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SOME ASYMMETRIC REVERSES OF YOUNG'S SCALAR AND OPERATOR INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper we obtain some new reverses of Young's scalar and operator inequalities. They are compared with some previous results due to Kittaneh-Manasrah, Furuichi-Tominaga, Zou et al. and Liao et al. Applications related to the Heinz mean for positive operators are given as well.

1. Introduction

We consider the *weighted arithmetic mean, geometric mean and harmonic mean* defined as

$$A_\nu(a,b) := (1-\nu)a + \nu b,\ G_\nu(a,b) := a^{1-\nu}b^\nu$$

and

$$\begin{aligned} H_\nu(a,b) &:= A_\nu^{-1}\left(a^{-1},b^{-1}\right) = \left[(1-\nu)a^{-1} + \nu b^{-1}\right]^{-1} \\ &= \frac{ab}{(1-\nu)b + \nu a}, \end{aligned}$$

where $a,\ b>0$ and $\nu \in [0,1]$. When $\nu = \frac{1}{2}$ we denote the arithmetic mean, geometric mean and harmonic mean as $A(a,b)$, $G(a,b)$ and $H(a,b)$.

We also define *Heinz mean* as

$$\check{H}_\nu(a,b) = \frac{1}{2}\left(G_\nu(a,b) + G_{1-\nu}(a,b)\right) = A\left(G_\nu(a,b), G_{1-\nu}(a,b)\right).$$

The famous *Young inequality* for scalars says that if $a,\ b>0$ and $\nu \in [0,1]$, then

$$H_\nu(a,b) \le G_\nu(a,b) \le A_\nu(a,b) \tag{1.1}$$

with equality if and only if $a=b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric-harmonic mean inequality.*

We also have the fundamental inequality for Heinz mean

$$G(a,b) \le \check{H}_\nu(a,b) \le A(a,b)$$

for any $a,\ b>0$ and $\nu \in [0,1]$.

We recall that *Specht's ratio* is defined by [19]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

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It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right) G_{\nu}(a,b) \leq A_{\nu}(a,b) \leq S\left(\frac{a}{b}\right) G_{\nu}(a,b), \tag{1.3}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [20] while the first one is due to Furuichi [9].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^{r}\left(\frac{a}{b}\right) G_{\nu}(a,b) \leq A_{\nu}(a,b) \leq K^{R}\left(\frac{a}{b}\right) G_{\nu}(a,b), \tag{1.5}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

The first inequality in (1.5) was obtained by Zou et al. in [21] while the second by Liao et al. [16]. We also have shown in [3] that it can also be obtained from a more general result for convex functions from [1].

Kittaneh and Manasrah [13], [14] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \leq A_{\nu}(a,b) - G_{\nu}(a,b) \leq R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.6}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.6) to an identity. We also have shown in [2] that it can also be obtained from a more general result for convex functions from [1].

For some operator versions of (1.6) see [13] and [14]. Other recent results for operators may be found in [4]-[8].

Motivated by the above results we establish in this paper some new additive and multiplicative reverses of Young's scalar and operator inequalities. To do these, we employ some new discovered inequalities for convex functions that are important in themselves as well. Applications for Heinz mean are also provided.

2. Preliminary Facts for Convex Functions

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then

$$f'_{-}(x) \leq f'_{+}(x) \leq f'_{-}(y) \leq f'_{+}(y),$$

which shows that both f'_{-} and f'_{+} are nondecreasing functions on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi(\mathring{I}) \subseteq \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \quad \text{for any } x, a \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $f'_+, f'_- \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \tag{2.2}$$

for every $x \in \mathring{I}$. In particular, φ is a nondecreasing function. If f is differentiable convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex function on I and $\varphi \in \partial f$. Then for any $x, y \in I$ and $p \in [0,1]$ we have*

$$\begin{aligned}
&(0 \leq)\, (1-p) f(x) + p f(y) - f((1-p)x + py) \\
&\leq \begin{cases} p\left[f(y) - f(x) - \varphi(x)(y-x)\right], \\ \\ (1-p)\left[\varphi(y)(y-x) - f(y) + f(x)\right], \end{cases}
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&(0 \leq)\, \frac{f(x) + f(y)}{2} - \frac{f((1-p)x + py) + f((1-p)y + px)}{2} \\
&\leq \frac{1}{2}\min\{p, 1-p\}\left[\varphi(y) - \varphi(x)\right](y-x) \leq \frac{1}{4}\left[\varphi(y) - \varphi(x)\right](y-x).
\end{aligned} \tag{2.4}$$

Proof. By the gradient inequality (2.1) we have

$$\begin{aligned}
f((1-p)x + py) &\geq f(x) + ((1-p)x + py - a)\varphi(x) \\
&= f(x) + p(y-x)\varphi(x)
\end{aligned}$$

for any $x, y \in I$ and $p \in [0,1]$.

This is equivalent to

$$-p(y-x)\varphi(x) \geq f(x) - f((1-p)x + py)$$

and by adding in both sides $pf(y) - pf(x)$, to

$$pf(y) - pf(x) - p(y-x)\varphi(x) \geq (1-p)f(x) + pf(y) - f((1-p)x + py)$$

for any $x, y \in I$ and $p \in [0,1]$, which is equivalent to the first branch of the inequality in (2.3).

If in this inequality we replace x by y we get

$$\begin{aligned}
&pf(x) + (1-p)f(y) - f((1-p)y + px) \\
&\leq p\left[f(x) - f(y) - \varphi(y)(x-y)\right]
\end{aligned} \tag{2.5}$$

for any $x, y \in I$ and $p \in [0,1]$.

If we replace in the same first branch of (2.3) p by $1-p$ we get

$$\begin{aligned}
&pf(x) + (1-p)f(y) - f(px + (1-p)y) \\
&\leq (1-p)\left[f(y) - f(x) - \varphi(x)(y-x)\right]
\end{aligned} \tag{2.6}$$

for any $x, y \in I$ and $p \in [0,1]$.

Moreover, if we replace in the first branch of (2.3) x by y and p by $1-p$ we get the second branch in (2.3).

If we add the first branch in (2.3) with (2.5) and divide by 2, then we get

$$(0 \leq) \frac{f(x)+f(y)}{2} - \frac{f((1-p)x+py)+f((1-p)y+px)}{2} \leq \frac{1}{2}p[\varphi(y)-\varphi(x)](y-x), \tag{2.7}$$

for any $x, y \in I$ and $p \in [0,1]$.

If we replace p by $1-p$ in (2.7) we get

$$(0 \leq) \frac{f(x)+f(y)}{2} - \frac{f((1-p)x+py)+f((1-p)y+px)}{2} \leq \frac{1}{2}(1-p)[\varphi(y)-\varphi(x)](y-x). \tag{2.8}$$

By (2.7) and (2.8) we get the first part of (2.4). The second part of (2.4) is obvious. □

Remark 1. *If for a $\varphi \in \partial f$, we have*

$$B_1(f,\varphi,I) := \sup_{(x,y)\in I^2}[f(y)-f(x)-\varphi(x)(y-x)] < \infty$$

and

$$B_2(f,\varphi,I) := \sup_{(x,y)\in I^2}[\varphi(y)(y-x)+f(x)-f(y)] < \infty$$

then

$$B_3(f,\varphi,I) := \sup_{(x,y)\in I^2}[[\varphi(y)-\varphi(x)](y-x)] < \infty$$

and

$$B_3(f,\varphi,I) \leq B_1(f,\varphi,I)+B_2(f,\varphi,I).$$

Moreover, we have from (2.3) that

$$(0 \leq)(1-p)f(x)+pf(y)-f((1-p)x+py) \leq \min\{pB_1(f,\varphi,I),(1-p)B_2(f,\varphi,I)\} \tag{2.9}$$

and from

$$(0 \leq) \frac{f(x)+f(y)}{2} - \frac{f((1-p)x+py)+f((1-p)y+px)}{2} \leq \frac{1}{2}\min\{p,1-p\}B_3(f,\varphi,I). \tag{2.10}$$

for any $x, y \in I$ and $p \in [0,1]$.

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}(0 \leq) \frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right) &\leq \frac{1}{2}\min\{f(y)-f(x)-\varphi(x)(y-x),\varphi(y)(y-x)-f(y)+f(x)\} \\ &\leq \frac{1}{4}[\varphi(y)-\varphi(x)](y-x)\end{aligned} \tag{2.11}$$

and

$$(0 \leq) \frac{f(x)+f(y)}{2} - \int_0^1 f((1-p)x+py)\,dp \leq \frac{1}{8}[\varphi(y)-\varphi(x)](y-x) \tag{2.12}$$

for any $x, y \in I$.

Proof. The first inequality in (2.11) follows by (2.3) for $p=\frac{1}{2}$. The second part of this inequality follows by the fact that for $a,b>0$, $\min\{a,b\}\leq\frac{1}{2}(a+b)$.

If we integrate the inequality (2.4) over $p\in[0,1]$ we get

$$(0\leq)\frac{f(x)+f(y)}{2} -\frac{1}{2}\left[\int_0^1 f((1-p)x+py)\,dp+\int_0^1 f((1-p)y+px)\,dp\right] \leq\frac{1}{2}[\varphi(y)-\varphi(x)](y-x)\int_0^1\min\{p,1-p\}\,dp, \tag{2.13}$$

for any $x,y\in I$.

Since

$$\int_0^1 f((1-p)x+py)\,dp=\int_0^1 f((1-p)y+px)\,dp$$

and

$$\int_0^1\min\{p,1-p\}\,dp=\int_0^1\left(\frac{1}{2}-\left|p-\frac{1}{2}\right|\right)dp =\frac{1}{2}-\int_0^1\left|p-\frac{1}{2}\right|dp=\frac{1}{4}$$

then by (2.13) we get the desired result (2.12). □

Remark 2. *If the function $f:I\to R$ is differentiable convex on $\mathring{I}$ and $x,\ y\in\mathring{I}$ then φ can be replaced with f' in all the results above.*

3. Scalar Inequalities for Weighted Means

We have:

Theorem 2. *For any $a,b>0$ and $\nu\in[0,1]$ we have the inequalities*

$$(0\leq)A_\nu(a,b)-G_\nu(a,b) \leq\min\left\{\nu a\left(\frac{b}{a}-1-\ln\frac{b}{a}\right),(1-\nu)b\left(\ln\frac{b}{a}-1+\frac{a}{b}\right)\right\} \tag{3.1}$$

and

$$(0\leq)A(a,b)-\check{H}_\nu(a,b) \leq\frac{1}{2}\min\{\nu,1-\nu\}(b-a)(\ln b-\ln a)\leq\frac{1}{4}(b-a)(\ln b-\ln a). \tag{3.2}$$

Proof. If we write the inequalities (2.3) and (2.4) for the convex function $f(x)=\exp(x)$ and real numbers x,y we get

$$(0\leq)A_\nu(\exp x,\exp y)-G_\nu(\exp x,\exp y) \leq\begin{cases}\nu[\exp y-\exp x-(y-x)\exp x],\\ \\ (1-\nu)[(y-x)\exp y-\exp y+\exp x],\end{cases} \tag{3.3}$$

and

$$(0 \leq) A_\nu (\exp x, \exp y) - \check{H}_\nu (\exp x, \exp y) \leq \frac{1}{2} \min\{\nu, 1-\nu\} (\exp y - \exp x)(y-x) \leq \frac{1}{4} (\exp y - \exp x)(y-x). \tag{3.4}$$

Let $x = \ln a$ and $y = \ln b$ in (3.3) and (3.4) to get

$$(0 \leq) A_\nu (a,b) - G_\nu (a,b) \leq \begin{cases} \nu [b - a - a(\ln b - \ln a)], \\ \\ (1-\nu)[b(\ln b - \ln a) - b + a], \end{cases}$$
$$= \begin{cases} \nu a \left(\frac{b}{a} - 1 - \ln \frac{b}{a}\right), \\ \\ (1-\nu) b \left(\ln \frac{b}{a} - 1 + \frac{a}{b}\right), \end{cases}$$

and

$$(0 \leq) A(a,b) - \check{H}_\nu (a,b) \leq \frac{1}{2} \min\{\nu, 1-\nu\} (b-a)(\ln b - \ln a) \leq \frac{1}{4}(b-a)(\ln b - \ln a).$$

□

Consider the function $D : (0,\infty) \to [0,\infty)$ defined by

$$D(h) := h - 1 - \ln h. \tag{3.5}$$

We have

$$D'(h) := 1 - \frac{1}{h} = \frac{h-1}{h}, \; D''(h) = \frac{1}{h^2},$$

which shows that the function D is decreasing on $(0,1)$, increasing on $(1,\infty)$ and convex on $(0,\infty)$.

If we tale $\frac{b}{a} = h \in (0,1)$, then

$$\begin{aligned} b\left(\ln \frac{b}{a} - 1 + \frac{a}{b}\right) &= ah\left(\ln h - 1 + \frac{1}{h}\right) \\ &= ah\left(\frac{1}{h} - 1 - \ln \frac{1}{h}\right) = ahD\left(\frac{1}{h}\right) \\ &= a\tilde{D}(h) \end{aligned}$$

where

$$\tilde{D}(h) := hD\left(\frac{1}{h}\right) = h \ln h - h + 1, \; h > 0. \tag{3.6}$$

We have

$$\tilde{D}'(h) = \ln h, \; \tilde{D}'' = \frac{1}{h},$$

which shows that the function $\tilde{D}$ is decreasing on $(0,1)$, increasing on $(1,\infty)$ and convex on $(0,\infty)$.

Consider also the function $\check{D} : (0,\infty) \to [0,\infty)$ defined by

$$\check{D}(h) := (h-1)\ln h. \tag{3.7}$$

We have

$$\check{D}'(h) = \ln h + 1 - \frac{1}{h}, \; \tilde{D}'' = \frac{1}{h} + \frac{1}{h^2},$$

which shows that the function $\check{D}$ is decreasing on $(0,1)$, increasing on $(1,\infty)$ and convex on $(0,\infty)$.

With these notations we have

Corollary 2. *For any $a,b>0$ and $\nu\in[0,1]$, if we put $h:=\frac{b}{a}$, then we have we have the inequalities*

$$(0\leq)\,A_\nu(a,b)-G_\nu(a,b)\leq a\min\left\{\nu D(h),(1-\nu)\tilde{D}(h)\right\} \tag{3.8}$$

and

$$(0\leq)\,A(a,b)-\check{H}_\nu(a,b)\leq\frac{1}{2}a\min\{\nu,1-\nu\}\check{D}(h)\leq\frac{1}{4}a\check{D}(h). \tag{3.9}$$

We observe that from (1.6) we also have the inequality

$$(0\leq)\,A_\nu(a,b)-G_\nu(a,b)\leq a\max\{\nu,1-\nu\}M(h) \tag{3.10}$$

where $h=\frac{b}{a}$ and $M(h)=\left(\sqrt{h}-1\right)^2$.

For $x>0$ and $y\in[0,1]$ we consider the functions

$$B_1(x,y):=\min\left\{yD(x),(1-y)\tilde{D}(x)\right\}$$

and

$$B_2(x,y):=\max\{\nu,1-\nu\}M(x)$$

as provided by the upper bounds in the inequalities (3.8) and (3.10).

The plot of the difference $D_1(x,y):=B_1(x,y)-B_2(x,y)$ in the box $[0,2]\times[0,1]$, see Figure 1 below, shows that it takes both positive and negative values meaning that neither of the upper bounds B_1 or B_2 is always best.

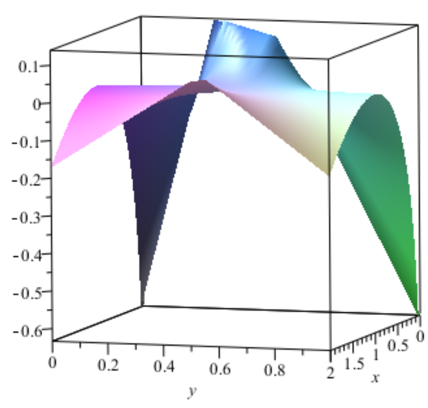


FIGURE 1. Plot of $D_1(x,y)$ in the box $[0,2]\times[0,1]$

Corollary 3. *Let $a,b>0$ with $\frac{b}{a}\in[m,M]$ for some $M>m>0$ and $\nu\in[0,1]$, then we have we have the inequalities*

$$A_\nu(a,b)-G_\nu(a,b) \le a\begin{cases}\min\left\{\nu D(m),(1-\nu)\tilde{D}(m)\right\} \text{ if } M<1;\\ \\ \min\left\{\nu\max\{D(m),D(M)\},(1-\nu)\max\left\{\tilde{D}(m),\tilde{D}(M)\right\}\right\}\\ \text{ if } m\le 1\le M\\ \\ \min\left\{\nu D(M),(1-\nu)\tilde{D}(M)\right\} \text{ if } 1<m,\end{cases} \tag{3.11}$$

and

$$\begin{aligned}A_\nu(a,b)-\check{H}_\nu(a,b) &\le \frac{1}{2}a\min\{\nu,1-\nu\}\begin{cases}\check{D}(m) \text{ if } M<1\\ \\ \max\left\{\check{D}(m),\check{D}(M)\right\} \text{ if } m\le 1\le M\\ \\ \check{D}(M) \text{ if } 1<m,\end{cases}\\ &\le \frac{1}{4}a\begin{cases}\check{D}(m) \text{ if } M<1\\ \\ \max\left\{\check{D}(m),\check{D}(M)\right\} \text{ if } m\le 1\le M\\ \\ \check{D}(M) \text{ if } 1<m.\end{cases}\end{aligned} \tag{3.12}$$

Proof. Using the properties of the functions D, $\tilde{D}$ and $\check{D}$ we have

$$\max_{h\in[m,M]}D(h)=\begin{cases}D(m) \text{ if } M<1\\ \max\{D(m),D(M)\} \text{ if } m\le 1\le M\\ D(M) \text{ if } 1<m,\end{cases}$$

and the same bounds for the other two functions $\tilde{D}$ and $\check{D}$.

By applying the inequalities (3.8) and (3.9) we then obtain the desired results. □

We have:

Theorem 3. *For any $a,b>0$ and $\nu\in[0,1]$ we have the inequalities*

$$\begin{aligned}(1\le)\,\frac{A_\nu(a,b)}{G_\nu(a,b)} &\le \min\left\{\left(\frac{\exp\left(\frac{b}{a}-1\right)}{\frac{b}{a}}\right)^{\nu},\left(\frac{\frac{b}{a}}{\exp\left(1-\frac{1}{\frac{b}{a}}\right)}\right)^{1-\nu}\right\}\end{aligned} \tag{3.13}$$

and

$$(1 \leq) \frac{G\left(A_{\nu}(a,b), A_{1-\nu}(a,b)\right)}{G(a,b)} \leq \exp\left[\min\{\nu, 1-\nu\}\frac{(b-a)^2}{2ab}\right] \leq \exp\left[\frac{(b-a)^2}{4ab}\right] \tag{3.14}$$

Proof. If we write the inequalities (2.3) and (2.4) for the convex function $f(t) = -\ln t$, $t > 0$, then we have

$$\begin{aligned}(0 \leq) \ln((1-\nu)a + \nu b) - (1-\nu)\ln a - \nu \ln b &\leq \begin{cases}\nu\left[\frac{1}{a}(b-a) - \ln b + \ln a\right], \\ (1-\nu)\left[-\frac{1}{b}(b-a) + \ln b - \ln a\right],\end{cases} \\ &= \begin{cases}\nu\left(\frac{b}{a} - 1 - \ln\frac{b}{a}\right), \\ (1-\nu)\left(\ln\frac{b}{a} - 1 + \frac{1}{\frac{b}{a}}\right),\end{cases} = \begin{cases}\ln\left(\frac{\exp\left(\frac{b}{a}-1\right)}{\frac{b}{a}}\right)^{\nu}, \\ \ln\left(\frac{\frac{b}{a}}{\exp\left(1-\frac{1}{\frac{b}{a}}\right)}\right)^{1-\nu},\end{cases}\end{aligned} \tag{3.15}$$

and

$$(0 \leq) \frac{\ln((1-\nu)a + \nu b) + \ln((1-\nu)b + \nu a)}{2} - \frac{\ln a + \ln b}{2} \leq \min\{\nu, 1-\nu\}\frac{(b-a)^2}{2ab} \leq \frac{(b-a)^2}{4ab}, \tag{3.16}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

These are clearly equivalent to the desired results (3.13) and (3.14). □

We have:

Corollary 4. *For any $a, b > 0$ and $\nu \in [0,1]$, if we put $h := \frac{b}{a}$, then we have we have the inequalities*

$$(1 \leq) \frac{A_{\nu}(a,b)}{G_{\nu}(a,b)} \leq \exp\left[\min\left\{\nu D(h), (1-\nu) D\left(\frac{1}{h}\right)\right\}\right] \tag{3.17}$$

and

$$\begin{aligned}(1 \leq) \frac{G\left(A_{\nu}(a,b), A_{1-\nu}(a,b)\right)}{G(a,b)} &\leq \exp\left[2\min\{\nu, 1-\nu\}\left[K(h) - 1\right]\right] \\ &\leq \exp\left[K(h) - 1\right],\end{aligned} \tag{3.18}$$

where K is Kantorovich's constant.

We consider the following upper bounds for the quantity $\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}$ as provided by the inequalities (2.1), (1.5) and (3.13)

$$C_1(x,y) := S(x), \tag{3.19}$$

$$C_2(x,y) := [K(x)]^{\max\{y, 1-y\}} \tag{3.20}$$

and

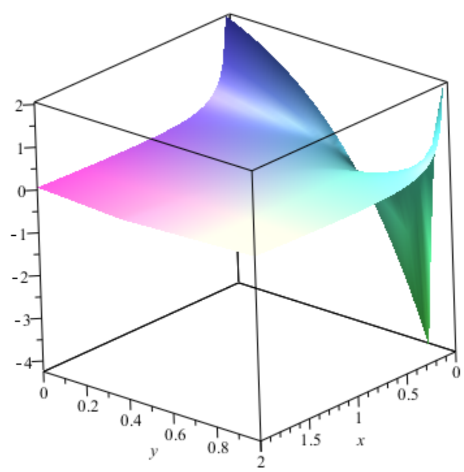
$$C_3(x,y) := \min\left\{ \left(\frac{\exp(x-1)}{x} \right)^y, \left(\frac{x}{\exp\left(1-\frac{1}{x}\right)} \right)^{1-y} \right\} \tag{3.21}$$

defined for $x > 0$ and $y \in [0,1]$.

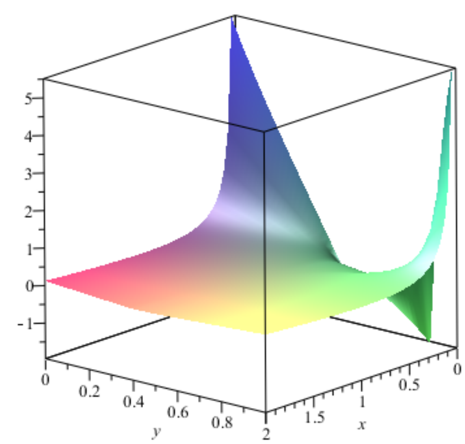
The plots of the differences

$$D_2(x,y) := C_1(x,y) - C_3(x,y) \text{ and } D_3(x,y) := C_2(x,y) - C_3(x,y)$$

in the box $[0,2] \times [0,1]$, which are depicted below, show that they take both positive and negative values, meaning that neither of the corresponding bounds are better always.



Plot of $D_2(x,y)$ in the box $[0,2] \times [0,1]$



Plot of $D_3(x,y)$ in the box $[0,2] \times [0,1]$

We observe that the function $\ell(t) := D\left(\frac{1}{t}\right) = \ln t - 1 + \frac{1}{t}$, $t > 0$ has the derivatives

$$\begin{aligned} \ell'(t) &= \frac{1}{t} - \frac{1}{t^2} = \frac{t-1}{t^2}, \\ \ell''(t) &= \frac{2-t}{t^3}, \end{aligned}$$

which shows that the function ℓ is decreasing on $(0,1)$, increasing on $(1,\infty)$, convex on $(0,2)$ and concave on $(2,\infty)$.

By the use of Corollary 4 we then have:

Corollary 5. *Let $a, b > 0$ with $\frac{b}{a} \in [m, M]$ for some $M > m > 0$ and $\nu \in [0,1]$, then we have we have the inequalities*

$$(3.22)\quad (1\le)\,\frac{A_\nu(a,b)}{G_\nu(a,b)}$$
$$\le\begin{cases}\exp\left[\min\left\{\nu D(m),(1-\nu)D\left(\frac{1}{m}\right)\right\}\right] & \text{if } M<1;\\[1ex] \exp\left[\min\left\{\nu\max\{D(m),D(M)\},(1-\nu)\max\left\{D\left(\frac{1}{m}\right),D\left(\frac{1}{M}\right)\right\}\right\}\right] \\ \quad \text{if } m\le 1\le M\\[1ex] \exp\left[\min\left\{\nu D(M),(1-\nu)D\left(\frac{1}{M}\right)\right\}\right] & \text{if } 1<m,\end{cases}$$

and

$$(3.23)\quad (1\le)\,\frac{G\left(A_\nu(a,b),A_{1-\nu}(a,b)\right)}{G(a,b)}$$
$$\le\exp\left[2\min\{\nu,1-\nu\}\begin{cases}[K(m)\ -1] & \text{if } M<1\\ [\max\{K(m),K(M)\}-1] & \text{if } m\le 1\le M\\ [K(M)-1] & \text{if } 1<m,\end{cases}\right]$$
$$\le\begin{cases}\exp[K(m)\ -1] & \text{if } M<1\\ \exp[\max\{K(m),K(M)\}-1] & \text{if } m\le 1\le M\\ \exp[K(M)-1] & \text{if } 1<m.\end{cases}.$$

4. Operator Inequalities

Throughout this section A, B are positive operators on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_\nu B:=(1-\nu)A+\nu B,\ \nu\in[0,1]$$

the *weighted operator arithmetic mean* and

$$A\sharp_\nu B:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},\ \nu\in[0,1]$$

the *weighted operator geometric mean.* When $\nu=\frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Let Φ be a continuous function defined on the interval J of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{J}$. Then by using the continuous functional calculus, we can define the noncommutative *perspective* $\mathcal{P}_\Phi(B,A)$ by setting

$$(4.1)\qquad \mathcal{P}_\Phi(B,A):=A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_\Phi(B,A)=A\Phi\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset\mathring{J}$.

We observe that if $\Phi(t)=t^\nu$ then

$$\mathcal{P}_\Phi(B,A)=A\sharp_\nu B.$$

Kamei and Fujii [11], [12] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators A and B, by

$$(4.2)\qquad S(A|B):=A^{\frac{1}{2}}\left(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [18].

Consider the logarithmic function ln. Then the relative operator entropy can be interpreted as the perspective of ln, namely

$$\mathcal{P}_{\ln}(B,A)=S(A|B).$$

If we consider the entropy function $\eta(t)=-t\ln t$, then it is well known that for any positive invertible operators A, B we have

$$S(A|B)=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}. \tag{4.3}$$

The function $f(t)=t\ln t=-\eta(t)$, $t>0$, is convex, then the perspective of this function is

$$\mathcal{P}_{(\cdot)\ln(\cdot)}(B,A)=-A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=-S(B|A),$$

where for the last equality we used (4.3) for A replacing B.

Theorem 4. *Let A, B be positive invertible operators and $\nu\in[0,1]$. Then we have*

$$(0\le)\,A\nabla_\nu B-A\sharp_\nu B\le\begin{cases}\nu\left(B-A-S(A|B)\right)\\[2ex](1-\nu)\left(A-B-S(B|A)\right)\end{cases} \tag{4.4}$$

and

$$\begin{aligned}(0\le)\,A\nabla B-\frac{A\sharp_\nu B+A\sharp_{1-\nu}B}{2}&\le-\frac{1}{2}\min\{\nu,1-\nu\}\left(S(B|A)+S(A|B)\right)\\&\le-\frac{1}{4}\left(S(B|A)+S(A|B)\right).\end{aligned} \tag{4.5}$$

Proof. We have from (3.1) for $a=1$ and $b=x>0$ that

$$(0\le)\,1-\nu+\nu x-x^\nu\le\begin{cases}\nu(x-1-\ln x)\\[2ex](1-\nu)(x\ln x-x+1)\end{cases},$$

for any $\nu\in[0,1]$.

If we use the continuous functional calculus, then we have for an operator $X>0$ that

$$(0\le)\,(1-\nu)1_H+\nu X-X^\nu\le\begin{cases}\nu(X-1_H-\ln X)\\[2ex](1-\nu)(X\ln X-X+1_H)\end{cases}, \tag{4.6}$$

for any $\nu\in[0,1]$.

Now, if we take in (4.6) $X=A^{-1/2}BA^{-1/2}$, then we have

$$\begin{aligned}&(0\le)\,(1-\nu)1_H+\nu A^{-1/2}BA^{-1/2}-\left(A^{-1/2}BA^{-1/2}\right)^\nu\\&\le\begin{cases}\nu\left(A^{-1/2}BA^{-1/2}-1_H-\ln A^{-1/2}BA^{-1/2}\right)\\[2ex](1-\nu)\left(A^{-1/2}BA^{-1/2}\ln A^{-1/2}BA^{-1/2}-A^{-1/2}BA^{-1/2}+1_H\right)\end{cases},\end{aligned} \tag{4.7}$$

for any $\nu\in[0,1]$.

If we multiply both sides of (4.7) by $A^{1/2}$ we get the desired result (4.4).

From the inequality (3.2) we have for $a = 1$ and $b = x > 0$ that

$$(0 \leq) \frac{x+1}{2} - \frac{x^{\nu} + x^{1-\nu}}{2} \leq \frac{1}{2} \min\{\nu, 1-\nu\} (x-1) \ln x \leq \frac{1}{4} (x-1) \ln x. \tag{4.8}$$

for any $\nu \in [0, 1]$.

If we use the continuous functional calculus, then we have for an operator $X > 0$ that

$$(0 \leq) \frac{X + 1_H}{2} - \frac{X^{\nu} + X^{1-\nu}}{2} \leq \frac{1}{2} \min\{\nu, 1-\nu\} (X \ln X - \ln X) \leq \frac{1}{4} (X \ln X - \ln X). \tag{4.9}$$

Now, if we take in (4.9) $X = A^{-1/2} B A^{-1/2}$, then we have

$$\begin{aligned} &(0 \leq) \frac{A^{-1/2} B A^{-1/2} + 1_H}{2} - \frac{\left(A^{-1/2} B A^{-1/2}\right)^{\nu} + \left(A^{-1/2} B A^{-1/2}\right)^{1-\nu}}{2} \\ &\leq \frac{1}{2} \min\{\nu, 1-\nu\} \left(A^{-1/2} B A^{-1/2} \ln A^{-1/2} B A^{-1/2} - \ln A^{-1/2} B A^{-1/2}\right) \\ &\leq \frac{1}{4} \left(A^{-1/2} B A^{-1/2} \ln A^{-1/2} B A^{-1/2} - \ln A^{-1/2} B A^{-1/2}\right). \end{aligned} \tag{4.10}$$

If we multiply both sides of (4.10) by $A^{1/2}$ we get the desired result (4.5). $\square$

Theorem 5. *Let A, B be positive invertible operators such that there exists $M > m > 0$ with the property that*

$$mA \leq B \leq MA \tag{4.11}$$

and $\nu \in [0, 1]$. Then we have

$$\begin{aligned} &(0 \leq) A \nabla_{\nu} B - A \sharp_{\nu} B \\ &\leq \begin{cases} \min\left\{\nu D(m), (1-\nu) \tilde{D}(m)\right\} A \text{ if } M < 1; \\ \min\left\{\nu \max\{D(m), D(M)\}, (1-\nu) \max\left\{\tilde{D}(m), \tilde{D}(M)\right\}\right\} A \\ \quad \text{if } m \leq 1 \leq M \\ \min\left\{\nu D(M), (1-\nu) \tilde{D}(M)\right\} A \text{ if } 1 < m. \end{cases} \end{aligned} \tag{4.12}$$

and

$$(0 \leq)\, A\nabla B - \frac{A\sharp_{\nu}B + A\sharp_{1-\nu}B}{2} \tag{4.13}$$
$$\leq \frac{1}{2}\min\{\nu, 1-\nu\}\begin{cases} \breve{D}(m)\, A \text{ if } M<1 \\ \\ \max\left\{\breve{D}(m), \breve{D}(M)\right\} A \text{ if } m \leq 1 \leq M \\ \\ \breve{D}(M)\, A \text{ if } 1<m, \end{cases}$$
$$\leq \frac{1}{4}\begin{cases} \breve{D}(m)\, A \text{ if } M<1 \\ \\ \max\left\{\breve{D}(m), \breve{D}(M)\right\} A \text{ if } m \leq 1 \leq M \\ \\ \breve{D}(M)\, A \text{ if } 1<m, \end{cases}$$

where the functions D, $\tilde{D}$ *and* $\breve{D}$ *are defined in (3.5)-(3.7).*

Proof. From the inequality (3.11) we have for $x \in [m, M] \subset (0, \infty)$ that

$$(0 \leq)\, 1-\nu+\nu x - x^{\nu} \tag{4.14}$$
$$\leq \begin{cases} \min\left\{\nu D(m), (1-\nu)\tilde{D}(m)\right\} \text{ if } M<1; \\ \\ \min\left\{\nu \max\{D(m), D(M)\}, (1-\nu)\max\left\{\tilde{D}(m), \tilde{D}(M)\right\}\right\} \\ \text{ if } m \leq 1 \leq M \\ \\ \min\left\{\nu D(M), (1-\nu)\tilde{D}(M)\right\} \text{ if } 1<m. \end{cases}$$

If X is a selfadjoint operator with $m1_H \leq X \leq M1_H$, then by (4.14) we have

$$(0 \leq)\, (1-\nu)\, 1_H + \nu X - X^{\nu} \tag{4.15}$$
$$\leq \begin{cases} \min\left\{\nu D(m), (1-\nu)\tilde{D}(m)\right\} \text{ if } M<1; \\ \\ \min\left\{\nu \max\{D(m), D(M)\}, (1-\nu)\max\left\{\tilde{D}(m), \tilde{D}(M)\right\}\right\} \\ \text{ if } m \leq 1 \leq M \\ \\ \min\left\{\nu D(M), (1-\nu)\tilde{D}(M)\right\} \text{ if } 1<m. \end{cases}$$

From the inequality (4.11) we have by multiplying both sides by $A^{-1/2}$ that $m1_H \leq A^{-1/2}BA^{-1/2} \leq M1_H$ and by writing (4.15) for $X = A^{-1/2}BA^{-1/2}$ we have

$$(0 \leq) (1-\nu) 1_H + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \tag{4.16}$$
$$\leq \begin{cases} \min\left\{\nu D(m), (1-\nu) \tilde{D}(m)\right\} \text{ if } M < 1; \\ \\ \min\left\{\nu \max\{D(m), D(M)\}, (1-\nu) \max\left\{\tilde{D}(m), \tilde{D}(M)\right\}\right\} \\ \text{ if } m \leq 1 \leq M \\ \\ \min\left\{\nu D(M), (1-\nu) \tilde{D}(M)\right\} \text{ if } 1 < m. \end{cases}$$

If we multiply both sides of (4.16) by $A^{1/2}$ we get the desired result (4.12).

By the inequality (3.12) we have for $x \in [m, M]$ that

$$(0 \leq) \frac{x+1}{2} - \frac{x^{\nu} + x^{1-\nu}}{2}$$
$$\leq \frac{1}{2} \min\{\nu, 1-\nu\} \begin{cases} \breve{D}(m) \text{ if } M < 1 \\ \\ \max\left\{\breve{D}(m), \breve{D}(M)\right\} \text{ if } m \leq 1 \leq M \\ \\ \breve{D}(M) \text{ if } 1 < m, \end{cases}$$
$$\leq \frac{1}{4} \begin{cases} \breve{D}(m) \text{ if } M < 1 \\ \\ \max\left\{\breve{D}(m), \breve{D}(M)\right\} \text{ if } m \leq 1 \leq M \\ \\ \breve{D}(M) \text{ if } 1 < m, \end{cases} .$$

which, by a similar argument produces the desired result (4.13). □

Finally, we have

Theorem 6. *With the assumptions of Theorem 5 we have*

$$(1 \leq) A\nabla_{\nu}B \tag{4.17}$$
$$\leq \begin{cases} \exp\left[\min\left\{\nu D(m), (1-\nu) D\left(\frac{1}{m}\right)\right\}\right] A\sharp_{\nu}B \textit{ if } M < 1; \\ \\ \exp\left[\min\left\{\nu \max\{D(m), D(M)\}, (1-\nu) \max\left\{D\left(\frac{1}{m}\right), D\left(\frac{1}{M}\right)\right\}\right\}\right] \\ \times A\sharp_{\nu}B \textit{ if } m \leq 1 \leq M \\ \\ \exp\left[\min\left\{\nu D(M), (1-\nu) D\left(\frac{1}{M}\right)\right\}\right] A\sharp_{\nu}B \textit{ if } 1 < m, \end{cases}$$

for any $\nu \in [0, 1]$.

The proof follows in a similar way by the inequality (3.22) and we omit the details.

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