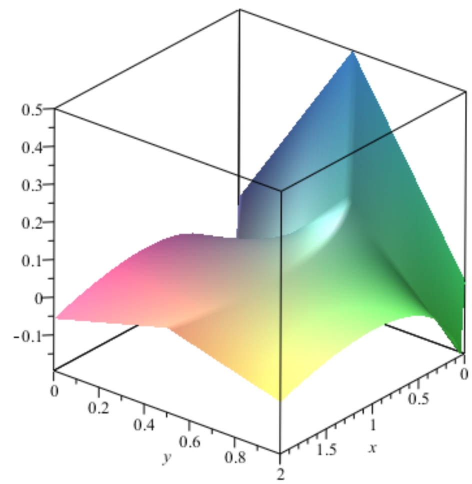


FIGURE 1. Plot of $D_1(x, y)$ in the box $[0, 2] \times [0, 1]$

We also observe that the inequality (2.11) provides the following upper bound

$$\left(\frac{A(a, b)}{G(a, b)}\right)^{2\max\{p, 1-p\}}$$

for the quantity

$$\frac{A_p(a, b)}{G_p(a, b)}$$

while the inequality (3.17) provides the upper bound

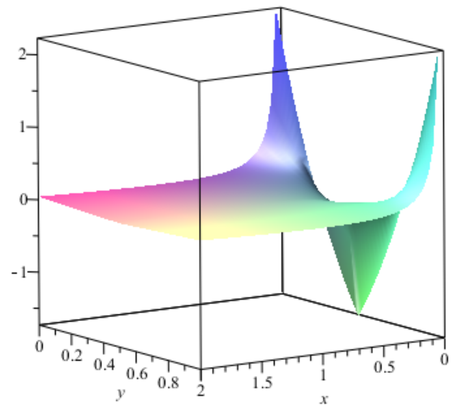
$$\left(\frac{I(a, b)}{G(a, b)}\right)^2$$

where $a, b > 0$ and $p \in [0, 1]$.

Let $a = 1$, $b = x > 0$ and $y = p \in [0, 1]$ in the above and consider the difference function

$$D_2(x, y) := \left(\frac{x+1}{2\sqrt{x}}\right)^{2\max\{y, 1-y\}} - \left(\frac{x^{\frac{x}{x-1}}}{e\sqrt{x}}\right)^2$$

The plot of the function $D_1(x, y)$ in the box $[0, 2] \times [0, 1]$, see Figure 2 below, shows that it takes both positive and negative values meaning that neither of the upper bounds is always best.



Plot of $D_2(x,y)$ in the box $[0,2]\times[0,1]$

4. Midpoint Type Integral Inequalities

We have:

Theorem 2. *Let X be a linear space and $C\subseteq X$ a convex subset in X, then for any convex function $f:C\to\mathbb{R}$ and $x,\ y\in C$ we have*

$$\begin{aligned}(4.1)\qquad &\frac{1}{2q(1-q)}\min\{1-q,q\}\\ &\times\left[\int_0^1 f((1-t)x+ty)\,dt-\frac{1}{1-2q}\int_q^{1-q}f((1-s)x+sy)\,ds\right]\\ &\le\int_0^1 f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\\ &\le\frac{1}{2q(1-q)}\max\{1-q,q\}\\ &\times\left[\int_0^1 f((1-t)x+ty)\,dt-\frac{1}{1-2q}\int_q^{1-q}f((1-s)x+sy)\,ds\right]\end{aligned}$$

or, equivalently

$$\begin{aligned}(4.2)\qquad &\frac{2q(1-q)}{\max\{1-q,q\}}\left[\int_0^1 f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right]\\ &\le\int_0^1 f((1-t)x+ty)\,dt-\frac{1}{1-2q}\int_q^{1-q}f((1-s)x+sy)\,ds\\ &\le\frac{2q(1-q)}{\min\{1-q,q\}}\left[\int_0^1 f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right]\end{aligned}$$

for any $q\in(0,1)$, $q\neq\frac{1}{2}$.

Proof. If we take in (2.2) $p=\frac{1}{2}$, then we have

$$\begin{aligned}\frac{1}{2q(1-q)}&\min\{1-q,q\}[(1-q)f(x)+qf(y)-f((1-q)x+qy)]\\&\le\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}[(1-q)f(x)+qf(y)-f((1-q)x+qy)]\end{aligned}\tag{4.3}$$

for any $x,\,y\in C$ and $q\in(0,1)$.

If we replace x by $(1-t)x+ty$ and y by $tx+(1-t)y$ in (4.3), then we get

$$\begin{aligned}&\frac{1}{2q(1-q)}\min\{1-q,q\}\\&\times[(1-q)f((1-t)x+ty)+qf(tx+(1-t)y)\\&-f((1-q)[(1-t)x+yt]+q[tx+(1-t)y])]\\&\le\left[\frac{f((1-t)x+ty)+f(tx+(1-t)y)}{2}-f\left(\frac{x+y}{2}\right)\right]\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}\\&\times[(1-q)f((1-t)x+ty)+qf(tx+(1-t)y)\\&-f((1-q)[(1-t)x+yt]+q[tx+(1-t)y])]\end{aligned}\tag{4.4}$$

for any $x,\,y\in C$, $t\in[0,1]$ and $q\in(0,1)$.

If we take the integral over $t\in[0,1]$ in (4.4) and take into account that

$$\int_0^1 f((1-t)x+ty)\,dt=\int_0^1 f(tx+(1-t)y)\,dt$$

we get

$$\begin{aligned}&\frac{1}{2q(1-q)}\min\{1-q,q\}\left[\int_0^1 f((1-t)x+ty)\,dt\right.\\&\left.-\int_0^1 f((1-q)[(1-t)x+ty]+q[tx+(1-t)y])\,dt\right]\\&\le\int_0^1 f((1-t)x+ty)\,dt-f\left(\frac{x+y}{2}\right)\\&\le\frac{1}{2q(1-q)}\max\{1-q,q\}\left[\int_0^1 f((1-t)x+ty)\,dt\right.\\&\left.-\int_0^1 f((1-q)[(1-t)x+ty]+q[tx+(1-t)y])\,dt\right]\end{aligned}\tag{4.5}$$

or any $x,\,y\in C$ and $q\in(0,1)$.

Observe that for any $x,\,y\in C$, $t\in[0,1]$ and $q\in(0,1)$ we have

$$\begin{aligned}&(1-q)[(1-t)x+ty]+q[tx+(1-t)y]\\&=[(1-q)(1-t)+qt]x+[(1-q)t+(1-t)q]y\end{aligned}$$

and by putting $s:=(1-q)t+(1-t)q$, for $q\ne\frac{1}{2}$ we have

$$[(1-q)(1-t)+qt]x+[(1-q)t+(1-t)q]y=(1-s)x+sy.$$

If $q \neq \frac{1}{2}$, then s is a change of variable, $ds = (1-2q)\,dt$ and we have for any x, $y \in C$ that

$$\int_0^1 f\left((1-q)\left[(1-t)\,x+ty\right]+q\left[tx+(1-t)\,y\right]\right)dt$$
$$=\frac{1}{1-2q}\int_q^{1-q} f\left((1-s)\,x+sy\right)ds.$$

On making use of (4.5) we get the desired result (4.1). □

Remark 3. *If we take $q=\frac{1}{4}$ in (4.2), then we get*

$$\frac{1}{2}\left[\int_0^1 f\left((1-t)\,x+ty\right)dt - f\left(\frac{x+y}{2}\right)\right] \tag{4.6}$$
$$\leq \int_0^1 f\left((1-t)\,x+ty\right)dt - 2\int_{1/4}^{3/4} f\left((1-s)\,x+sy\right)ds$$
$$\leq \frac{3}{2}\left[\int_0^1 f\left((1-t)\,x+ty\right)dt - f\left(\frac{x+y}{2}\right)\right]$$

for any x, $y \in C$.

Let $(X, \|\cdot\|)$ be a real or complex normed linear space and $r \geq 1$. Then by (4.2) we have the inequalities

$$\frac{2q\,(1-q)}{\max\{1-q,q\}}\left[\int_0^1 \|(1-t)\,x+ty\|^r\,dt - \left\|\frac{x+y}{2}\right\|^r\right] \tag{4.7}$$
$$\leq \int_0^1 \|(1-t)\,x+ty\|^r\,dt - \frac{1}{1-2q}\int_q^{1-q} \|(1-s)\,x+sy\|^r\,ds$$
$$\leq \frac{2q\,(1-q)}{\min\{1-q,q\}}\left[\int_0^1 \|(1-t)\,x+ty\|^r\,dt - \left\|\frac{x+y}{2}\right\|^r\right]$$

for any x, $y \in X$.

Remark 4. *If the function f is defined on the real interval I and a, $b \in I$, $a \neq b$ then by (4.2) and (3.2) we have*

$$\frac{2q\,(1-q)}{\max\{1-q,q\}}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right] \tag{4.8}$$
$$\leq \frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{(1-2q)\,(b-a)}\int_{(1-q)a+qb}^{qa+(1-q)b} f(v)\,dv$$
$$\leq \frac{2q\,(1-q)}{\min\{1-q,q\}}\left[\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right]$$

for any $q \in (0,1)$, $q \neq \frac{1}{2}$.

If we take the convex function $f : \mathbb{R} \to (0, \infty)$, $f(x) = \exp(\alpha x)$, with $\alpha \neq 0$, in (4.8) then we have from (4.8) (for $a = x$, $b = y$) that

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q,q\}}\left[E_{\alpha}(x,y)-\exp(\alpha A(x,y))\right] \\ &\leq E_{\alpha}(x,y)-E_{\alpha}\left(A_{1-q}(x,y),A_{q}(x,y)\right) \\ &\leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[E_{\alpha}(x,y)-\exp(\alpha A(x,y))\right] \end{aligned} \tag{4.9}$$

for any x, $y \in \mathbb{R}$, $\alpha \neq 0$ and $q \in (0,1)$.

Now, if we take in (4.9) $x = \ln a$ and $y = \ln b$ for a, $b > 0$, then we get

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q,q\}}\left[L(a^{\alpha},b^{\alpha})-G^{\alpha}(a,b)\right] \\ &\leq L(a^{\alpha},b^{\alpha})-L\left(G_{1-q}^{\alpha}(a,b),G_{q}^{\alpha}(a,b)\right) \\ &\leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[L(a^{\alpha},b^{\alpha})-G^{\alpha}(a,b)\right]. \end{aligned} \tag{4.10}$$

for any $\alpha \neq 0$ and $q \in (0,1)$.

If we take $\alpha = 1$ in (4.10), then we get

$$\begin{aligned} &\frac{2q(1-q)}{\max\{1-q,q\}}\left[L(a,b)-G(a,b)\right] \\ &\leq L(a,b)-L\left(G_{1-q}(a,b),G_{q}(a,b)\right) \\ &\leq \frac{2q(1-q)}{\min\{1-q,q\}}\left[L(a,b)-G(a,b)\right], \end{aligned} \tag{4.11}$$

for any for a, $b > 0$ and $q \in (0,1)$.

If we take in (4.8) $f(x) = -\ln x$, then we obtain

$$\left(\frac{A(a,b)}{I(a,b)}\right)^{\frac{2q(1-q)}{\max\{1-q,q\}}} \leq \frac{I\left(A_q(a,b),A_{1-q}(a,b)\right)}{I(a,b)} \leq \left(\frac{A(a,b)}{I(a,b)}\right)^{\frac{2q(1-q)}{\min\{1-q,q\}}} \tag{4.12}$$

for any for a, $b > 0$ and $q \in (0,1)$.

5. Applications for Young's Operator Inequalities

Throughout this section A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_{\nu}B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean* and

$$A\sharp_{\nu}B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},$$

the *weighted operator geometric mean.* When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{5.1}$$

with equality if and only if $a = b$. The inequality (5.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [42]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{5.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu}, \tag{5.3}$$

where $a,\ b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$.

The second inequality in (5.3) is due to Tominaga [43] while the first one is due to Furuichi [21].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{5.4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu} \tag{5.5}$$

where $a,\ b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

The first inequality in (5.5) was obtained by Zou et al. in [44] while the second by Liao et al. [31].

In [12] we proved the following reverses of Young's inequality

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{5.6}$$

and

$$1 \le \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{5.7}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$, where K is Kantorovich's constant defined by (5.4).

From (3.13) we have the additive inequality

$$\begin{aligned} 2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[A(a,b) - L(a,b)\right] &\le A_\nu(a,b) - G_\nu(a,b) \\ &\le 2\left[A(a,b) - L(a,b)\right], \end{aligned} \tag{5.8}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$ and from (3.17) we have the multiplicative inequalities

$$\left(\frac{I(a,b)}{G(a,b)}\right)^{2\frac{\nu(1-\nu)}{\nu^2-\nu+1}} \le \frac{A_\nu(a,b)}{G_\nu(a,b)} \le \left(\frac{I(a,b)}{G(a,b)}\right)^2, \tag{5.9}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$.

We consider the functions $L:(0,\infty)\to(0,\infty)$ defined by

$$L(x):=\begin{cases} L(x,1) \text{ if } x\neq 1 \\ \\ 1 \text{ if } x\neq 1\end{cases}=\begin{cases} \frac{x-1}{\ln x} \text{ if } x\neq 1 \\ \\ 1 \text{ if } x\neq 1\end{cases}$$

and $I:(0,\infty)\to(0,\infty)$ defined by

$$I(x):=\begin{cases} I(x,1) \text{ if } x\neq 1 \\ \\ 1 \text{ if } x\neq 1\end{cases}=\begin{cases} \frac{x^{\frac{x}{x-1}}}{e} \text{ if } x\neq 1 \\ \\ 1 \text{ if } x\neq 1.\end{cases}$$

These functions are continuous on $(0,\infty)$.

If A, B are positive invertible operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then by making use of continuous functional calculus, we can consider the operators

$$A\flat_L B := A^{1/2}L\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$

and

$$A\flat_I B := A^{1/2}I\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

Since, by the well known inequalities for means we have

$$G(x,1)\le L(x,1)\le I(x,1)\le A(x,1)$$

for any $x>0$, we conclude that the following operator inequalities also hold

$$A\sharp B\le A\flat_L B\le A\flat_I B\le A\nabla B \tag{5.10}$$

for any A, B positive invertible operators.

Proposition 1. *For for any A, B positive invertible operators we have*

$$2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[A\nabla B-A\flat_L B\right]\le A\nabla_\nu B-A\sharp_\nu B\le 2\left[A\nabla B-A\flat_L B\right] \tag{5.11}$$

for any $\nu\in[0,1]$.

Proof. From the inequality (5.8) we have for any $x>0$ that

$$2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[\frac{x+1}{2}-L(x)\right]\le 1-\nu+\nu x-x^\nu\le 2\left[\frac{x+1}{2}-L(x)\right] \tag{5.12}$$

or any $\nu\in[0,1]$.

Using the functional calculus for the positive operator X we have

$$\begin{aligned}2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[\frac{X+1_H}{2}-L(X)\right]&\le(1-\nu)1_H+\nu X-X^\nu \\ &\le 2\left[\frac{X+1_H}{2}-L(X)\right],\end{aligned} \tag{5.13}$$

for any $\nu\in[0,1]$.

Now, if we take $X=A^{-1/2}BA^{-1/2}$ in (5.13), then we get

$$\begin{aligned}&2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\left[\frac{A^{-1/2}BA^{-1/2}+1_H}{2}-L\left(A^{-1/2}BA^{-1/2}\right)\right] \\ &\le(1-\nu)1_H+\nu A^{-1/2}BA^{-1/2}-\left(A^{-1/2}BA^{-1/2}\right)^\nu \\ &\le 2\left[\frac{A^{-1/2}BA^{-1/2}+1_H}{2}-L\left(A^{-1/2}BA^{-1/2}\right)\right],\end{aligned} \tag{5.14}$$

for any $\nu \in [0,1]$.

By multiplying both sides of (5.14) with $A^{1/2}$ we deduce the desired result (5.11). $\square$

We consider the functions Υ, $\digamma : (0,\infty) \to (0,\infty)$ by

$$\Upsilon(t) := \frac{t+1}{2} - L(t) = \begin{cases} \frac{t+1}{2} - \frac{t-1}{\ln t} \text{ if } t \neq 1 \\ \\ 0 \text{ if } t = 1 \end{cases} \tag{5.15}$$

and

$$\digamma(t) := \left(\frac{I(t)}{\sqrt{t}}\right)^2 = \begin{cases} \left(\frac{t^{\frac{t}{t-1}}}{e\sqrt{t}}\right)^2 \text{ if } t \neq 1 \\ \\ 1 \text{ if } t = 1. \end{cases} \tag{5.16}$$

After conducting some numerical experiments, we can consider the following conjecture for which we do not have yet an analytic proof:

Conjecture 1. *The functions Υ and $\digamma$ are strictly decreasing on $(0,1)$, strictly increasing on $(1,\infty)$ and strictly convex on $(0,\infty)$. The global minimum for Υ is 0 reached for $t=1$ and the global minimum of $\digamma$ is 1 reached for $t=1$.*

Finally, we have the following result.

Proposition 2. *Let A, B positive invertible operators such that*

$$mA \le B \le MA \tag{5.17}$$

for some constants m, M with $M > m > 0$. Then we have

$$2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\inf_{t\in[m,M]}\Upsilon(t)A \le A\nabla_\nu B - A\sharp_\nu B \le 2\sup_{t\in[m,M]}\Upsilon(t)A \tag{5.18}$$

and

$$\left(\inf_{t\in[m,M]}\digamma(t)\right)^{\frac{\nu(1-\nu)}{\nu^2-\nu+1}} A\sharp_\nu B \le A\nabla_\nu B \le \sup_{t\in[m,M]}\digamma(t)A\sharp_\nu B. \tag{5.19}$$

Proof. From the inequality on (5.12) we have

$$\begin{aligned} 2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\inf_{t\in[m,M]}\left[\frac{x+1}{2} - L(x)\right] &\le 1-\nu+\nu x - x^\nu \\ &\le 2\sup_{t\in[m,M]}\left[\frac{x+1}{2} - L(x)\right] \end{aligned} \tag{5.20}$$

for any $x \in [m,M]$.

If the condition (5.17) holds, then by multiplying both sides with $A^{-1/2}$ we get $m1_H \le A^{-1/2}BA^{-1/2} \le 1_H M$. Then by (5.20) we have

$$\begin{aligned} &2\frac{\nu(1-\nu)}{\nu^2-\nu+1}\inf_{t\in[m,M]}\Upsilon(t) \\ &\le (1-\nu)1_H + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^\nu \\ &\le 2\sup_{t\in[m,M]}\Upsilon(t) \end{aligned} \tag{5.21}$$

for any $\nu \in [0,1]$.

By multiplying both sides of this inequality with $A^{1/2}$ we deduce the desired result (5.18).

From the inequality (5.9) we have

$$(\digamma(x))^{\frac{\nu(1-\nu)}{\nu^2-\nu+1}} \le \frac{A_\nu(x,1)}{G_\nu(x,1)} \le \digamma(x),$$

for any $x > 0$ and for any $\nu \in [0,1]$.

This implies that

$$\left(\inf_{t\in[m,M]} \digamma(t)\right)^{\frac{\nu(1-\nu)}{\nu^2-\nu+1}} G_\nu(x,1) \le A_\nu(x,1) \le \sup_{t\in[m,M]} \digamma(t)\, G_\nu(x,1),$$

for any $x \in [m,M]$ and for any $\nu \in [0,1]$.

Now, on making use of a similar argument as above, we get the desired result (5.19). □

Remark 5. *If the above conjecture is true, then the bounds in Proposition 2 have a simpler form by taking into account that*

$$\inf_{t\in[m,M]} \Upsilon(t) = \begin{cases} \Upsilon(M) & \text{if } M<1 \\ 0 & \text{if } m \le 1 \le M \\ \Upsilon(m) & \text{if } 1<m, \end{cases}$$

$$\sup_{t\in[m,M]} \Upsilon(t) = \begin{cases} \Upsilon(m) & \text{if } M<1 \\ \max\{\Upsilon(m), \Upsilon(M)\} & \text{if } m \le 1 \le M \\ \Upsilon(M) & \text{if } 1<m, \end{cases}$$

$$\inf_{t\in[m,M]} \digamma(t) = \begin{cases} \digamma(M) & \text{if } M<1 \\ 1 & \text{if } m \le 1 \le M \\ \digamma(m) & \text{if } 1<m, \end{cases}$$

and

$$\sup_{t\in[m,M]} \digamma(t) = \begin{cases} \digamma(m) & \text{if } M<1 \\ \max\{\digamma(m), \digamma(M)\} & \text{if } m \le 1 \le M \\ \digamma(M) & \text{if } 1<m. \end{cases}$$

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