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SEVERAL APPLICATIONS OF YOUNG-TYPE AND HOLDER'S INEQUALITIES

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Abstract. The aim of this paper is to present several applications of several new Young-type and Holder's inequalities given by Alzer, H., Fonseca, C. M. and Kovacec, A. to isotonic linear functional, power series and inner product.

1. Introduction

The famous Young's inequality, as a classical result, state that:

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

when a and b are positive numbers, $a \neq b$ and $\nu \in (0,1)$.

In these years, there are many interesting generalizations of this well-known inequality and its reverse, see for example [11, 12, 9, 8, 1] many others and references therein.

As in [1], we consider $A_{\nu}(a,b) = \nu a + (1-\nu)b$, and $G_{\nu}(a,b) = a^{\nu}b^{1-\nu}$. The following result, given in [8] is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah, [11], [12].

Proposition 1. *For all $a, b > 0$ we have*

$$3\nu\left(A_{\frac{1}{3}}(a,b) - G_{\frac{1}{3}(a,b)}\right) \leq A_{\nu}(a,b) - G_{\nu}(a,b)$$

if $0 < \nu \leq \frac{1}{3}$, and

$$3\nu(1-\nu)\left(A_{\frac{2}{3}}(a,b) - G_{\frac{2}{3}(a,b)}\right) \leq A_{\nu}(a,b) - G_{\nu}(a,b)$$

if $\frac{1}{3} \leq \nu < 1$.

More recently, in [1] are given new results which extend many generalizations of Young's inequality given before. We recall these results below in order to use them in the next sections.

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Theorem 1. *Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

Theorem 2. *Let $\nu \in (0,1.)$ For all real numbers a, b with $0 < a < b$ we have*

$$\frac{\nu(1-\nu)}{2} a \log^2\left(\frac{b}{a}\right) < A_{\nu}(a,b) - G_{\nu}(a,b) < \frac{\nu(1-\nu)}{2} b \log^2\left(\frac{b}{a}\right)$$

and

$$\exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{a}{b}\right)^2\right) < \frac{A_{\nu}(a,b)}{G_{\nu}(a,b)} < \exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{b}{a}\right)^2\right).$$

In each inequality, the factor $\frac{\nu(1-\nu)}{2}$ is the best possible.

2. **The Young-type and Holder's inequalities for isotonic linear functional**

The following important definition is given in [2], [6], [3] and it is necessary to recall it here.

Definition 1. *Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:*
(L1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.
(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.
An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:
(A1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.
(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.
The mapping A is said to be normalised if
(A3) $A(\mathbf{1}) = 1$.

Lemma 1. *([10], Corollary 3.3) If f is Δ- integrable on $[a,b)$ then for an arbitrary positive number α the function $|f|^{\alpha}$ is Δ-integrable on $[a,b)$.*

Lemma 2. *([10], Theorem 3.6) Let f and g be Δ-integrable functions on $[a,b)$. then their product fg is Δ-integrable on $[a,b)$.*

The following results are applications of recent Young-type inequalities given in [1] in the case if isotonic linear functionals and also Holder-type inequalities in the same case using methods presented in [7],[6].

Also, are presented as particular cases these inequalities for the cases of the time scale Cauchy delta, Cauchy navbla, α- diamond, multiple Riemann, and multiple Lebesque integrals.

Theorem 3. *L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If $f^p,\ g^q,\ fg,\ f^p \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right),\ g^q \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right) \in L,\ A(f^p) > 0,\ A(g^q) > 0,$ f and g are positive functions and $0 < \frac{f^p}{A(f^p)} < \frac{g^q}{A(g^q)}$ then:*

$$\frac{1}{2pq}\frac{1}{A(f^p)}A\left[f^p \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right)\right] <$$
$$< 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)} <$$
$$< \frac{1}{2pq}\frac{1}{A(g^q)}A\left[g^q \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right)\right]$$

and

$$A\left(fg \exp\frac{1}{2pq}\left(1-\frac{f^p}{g^q}\frac{A(g^q)}{A(f^p)}\right)^2\right) <$$
$$< A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q) <$$
$$< A\left(fg \exp\frac{1}{2pq}\left(1-\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right)^2\right),$$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, if in this second case, instead of $f^p \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right)$, $g^q \log^2\left(\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right) \in L$ we have $fg \exp\frac{1}{2pq}\left(1-\frac{f^p}{g^q}\frac{A(g^q)}{A(f^p)}\right)^2$, $fg \exp\frac{1}{2pq}\left(1-\frac{g^q}{f^p}\frac{A(f^p)}{A(g^q)}\right)^2 \in L$.

Proof. We take into account first inequality from Theorem 2,

$$\frac{\nu(1-\nu)}{2} a \log^2\left(\frac{b}{a}\right) < A_\nu(a,b) - G_\nu(a,b) < \frac{\nu(1-\nu)}{2} b \log^2\left(\frac{b}{a}\right)$$

where we replace a by $\frac{f^p(x)}{A(f^p)}$ and b by $\frac{g^q(x)}{A(g^q)}$ and $\nu = \frac{1}{p}$.

Then we obtain:

$$\frac{1}{pq}\frac{f^p(x)}{A(f^p)}\log^2\left(\frac{g^q(x)}{f^p(x)}\frac{A(f^p)}{A(g^q)}\right) < \frac{1}{p}\frac{f^p(x)}{A(f^p)} + \frac{1}{q}\frac{g^q(x)}{A(g^q)} - \frac{fg(x)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q) <}$$
$$< \frac{1}{pq}\frac{g^q(x)}{A(g^q)}\log^2\left(\frac{g^q(x)}{f^p(x)}\frac{A(f^p)}{A(g^q)}\right)$$

for any $x \in E$.

Now we take the functional A in previous inequality and we get the desired inequality.

For the second inequality the same method will be used.

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Theorem 4. *If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If $f^p,\ g^q,\ fg,\ f^{p\tau},\ g^{q(1-\tau)} \in L$, $A(f^p) > 0,\ A(g^q) > 0$, $p\tau > 1$, $\tau < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f and g are positive functions then:*

$$\frac{1}{p\tau}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right] < 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q)} <$$
$$< \frac{1}{q(1-\tau)}\left[1 - \frac{A(f^{p\tau})A(g^{q(1-\tau)})}{A^{\tau}(f^p)A^{1-\tau}(g^q)}\right].$$

Moreover, if $A, B : L \to \mathbf{R}$ are two normalised isotonic functionals such that the conditions $f^{p\tau}g^{q(1-\tau)},\ f^{p(1-\tau)}g^{q\tau},\ f^{\frac{p}{q}}g^{\frac{q}{p}} \in L$ appear instead of $f^{p\tau},\ g^{q(1-\tau)} \in L$ then we have the following inequality:

$$\frac{1}{p\tau}[\tau A(f^p)B(g^q) + (1-\tau)A(g^q)B(f^p) - A(f^{p\tau}g^{q(1-\tau)})B(f^{p(1-\tau)}g^{q\tau})] <$$
$$< \frac{1}{p}A(f^p)B(g^q) + \frac{1}{q}A(g^q)B(f^p) - A(fg)B(f^{\frac{p}{q}}g^{\frac{q}{p}}) <$$
$$< \frac{1}{p\tau}[\tau A(f^p)B(g^q) + (1-\tau)A(g^q)B(f^p) - A(f^{p\tau}g^{q(1-\tau)})B(f^{p(1-\tau)}g^{q\tau})].$$

A particular case will be obtained when, $A = B$,

$$\frac{1}{p\tau}[A(f^p)A(g^q) - A(f^{p\tau}g^{q(1-\tau)})A(f^{p(1-\tau)}g^{q\tau})] <$$
$$< A(f^p)A(g^q) - A(fg)A(f^{\frac{p}{q}}g^{\frac{q}{p}}) <$$
$$< \frac{1}{p\tau}[A(f^p)A(g^q) - A(f^{p\tau}g^{q(1-\tau)})A(f^{p(1-\tau)}g^{q\tau})].$$

Proof. If we use in inequality from Theorem 1, $a = \frac{f^p(x)}{g^q(x)}$ and $b = \frac{f^p(y)}{g^q(y)}$ then we get

$$\frac{\nu}{\tau}\left[\tau\frac{f^p(x)}{g^q(x)} + (1-\tau)\frac{f^p(y)}{g^q(y)} - \frac{f^{p\tau}(x)}{g^{q\tau}(x)}\frac{f^{p(1-\tau)}(y)}{g^{q(1-\tau)}(y)}\right] <$$
$$< \nu\frac{f^p(x)}{g^q(x)} + (1-\nu)\frac{f^p(y)}{g^q(y)} - \frac{f^{p\nu}(x)}{g^{q\nu}(x)}\frac{f^{p(1-\nu)}(y)}{g^{q(1-\nu)}(y)} <$$
$$< \frac{1-\nu}{1-\tau}\left[\tau\frac{f^p(x)}{g^q(x)} + (1-\tau)\frac{f^p(y)}{g^q(y)} - \frac{f^{p\tau}(x)}{g^{q\tau}(x)}\frac{f^{p(1-\tau)}(y)}{g^{q(1-\tau)}(y)}\right],$$

for any $x, y \in E$.

We multiply last inequality by $g^q(x)g^q(y) > 0$ and we get

$$\frac{\nu}{\tau}\left[\tau f^p(x)g^q(y) + (1-\tau)f^p(y)g^q(x) - \frac{f^{p\tau}(x)}{g^{q\tau}(x)}\frac{f^{p(1-\tau)}(y)}{g^{q(1-\tau)}(y)}g^q(x)g^q(y)\right] <$$
$$< \nu f^p(x)g^q(y) + (1-\nu)f^p(y)g^q(x) - \frac{f^{p\nu}(x)}{g^{q\nu}(x)}\frac{f^{p(1-\nu)}(y)}{g^{q(1-\nu)}(y)}g^q(x)g^q(y) <$$
$$< \frac{1-\nu}{1-\tau}\left[\tau f^p(x)g^q(y) + (1-\tau)f^p(y)g^q(x) - \frac{f^{p\tau}(x)}{g^{q\tau}(x)}\frac{f^{p(1-\tau)}(y)}{g^{q(1-\tau)}(y)}g^q(x)g^q(y)\right],$$

for any $x, y \in E$.

Fix $y \in E$ as in [7] and we have in the order of L that

$$\frac{\nu}{\tau}\left[\tau f^p g^q(y) + (1-\tau)f^p(y)g^q - f^{p\tau}f^{p(1-\tau)}(y)g^{q(1-\tau)}g^{q\tau}(y)\right] <$$

$$< \nu f^p g^q(y) + (1-\nu) f^p(y) g^q - f^{p\nu} f^{p(1-\nu)}(y) g^{q(1-\nu)} g^{q\nu}(y) <$$

$$< \frac{1-\nu}{1-\tau}\left[\tau f^p g^q(y) + (1-\tau) f^p(y) g^q - f^{p\tau} f^{p(1-\tau)}(y) g^{q(1-\tau)} g^{q\tau}(y)\right],$$

If we take now the functional A in previous inequality, we have,

$$\frac{1}{p\tau}\left[\tau A(f^p) g^q(y) + (1-\tau) f^p(y) A(g^q) - A(f^{p\tau} g^{q(1-\tau)}) f^{p(1-\tau)}(y) g^{q\tau}(y)\right] <$$

$$< \frac{1}{p} A(f^p) g^q(y) + \frac{1}{q} f^p(y) A(g^q) - A(fg) f^{\frac{p}{q}}(y) g^{\frac{q}{p}}(y) <$$

$$< \frac{1}{q(1-\tau)}\left[\tau A(f^p) g^q(y) + (1-\tau) f^p(y) A(g^q) - A(f^{p\tau} g^{q(1-\tau)}) f^{p(1-\tau)}(y) g^{q\tau}(y)\right],$$

for any $y \in E$. Then we write this inequality in the order of L and then we take the functional B in this inequality obtaining the desired result.

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Theorem 5. *Let* $p\tau > 1,\ \tau < 1$ *with* $p > 0, \tau > 0$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* L *satisfy conditions L1, L2 and A satisfy A1, A2 on the set* E. *If* $f^p,\ g^q,\ fg,\ f^{p\tau} g^{q(1-\tau)},\ f^{\frac{pk}{n}} g^{q(1-\frac{k}{n})} \in L,\ k = \overline{0,n},\ A(f^p) > 0,\ A(g^q) > 0$ *and* f *and* g *are positive functions then:*

$$\frac{1}{(p\tau)^n}\left[A\left(\left(\tau \frac{f^{\frac{p}{n}}}{A^{\frac{1}{n}}(f^p)} + (1-\tau)\frac{g^{\frac{q}{n}}}{A^{\frac{1}{n}}(g^q)}\right)^n\right) - \frac{A(f^{p\tau} g^{q(1-\tau)})}{A^{\tau}(f^p) A^{1-\tau}(g^q)}\right] <$$

$$< A\left[\left(\frac{1}{p}\frac{f^{\frac{p}{n}}}{A^{\frac{1}{n}}(f^p)} + \frac{1}{q}\frac{g^{\frac{q}{n}}}{A^{\frac{1}{n}}(g^q)}\right)^n\right] - \frac{A(fg)}{A^{\frac{1}{p}}(f^p) A^{\frac{1}{q}}(g^q)} <$$

$$< \frac{1}{(q(1-\tau))^n}\left[A\left(\left(\tau \frac{f^{\frac{p}{n}}}{A^{\frac{1}{n}}(f^p)} + (1-\tau)\frac{g^{\frac{q}{n}}}{A^{\frac{1}{n}}(g^q)}\right)^n\right) - \frac{A(f^{p\tau} g^{q(1-\tau)})}{A^{\tau}(f^p) A^{1-\tau}(g^q)}\right]$$

or

$$\frac{1}{(p\tau)^n}\left[\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\tau^k (1-\tau)^{n-k}}{A^{\frac{k}{n}}(f^p) A^{1-\frac{k}{n}}(g^q)} A\left(f^{\frac{pk}{n}} g^{q(1-\frac{k}{n})}\right) - \frac{A(f^{p\tau} g^{q(1-\tau)})}{A^{\tau}(f^p) A^{1-\tau}(g^q)}\right] <$$

$$< \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{1}{p^k q^{n-k}}\frac{A(f^{\frac{pk}{n}} g^{q(1-\frac{k}{n})})}{A^{\frac{k}{n}}(f^p) A^{1-\frac{k}{n}}(g^q)} - \frac{A(fg)}{A^{\frac{1}{p}}(f^p) A^{\frac{1}{q}}(g^q)} <$$

$$< \frac{1}{(q(1-\tau))^n}\left[\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\tau^k (1-\tau)^{n-k}}{A^{\frac{k}{n}}(f^p) A^{1-\frac{k}{n}}(g^q)} A\left(f^{\frac{pk}{n}} g^{q(1-\frac{k}{n})}\right) - \frac{A(f^{p\tau} g^{q(1-\tau)})}{A^{\tau}(f^p) A^{1-\tau}(g^q)}\right]$$

Proof. The proof will be as in Theorem 3 and Theorem 4.

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Theorem 6. *Let $p\tau > 1,\ \tau < 1$ with $p > 0, \tau > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If $f^p,\ g^q,\ fg,\ f^{p\tau} g^{q(1-\tau)}$, $f^{p(1-\frac{k}{n})} g^{q\frac{k}{n}},\ f^{p(1-\tau)} g^{q\tau},\ f^{\frac{pk}{n}} g^{q(1-\frac{k}{n})} \in L,\ \ k = \overline{0,n},\ A(f^p) > 0,\ A(g^q) > 0$ and f and g are positive functions then we have:*

$$\left(\frac{\nu}{\tau}\right)^n [\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \tau^{n-k}(1-\tau)^k A(f^{p(1-\frac{k}{n})} g^{q\frac{k}{n}}) B(f^{p\frac{k}{n}} g^{q(1-\frac{k}{n})}) - A(f^{p\tau} g^{q(1-\tau)}) B(f^{p(1-\tau)} g^{q\tau})] <$$

$$< \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \frac{1}{p^{n-k} g^k} A(f^{p(1-\frac{k}{n})} g^{q\frac{k}{n}}) B(f^{p\frac{k}{n}} g^{q(1-\frac{k}{n})}) - A(fg) B(f^{\frac{p}{q}} g^{\frac{q}{p}}) <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n [\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) \tau^{n-k}(1-\tau)^k A(f^{p(1-\frac{k}{n})} g^{q\frac{k}{n}}) B(f^{p\frac{k}{n}} g^{q(1-\frac{k}{n})}) -$$

$$- A(f^{p\tau} g^{q(1-\tau)}) B(f^{p(1-\tau)} g^{q\tau})]$$

In [8] Feng improved the left-hand side of the Kittaneh-Manasrah inequality [11, 12].

Proposition 2. *If L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E and if $f^p,\ g^q,\ fg \in L,\ A(f^p) > 0,\ A(g^q) > 0,\ f$ and g are positive functions and $\frac{1}{p} + \frac{1}{q} = 1$ then:*

(i) If $p \geq 3$ and $f^{\frac{p}{3}} g^{\frac{2q}{3}} \in L$

$$\frac{3}{p}\left(1 - \frac{A(f^{\frac{p}{3}} g^{\frac{2}{3}q})}{A^{\frac{1}{3}}(f^p) A^{\frac{2}{3}}(g^q)}\right) \leq 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p) A^{\frac{1}{q}}(g^q)}.$$

(ii) If $1 < p \leq \frac{3}{2}$ and $f^{\frac{2p}{3}} g^{\frac{q}{3}} \in L$

$$\frac{3}{q}\left(1 - \frac{A(f^{2\frac{p}{3}} g^{\frac{q}{3}})}{A^{\frac{2}{3}}(f^p) A^{\frac{1}{3}}(g^q)}\right) \leq 1 - \frac{A(fg)}{A^{\frac{1}{p}}(f^p) A^{\frac{1}{q}}(g^q)}.$$

(iii) If $p > 3$ and $f^{\frac{p}{3}} g^{\frac{2q}{3}},\ f^{\frac{2p}{3}} g^{\frac{q}{3}},\ f^{\frac{p}{q}} g^{\frac{q}{p}} \in L$ then we have:

$$\left(\frac{2}{p} - \frac{1}{q}\right) A(g^q) B(f^p) \leq \frac{3}{p} A(f^{\frac{p}{3}} g^{\frac{2q}{3}}) B(f^{\frac{2p}{3}} g^{\frac{q}{3}}) - A(fg) B(f^{\frac{p}{q}} g^{\frac{q}{p}}).$$

Moreover, if $A = B$ then previous inequality becomes:

$$\left(\frac{2}{p} - \frac{1}{q}\right) A(g^q) A(f^p) \leq \frac{3}{p} A(f^{\frac{p}{3}} g^{\frac{2q}{3}}) A(f^{\frac{2p}{3}} g^{\frac{q}{3}}) - A(fg) A(f^{\frac{p}{q}} g^{\frac{q}{p}})$$

(iv) If $1 < p \leq \frac{3}{2}$ and $g^{\frac{q}{3}} f^{\frac{2p}{3}},\ g^{\frac{2q}{3}} f^{\frac{p}{3}},\ f^{\frac{p}{q}} g^{\frac{q}{p}} \in L$ then we have:

$$\left(\frac{2}{q} - \frac{1}{p}\right) B(g^q) A(f^p) \leq \frac{3}{q} A(g^{\frac{q}{3}} f^{\frac{2p}{3}}) B(g^{\frac{2q}{3}} f^{\frac{p}{3}}) - A(fg) B(f^{\frac{p}{q}} g^{\frac{q}{p}}).$$

Moreover, if $A = B$ then previous inequality becomes:

$$\left(\frac{2}{q} - \frac{1}{p}\right) A(g^q) A(f^p) \leq \frac{3}{q} A(g^{\frac{q}{3}} f^{\frac{2p}{3}}) A(g^{\frac{2q}{3}} f^{\frac{p}{3}}) - A(fg) A(f^{\frac{p}{q}} g^{\frac{q}{p}}).$$

Consequence 1. *Theorem 4, 5 and Proposition 2 can be rewritten for the time scale Cauchy delta, Cauchy nabla, α-diamond, multiple Riemann, and multiple Lebesque integrals.*

Remark 1. *For example, first inequality of Theorem 4 can be rewritten like below:*
(i) Let $f, g \in C_{rd}([a,b], \mathbf{R})$ be positive functions,

$$\frac{1}{p\tau}\left[1-\frac{\int_a^b f^{p\tau}(x)\Delta x \int_a^b g^{q(1-\tau)}(x)\Delta x}{\left(\int_a^b f^p(x)\Delta x\right)^{\tau}\left(\int_a^b g^q(x)\Delta x\right)^{1-\tau}}\right] < 1-\frac{\int_a^b f(x)g(x)\Delta x}{\left(\int_a^b f^p(x)\Delta x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\Delta x\right)^{\frac{1}{q}}} <$$

$$< \frac{1}{q(1-\tau)}\left[1-\frac{\int_a^b f^{p\tau}(x)\Delta x \int_a^b g^{q(1-\tau)}(x)\Delta x}{\left(\int_a^b f^p(x)\Delta x\right)^{\tau}\left(\int_a^b g^q(x)\Delta x\right)^{1-\tau}}\right],$$

where p, τ and q as in Theorem 4.
(ii) Let $f, g \in C_{ld}([a,b], \mathbf{R})$ be positive functions,

$$\frac{1}{p\tau}\left[1-\frac{\int_a^b f^{p\tau}(x)\nabla x \int_a^b g^{q(1-\tau)}(x)\nabla x}{\left(\int_a^b f^p(x)\nabla x\right)^{\tau}\left(\int_a^b g^q(x)\nabla x\right)^{1-\tau}}\right] < 1-\frac{\int_a^b f(x)g(x)\nabla x}{\left(\int_a^b f^p(x)\nabla x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)\nabla x\right)^{\frac{1}{q}}} <$$

$$< \frac{1}{q(1-\tau)}\left[1-\frac{\int_a^b f^{p\tau}(x)\nabla x \int_a^b g^{q(1-\tau)}(x)\nabla x}{\left(\int_a^b f^p(x)\nabla x\right)^{\tau}\left(\int_a^b g^q(x)\nabla x\right)^{1-\tau}}\right],$$

where p, τ and q as in Theorem 4.
(iii) Let $f, g : [a,b] \to \mathbf{R}$ be $\diamond_\alpha$-integrable and positive functions

$$\frac{1}{p\tau}\left[1-\frac{\int_a^b f^{p\tau}(x)) \diamond_\alpha x \int_a^b g^{q(1-\tau)}(x)) \diamond_\alpha x}{\left(\int_a^b f^p(x)) \diamond_\alpha x\right)^{\tau}\left(\int_a^b g^q(x)) \diamond_\alpha x\right)^{1-\tau}}\right] <$$

$$< 1-\frac{\int_a^b f(x)g(x)) \diamond_\alpha x}{\left(\int_a^b f^p(x)) \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b g^q(x)) \diamond_\alpha x\right)^{\frac{1}{q}}} <$$

$$< \frac{1}{q(1-\tau)}\left[1-\frac{\int_a^b f^{p\tau}(x)) \diamond_\alpha x \int_a^b g^{q(1-\tau)}(x)) \diamond_\alpha x}{\left(\int_a^b f^p(x)) \diamond_\alpha x\right)^{\tau}\left(\int_a^b g^q(x)) \diamond_\alpha x\right)^{1-\tau}}\right],$$

where p, τ and q as in Theorem 4.

3. **The Young-type inequalities for inner product**

First, as in [6], it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(H; < .,. >)$. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$
Using this notation, as in [6] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{1}$$

in the operator order of $B(H)$.

Using the definition from [6], we say that the functions $f, g : [a, b] \to \mathbf{R}$ are *syncronous* (*asyncronous*) on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq) 0$$

for each $t, s \in [a, b]$.

Consequence 2. *Let $A \in B(H)$ and $B \in B(H)$ be two selfadjoint operators on H and $\nu \in (0, 1)$.*
(i) If $0 < \nu \leq \frac{1}{3}$ then we have

$$3\nu \left[\frac{1}{3} < Ax, x >< y, y > + \frac{2}{3} < x, x >< By, y > - < A^{\frac{1}{3}} x, x >< B^{\frac{2}{3}} y, y > \right] \leq$$
$$\leq \nu < Ax, x >< y, y > + (1 - \nu) < x, x >< By, y > - < A^{\nu} x, x >< B^{1-\nu} y, y >$$

(ii) If $\frac{3}{2} \leq \nu < 1$ then we obtain

$$3(1-\nu) \left[\frac{2}{3} < Ax, x >< y, y > + \frac{1}{3} < x, x >< By, y > - < A^{\frac{2}{3}} x, x >< B^{\frac{1}{3}} y, y > \right] \leq$$
$$\leq \nu < Ax, x >< y, y > + (1 - \nu) < x, x >< By, y > - < A^{\nu} x, x >< B^{1-\nu} y, y >$$

(iii) If $\lambda = 1$ and $0 < \nu < \tau < 1$ then we have

$$\frac{\nu}{\tau} [\tau < Ax, x >< y, y > + (1 - \tau) < x, x >< By, y > - < A^{\tau} x, x >< B^{1-\tau} y, y >] <$$
$$< \nu < Ax, x >< y, y > + (1 - \nu) < x, x >< By, y > - < A^{\nu} x, x >< B^{1-\nu} y, y ><$$
$$< \frac{1-\nu}{1-\tau} [\tau < Ax, x >< y, y > + (1-\tau) < x, x >< By, y > - < A^{\tau} x, x >< B^{1-\tau} y, y >]$$

for each $x, y \in H$.
(iv) Moreover, when $B \in B^*(K)$ is an selfadjoint operator on pseudo-Hilbert space K, see [4] and [5] , instead of a selfadjoint operator on H and $\lambda = 1$ and $0 < \nu < \tau < 1$ then we have

$$\frac{\nu}{\tau} [\tau < Ax, x > [y, y] + (1 - \tau) < x, x > [By, y] - < A^{\tau} x, x > [B^{1-\tau} y, y]] <$$
$$< \nu < Ax, x > [y, y] + (1 - \nu) < x, x > [By, y] - < A^{\nu} x, x > [B^{1-\nu} y, y] <$$

$$< \frac{1-\nu}{1-\tau}[\tau < Ax, x > [y, y] + (1-\tau) < x, x > [By, y]- < A^{\tau}x, x[< B^{1-\tau}y, y]]$$

for each $x \in H$ and $y \in K$.

Proof. The demonstration will be as in [6] and we prove only (i).

We fix a and apply the property (1) for the first inequality from Proposition 1, obtaining:

$$3\nu[\frac{1}{3}A + \frac{2}{3}Ib - A^{\frac{1}{3}}b^{\frac{2}{3}}] \leq \nu A + (1-\nu)Ib - A^{\nu}b^{(1-\nu)}$$

for each $b \in \mathbf{R}$. Then we obtain for each $x \in H$ the following inequality:

$$3\nu[\frac{1}{3} < Ax, x > + \frac{2}{3}b < x, x > - < A^{\frac{1}{3}}x, x > b^{\frac{2}{3}}] \leq$$

$$\leq \nu < Ax, x > + (1-\nu)b < x, x > - < A^{\nu}x, x > b^{1-\nu}$$

and now we will apply again the property (1), this time for b and we get

$$3\nu[\frac{1}{3} < Ax, x > I + \frac{2}{3}B < x, x > - < A^{\frac{1}{3}}x, x > B^{\frac{2}{3}}] \leq$$

$$\leq \nu < Ax, x > I + (1-\nu)B < x, x > - < A^{\nu}x, x > B^{1-\nu}$$

which is equivalent to:

$$3\nu[\frac{1}{3} < Ax, x >< y, y > + \frac{2}{3} < By, y >< x, x > - < A^{\frac{1}{3}}x, x >< B^{\frac{2}{3}}y, y >] \leq$$

$$\leq \nu < Ax, x >< y, y > + (1-\nu) < By, y >< x, x > - < A^{\nu}x, x >< B^{1-\nu}y, y > .$$

∎

Remark 2. *Let $A \in B(H)$ and $B \in B(H)$ be two selfadjoint operators on H and $\nu \in (0, 1)$.*

(i) If we consider in Proposition 2 the elements $x, y \in H$ with $||x|| = 1$ and $||y|| = 1$ then we obtain the following inequalities:

$$\frac{\nu(1-\nu)}{2}[< Ax, x >< \log^2 By, y > + < A\log^2 Ax, x > -$$

$$-2 < A\log Ax, x >< \log By, y >] <$$

$$< \nu < Ax, x > + (1-\nu) < By, y > - < A^{\nu}x, x >< B^{1-\nu}y, y ><$$

$$< \frac{\nu(1-\nu)}{2}[< B\log^2 By, y > + < \log^2 Ax, x >< By, y > -$$

$$-2 < \log Ax, x >< B\log By, y >],$$

(ii) If we consider in Consequence 2 the elements $x, y \in H$ with $||x|| = 1$ and $||y|| = 1$ then we obtain the following inequalities:

(a) If $p \geq 3$ then

$$3\nu\left[\frac{1}{3} < Ax, x > + \frac{2}{3} < By, y > - < A^{\frac{1}{3}}x, x >< B^{\frac{2}{3}}y, y >\right] \leq$$

$$\leq \nu < Ax, x > + (1-\nu) < By, y > - < A^{\nu}x, x >< B^{1-\nu}y, y > .$$

(b) If $1 < p \leq \frac{3}{2}$ then

$$3(1-\nu)\left[\frac{2}{3} < Ax, x > + \frac{1}{3} < By, y > - < A^{\frac{2}{3}}x, x >< B^{\frac{1}{3}}y, y >\right] \leq$$

$$\leq \nu < Ax, x > + (1-\nu) < By, y > - < A^{\nu}x, x >< B^{1-\nu}y, y > .$$

(c) If $\lambda = 1$ *and* $0 < \nu < \tau < 1$ *then*

$$\frac{\nu}{\tau}[\tau < Ax, x > +(1-\tau) < By, y > - < A^{\tau}x, x >< B^{1-\tau}y, y >] <$$
$$< \nu < Ax, x > +(1-\nu) < By, y > - < A^{\nu}x, x >< B^{1-\nu}y, y ><$$
$$< \frac{1-\nu}{1-\tau}[\tau < Ax, x > +(1-\tau) < By, y > - < A^{\tau}x, x >< B^{1-\tau}y, y >].$$

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