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New Generalization of Discrete Montgomery Identity with Applications

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Abstract

In this paper, a discrete version of the well-known Montgomery's identity is generalized, and a refinement of an inequality derived by B.G. Pachpatte in 2007 is presented. Finally, the results obtained are applied for expanding a complex multinomial formula in different way of the classical expansion.

Key words: Montgomery identity, Gruss inequality, Multinomial expansion, Refinements, Generalizations.

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1 Introduction

In [4] appeared the following discrete version of the well-known Montgomery's identity. It which will be used later on to prove our main results is stated as follows

$$x_k = 1/n \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} D_n(k,i) \triangle x_i,$$

where $\{x_k\}$ for $1 \le k \le n$ is a finite sequence of real numbers, $\triangle x_i = x_{i+1} - x_i$, and where $D_n(k,i) = i/n$ for $1 \le i \le k-1$ and $D_n(k,i) = i/n-1$ for $k \le i \le n$. Putting $H_n(k) = \sum_{i=1}^{n-1} |D_n(k,i)|$ for all $1 \le k \le n$, and taking into account the computation from [5], yields

$$\sum_{k=1}^{n} H_n(k) = \sum_{k=1}^{n} \left(\frac{1}{n} \left[\frac{n^2-1}{4} - \left(k - \frac{n+1}{2} \right)^2 \right] \right) = \frac{n^2-1}{6},$$

as can be immediately checked.

The aim of this paper is to generalize this discrete version of Montgomery's identity and to use it to refine and inequality derived by B.G. Pachpatte in 2007 and published in [3]. Furthermore, as an application of the results obtained we have expanded in a different way of the usual expansions a classical multinomial formulae. Multinomial expansions have many applications in probability, stochastic processes and number theory ([1],[2]).

2 Generalization of Discrete Montgomery Identity

In the following a generalization of discrete Montgomery's identity for n sequences of numbers, namely, $\{x_{1k}\},\{x_{2k}\},\ldots,\{x_{nk}\}$ for $1 \le k \le n$, is given.

Theorem 1 *Let $\{x_{1k}\},\{x_{2k}\},...,\{x_{mk}\}$, $m \ge 2$, $k = 1,...,n$ be finite sequences of real numbers, then*

$$\begin{aligned}\prod_{i=1}^{m} x_{ik} &= \frac{1}{n^{m-1}}x_{mk}\prod_{j=1}^{m-1}\sum_{i=1}^{n}x_{ji} + \frac{1}{n^{m-2}}x_{mk}X_{m-1k}\prod_{j=1}^{m-2}\sum_{i=1}^{n}x_{ji} + ...\\ &+ \frac{1}{n}x_{mk}x_{m-1k}\cdot ...\cdot x_{3k}X_{2k}\sum_{i=1}^{n}x_{1i} + x_{mk}x_{m-1k}\cdot ...\cdot x_{2k}X_{1k},\end{aligned}$$

for $k = 1,...,n$, where $X_{jk} = \sum_{i=1}^{n-1} D_n(k,i)\triangle x_{ji}$ for $j = 1,...,m-1$.

Proof. To prove the preceding the identity by argue by induction. First, multiplying Montgomery's identity for $\{x_{1k}\}$, $1 \le k \le n$ by x_{2k}, yields

$$x_{2k}x_{1k} = \frac{1}{n}x_{2k}\sum_{i=1}^{n}x_{1i} + x_{2k}X_{1k},$$

which is exactly the identity claimed for the case when $m = 2$. Now, we assume that the identity holds for a positive integer m, and we have to proof that it also holds for $m+1$. Indeed, multiplying the identity for m sequences of real numbers by x_{m+1k}, yields

$$\begin{aligned}\prod_{i=1}^{m+1} x_{ik} &= \frac{1}{n^{m-1}}x_{m+1k}x_{mk}\prod_{j=1}^{m-1}\sum_{i=1}^{n}x_{ji} + \frac{1}{n^{m-2}}x_{m+1k}x_{mk}X_{m-1k}\prod_{j=1}^{m-2}\sum_{i=1}^{n}x_{ji}\\ &+ ... + \frac{1}{n}x_{m+1k}x_{mk}\cdot ...\cdot x_{3k}X_{2k}\sum_{i=1}^{n}x_{1i} + x_{m+1k}x_{mk}\cdot ...\cdot x_{2k}X_{1k}\end{aligned}$$

Taking into account the identity for two sequences of real numbers, which is true, applying it to the sequences $\{x_{m+1k}\},\{x_{mk}\}$, $(1 \le k \le n)$, and changing the first term of the RHS of previous identity, we obtain for all $1 \le k \le n$

$$\begin{aligned}\prod_{i=1}^{m+1} x_{ik} &= \frac{1}{n^{m}}x_{m+1k}\prod_{j=1}^{m}\sum_{i=1}^{n}x_{ji} + \frac{1}{n^{m-1}}x_{m+1k}X_{mk}\prod_{j=1}^{m-1}\sum_{i=1}^{n}x_{ji} + ...\\ &+ \frac{1}{n}x_{m+1k}x_{mk}\cdot ...\cdot x_{3k}X_{2k}\sum_{i=1}^{n}x_{1i} + x_{m+1k}x_{mk}\cdot ...\cdot x_{2k}X_{1k}.\end{aligned}$$

So, by the Principle of Mathematical Induction, the proof is complete. □

An immediate consequence of the previous result is the following

Corollary 1 *Let $\{x_{1k}\}, \{x_{2k}\}, ..., \{x_{mk}\}$, $m \geq 2$, $k = 1, ..., n$ be finite sequences of real numbers, then*

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{m} x_{ik} &= \frac{1}{n^m}\prod_{i=1}^{m}\sum_{k=1}^{n} x_{ik} + \frac{1}{n^{m-1}}\prod_{i=1}^{m-2}\sum_{k=1}^{n} x_{ik} \cdot \sum_{k=1}^{n}(x_{mk}X_{m-1k}) \\
&+ ... + \frac{1}{n^2}\sum_{k=1}^{n} x_{1k} \cdot \sum_{k=1}^{n}(x_{mk}x_{m-1k} \cdot ... \cdot x_{3k}X_{2k}) \\
&+ \frac{1}{n}\sum_{k=1}^{n}(x_{mk}x_{m-1k} \cdot ... \cdot x_{2k}X_{1k}),
\end{aligned}$$

where $X_{jk} = \sum_{i=1}^{n-1} D_n(k,i)\triangle x_{ji}$ *for* $j = 1, ..., m-1$.

Proof. Adding up the identities of Theorem 1 for $k = 1, ..., n$, and finally multiplying by $1/n$ the result follows. □

3 Gruss like Inequalities

Hereafter, we give a direct refinement of the first inequality of Theorem 2.2 from [3] derived by B.G. Pachpatte, using discrete Montgomery identity for two sequences of numbers. It is stated as follows

Theorem 2 *Let $\{x_k\}, \{y_k\}$ $(1 \leq k \leq n)$ be two finite sequences of real numbers such that $\max_{1\leq k\leq n-1}\{|\triangle x_k|\} = A$ and $\max_{1\leq k\leq n-1}\{|\triangle y_k|\} = B$, where A, B are nonnegative constants. Then the following inequality holds*

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k y_k - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)\right| \leq \min\left\{\frac{A}{n}\sum_{k=1}^{n}|y_k|H_n(k), \frac{B}{n}\sum_{k=1}^{n}|x_k|H_n(k)\right\}.$$

Proof. Writing the identity of Montgomery for each element x_k of the sequence $\{x_k\}$ $(1 \leq k \leq n)$, and multiplying it by y_k, we get

$$x_k y_k - \frac{1}{n} y_k \sum_{i=1}^{n} x_i = y_k \sum_{i=1}^{n-1} D_n(k,i)\triangle x_i.$$

Adding up the previous identities for $1 \leq k \leq n$, we obtain

$$\frac{1}{n}\sum_{k=1}^{n} x_k y_k - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) = \frac{1}{n}\sum_{k=1}^{n} y_k \sum_{i=1}^{n-1} D_n(k,i)\triangle x_i$$

and

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k y_k - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)\right| \leq \frac{A}{n}\sum_{k=1}^{n}|y_k|H_n(k)$$

after taking the absolute value in both sides of the preceding identity.

Likewise, a similar inequality is obtained if we carry out the same procedure using the elements of the sequence $\{y_k\}$ $(1 \leq k \leq n)$, multiplied by the corresponding x_k. Then, we get

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k y_k - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)\right| \leq \frac{B}{n}\sum_{k=1}^{n}|x_k|H_n(k).$$

Finally, using the above results the inequality claimed in the statement follows and the proof is complete. □

Observe that the inequality presented in Theorem 2 is a refinement of the result given in [3] by Pachpatte.

A similar result is the following

Theorem 3 *Let $\{x_k\}, \{y_k\}$ for $k = 1, ..., n$ be two finite sequences of real numbers such that $\max_{1\le k\le n}\{|x_k|\} = A$ and $\max_{1\le k\le n}\{|y_k|\} = B$, where A, B are nonnegative constants. Then the following inequality holds*

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k y_k - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right)\right| \le \frac{2AB}{n}\sum_{k=1}^{n} H_n(k).$$

Proof. Since $\max_{1\le k\le n}\{|x_k|\} = A$ and $\max_{1\le k\le n}\{|y_k|\} = B$, then on account of the modulus properties we obtain $|\triangle x_k| \le 2A$ and $|\triangle y_k| \le 2B$, for $k = 1, ..., n-1$. Now following the steps of the proof of Theorem 2 we get the result we wanted to prove. □

Next taking into account the above results, we give some generalizations of the discrete Gruss type inequality.

Theorem 4 *Let $\{x_{1k}\}, \{x_{2k}\}, ..., \{x_{mk}\}$, $m \ge 2$, $k = 1, ..., n$ be finite sequences of real numbers, such that $\max_{1\le k\le n-1}\{|\triangle x_{ik}|\} = A_i$, where the $A_i's$ are nonnegative constants, for $i = 1, ..., m-1$. Then the following inequality holds*

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{m} x_{ik} - \prod_{i=1}^{m}\frac{1}{n}\sum_{k=1}^{n} x_{ik}\right| &\le \frac{A_{m-1}}{n^{m-1}}\prod_{i=1}^{m-2}\sum_{k=1}^{n}|x_{ik}|\cdot\sum_{k=1}^{n}(|x_{mk}|H_n(k)) + ...\\ &+ \frac{A_2}{n^2}\sum_{k=1}^{n}|x_{1k}|\cdot\sum_{k=1}^{n}\left(\prod_{i=3}^{m}|x_{ik}|\cdot H_n(k)\right)\\ &+ \frac{A_1}{n}\sum_{k=1}^{n}\left(\prod_{i=2}^{m}|x_{ik}|\cdot H_n(k)\right).\end{aligned}$$

Proof. Applying the modulus property to the previous identity and taking into account that

$$|X_{jk}| = \left|\sum_{i=1}^{n-1} D_n(k,i)\triangle x_{ji}\right| \le \sum_{i=1}^{n-1}|D_n(k,i)|\,|\triangle x_{ji}| \le A_j\sum_{i=1}^{n-1}|D_n(k,i)| = A_j H_n(k),$$

for $j = 1, ..., m-1$, we get the result stated. □

Finally, changing the conditions as we have done before, we obtain another generalization of the discrete Gruss like inequality.

Theorem 5 *Let* $\{x_{1k}\}, \{x_{2k}\}, ..., \{x_{mk}\}$, $m \geq 2$, $k = 1, ..., n$ *be finite sequences of real numbers, such that* $\max_{1 \leq k \leq n}\{|x_{ik}|\} = A_i$, *where the* $A_i' s$ *are nonnegative constants, for* $i = 1, ..., m$. *Then the following inequality holds*

$$\left| \frac{1}{n} \sum_{k=1}^{n} \prod_{i=1}^{m} x_{ik} - \prod_{i=1}^{m} \frac{1}{n} \sum_{k=1}^{n} x_{ik} \right| \leq \frac{2 \prod_{i=1}^{m} A_i}{n} \sum_{k=1}^{n} H_n(k).$$

Proof. From $\max_{1 \leq k \leq n}\{|x_{ik}|\} = A_i$, $i = 1, ..., m$, and taking into account the modulus properties we obtain $|\triangle x_{ik}| \leq 2A_i$, $k = 1, ..., n-1$ and $i = 1, ..., m$. Furthermore, applying the modulus property to the identity from Theorem 4 we get the wanted result. □

4 A multinomial expansion

In the following we will apply the results presented so far to expand a complex formula. For general purposes, let us consider the following formula

$$\Gamma = \left(\sum_{k=1}^{n} f_k \right)^m = \sum_{n_1 + n_2 + ... + n_m = n} \binom{n}{n_1, n_2, \ldots, n_m} f_1^{n_1} f_2^{n_2} \cdots f_m^{n_m},$$

where m and n are positive integers, $n_1, n_2, \ldots, n_m$ are nonnegative integers, and f_k is an arbitrary expression. A complete set of multinomial expansions was presented in [2], which shows the detailed structure of each decomposed term. Here we present a different way to expand Γ based on the identity given in Corollary 1. It is stated in the following

Theorem 6 *Let* m, n *be positive integers, with* $m \geq 2$ *and* f_k, $1 \leq k \leq n$ *be arbitrary expressions, then*

$$\begin{aligned} \Gamma &= n^{m-1} \sum_{k=1}^{n} f_k^m - \left[n \left(\sum_{k=1}^{n} f_k \right)^{m-2} \cdot \sum_{k=1}^{n} f_k X_k + ... \right. \\ &+ \left. n^{m-2} \sum_{k=1}^{n} f_k \cdot \sum_{k=1}^{n} f_k^{m-2} X_k + n^{m-1} \sum_{k=1}^{n} f_k^{m-1} X_k \right], \end{aligned}$$

where $X_k = \sum_{i=1}^{n-1} D_n(k,i) \triangle f_i$.

Proof. It follows directly from Corollary 1, where all the elements of the sequence of numbers are f_k, $1 \leq k \leq n$. □

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