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SOME q-ANALOGUES OF HERMITE-HADAMARD INEQUALITY FOR s-CONVEX FUNCTIONS IN THE SECOND SENSE AND RELATED ESTIMATES

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Abstract. In this paper, quantum analogue of the famous Hermite-Hadamard's inequality for s-convex functions is presented. Some quantum estimates for the right side of the q-analogue of the Hermite-Hadamard inequality by using the s-convexity of the absolute value of the q-derivatives are obtained. Inequalities of Hermite-Hadamard type for the products of convex and s-convex functions using quantum calculus are proved as well.

1. Introduction

The study of calculus without limits is known as quantum calculus or q-calculus. The famous mathematician Euler initiated the study q-calculus in the eighteenth century by introducing the parameter q in Newton's work of infinite series. In the nineteenth century, many outstanding results such as Jacobi's triple product identity and the theory of q-hypergeometric functions were obtained. In early twentieth century, Jackson [7] has started a symmetric study of q-calculus and introduced q-definite integrals. The subject of quantum calculus has numerous applications in different areas of mathematics and physics such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, quantum theory, mechanics and in theory of relativity. This subject has received outstanding attention by many researchers and hence it is considered as an incorporative subject between mathematics and physics. Interested readers are referred to [4, 5, 8] for some current advances in the theory of quantum calculus and theory of inequalities in quantum calculus.

Recall that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

The following remarkable result is considered a necessary and sufficient condition for a function $f : I \subset \mathbb{R} \to \mathbb{R}$ to be convex on $[a,b]$, where $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities in (1.1) are known as Hermite-Hadamard inequalities. Theory of inequalities and theory of convex functions have been observed to be profoundly

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dependent on each other and consequently a vast literature on inequalities has been produced by a number of researchers by using convex functions, see [1, 2].

In the paper [6], Hudzik and Maligranda considered the class of s-convex functions in the second sense as a generalization of the class of convex functions. This class is defined as follows:

Definition 1. [6] *A function $f : [0, \infty) \to \mathbb{R}$ is said to be an s-convex function in the second sense if the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all x, $y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, \infty]$. The class of all s-convex functions in the second sense is denoted by K_s^2.

It has be shown in [6] that all functions in the class K_s^2 are non-negative for $s \in (0, 1)$.

Dragomir and Fitzpatrick [3] proved the following result as a variant of (1.1) for s-convex functions in the second sense.

Theorem 1. [3] *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let a, $b \in [0, \infty)$, $a < b$. If $f \in L([a, b])$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.2}$$

The constant $\frac{1}{s+1}$ is the best possible in the second inequality.

In a very fresh article, Tariboon et al. [12, 13] introduced the concept of quantum derivatives and quantum integrals on finite intervals and developed various quantum analogues for Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Grüss, Grüss-Čebyšev and other integral inequalities using classical convexity. Most recently, Noor et al. [10, 11] and Zhuang et al. [15] have contributed to the ongoing research and have developed some integral inequalities which provide quantum estimates for the right part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and q-differentiable quasi-convex functions.

Inspired by the recent progress in the field quantum calculus, our aim is to establish a variant of (1.2) in quantum calculus. Furthermore, we will also prove some new quantum estimates by using the s-convexity of the absolute value of the q-derivatives.

2. Preliminaries

In this section we recall some q-calculus essentials over finite intervals.

Let $J = [a, b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$, q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ is given in the following definition.

Definition 2. [12] *Let $f : J \to \mathbb{R}$ be a continuous function and let $x \in J$. Then q-derivative of f at x is defined by the expression*

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, x \neq a. \tag{2.1}$$

Since $f : J \to \mathbb{R}$ is a continuous function, thus we have ${}_aD_q f(a) = \lim_{x \to a} {}_aD_q f(x)$. The function f is said to be q-differentiable on J if ${}_aD_q f(x)$ exists for all $x \in J$.

If $a = 0$ in (2.1), then ${}_0D_q f(x) = D_q f(x)$, where $D_q f(x)$ is the well-known q-derivative of f defined by the expression

$$D_q f(x) = \frac{f(qx) - f(x)}{(1-q)x}, x \neq 0. \tag{2.2}$$

For more details on q-derivative given above by (2.2), we refer the reader to [8].

Definition 3. [12] *$f : J \to \mathbb{R}$ be a continuous function. A second-order q-derivative on J is denoted as ${}_aD_q^2 f$, provided ${}_aD_q f$ is q-differentiable on J is defined as ${}_aD_q^2 f = {}_aD_q({}_aD_q f) : J \to \mathbb{R}$. Similarly higher order q-derivative on J is defined by ${}_aD_q^n f = {}_aD_q\left({}_aD_q^{n-1} f\right) : J \to \mathbb{R}$.*

The following result is very important to evaluate q-derivatives.

Lemma 1. [12] *Let $\alpha \in \mathbb{R}$ and $0 < q < 1$, we have*

$${}_aD_q (x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}.$$

One can find further properties of q-derivatives in [14].

Definition 4. [12] *Suppose that $f : J \to \mathbb{R}$ is a continuous function. Then the definite q-integral on J is defined by*

$$\int_a^x f(x)_a \, d_q x = (x-a)(1-q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \tag{2.3}$$

for $x \in J$. If $c \in (a, x)$, then the definite q-integral on J is defined as

$$\begin{aligned}\int_c^x f(x)_a \, d_q x &= \int_a^x f(x)_a \, d_q x - \int_a^c f(x)_a \, d_q x \\ &= (x-a)(1-q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\quad + (c-a)(1-q)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

If $a = 0$ in (2.3), then we get the classical q-definite integral defined by (see [4]*)*

$$\int_0^x f(x)_0 \, d_q x = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x), x \in [0, \infty).$$

Definition 5. [8] *For any real number α*

$$[\alpha] = \frac{1-q^{\alpha}}{1-q}.$$

It is clear that if n is a natural number, then $[n] = 1 + q + \cdots + q^{n-1}$.

The following results hold about definite q-integrals.

Theorem 2. [14] *Let $f : J \to \mathbb{R}$ be a continuous function. Then*

(1) ${}_aD_q \int_a^x f(t)_a \, d_q t = f(x)$
(2) $\int_c^x {}_aD_q f(t)_a \, d_q t = f(x) - f(c)$, $c \in (a, x)$.

Theorem 3. [14] *Suppose that $f, g : J \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

(1) $\int_a^x [f(t) + g(t)]_a \, d_q t = \int_a^x f(t)_a \, d_q t + \int_a^x g(t)_a \, d_q t$;
(2) $\int_a^x \alpha f(t)_a \, d_q t = \alpha \int_a^x f(t)_a \, d_q t$;
(3) $\int_c^x f(t)_a D_q g(t)_a \, d_q t = f(t) g(t)|_c^x - \int_c^x g(qt + (1-q)a)_a D_q f(t)_a \, d_q t$, $c \in (a, x)$.

The following is a valuable results to evaluate definite q-integrals.

Lemma 2. [12] *For $\alpha \in \mathbb{R} \backslash \{-1\}$ and $0 < q < 1$, the following formula holds:*

$$\int_a^x (x-a)^\alpha \ {}_a d_q x = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}.$$

3. Main Results

The following result provides a q-analogue of the Hermite-Hadamard type inequality for s-convex functions.

Theorem 4. *Suppose that $f : [0, \infty) \to [0, \infty)$ is a continuous s-convex function in the second sense, where $0 < s < 1$, $0 < q < 1$ and $a, b \in [0, \infty)$, $a < b$. If f is q-integrable on $[a, b]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)_a \, d_q x \le \frac{f(a) + f(b)}{[s+1]}. \tag{3.1}$$

Proof. By the s-convexity f, we have

$$f((1-t)a + tb) \le (1-t)^s f(a) + t^s f(b)$$

for all $t \in [0, 1]$. By q-integration of the above inequality over the interval $[0, 1]$, one gets

$$\int_0^1 f((1-t)a + tb)_0 \, d_q t \le f(a) \int_0^1 (1-t)_0^s \, d_q t + f(b) \int_0^1 t_0^s d_q t.$$

Since

$$\int_0^1 (1-t)_0^s \, d_q t = \int_0^1 t^s \ {}_0 d_q t = \frac{1-q}{1-q^{s+1}} = \frac{1}{[s+1]}$$

and

$$\begin{aligned} \int_0^1 f((1-t)a + tb)_0 \, d_q t &= \int_a^b f(x)_a \, d_q \left(\frac{x-a}{b-a}\right) \\ &= \frac{1}{b-a} \int_a^b f(x)_a \, d_q x. \end{aligned}$$

Hence

$$\frac{1}{b-a} \int_a^b f(x)_a \, d_q x \le \frac{1-q}{1-q^{s+1}} (f(a) + f(b)). \tag{3.2}$$

Again by using the s-convexity of f, we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{(1-t)a + tb + ta + (1-t)b}{2}\right) \\ &\le 2^{-s} f((1-t)a + tb) + 2^{-s} f(ta + (1-t)b). \end{aligned}$$

By q-integration of the above inequality over the interval $[0,1]$ and by using the change of variables techniques, we obtain

$$f\left(\frac{a+b}{2}\right) \leq 2^{-s}\int_0^1 f\left((1-t)\,a+tb\right)_0 d_qt + 2^{-s}\int_0^1 f\left(ta+(1-t)\,b\right)_0 d_qt = \frac{2^{-s+1}}{b-a}\int_a^b f\left(x\right)_a d_qx. \tag{3.3}$$

Combining (3.2) and (3.3), we get (3.1). □

Remark 1. *It is to be noted that $s=1$, (3.1) becomes the inequality proved in* [13, Theorem 3.2, page 5].

Although the following Lemma has been proved in [10] and [12] but we will prove it by using (3) of Theorem 3.

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with ${}_aD_q$ be continuous and q-integrable on $[a,b]$, a, $b \in I^\circ$, where $0<q<1$, then*

$$\Upsilon_q\left(a,b\right)\left(f\right) := \frac{1}{b-a}\int_a^b f\left(x\right)\ {}_ad_qx - \frac{qf\left(a\right)+f\left(b\right)}{q+1} = \frac{q\left(b-a\right)}{1+q}\int_0^1 \left(1-\left(1+q\right)t\right)\ {}_aD_qf\left(\left(1-t\right)a+tb\right)\ {}_0d_qt. \tag{3.4}$$

Proof. By making use of the change of variables $1-(1+q)\,t = x$ and using (3) of Theorem 3, we have

$$\begin{aligned}
&\int_0^1 \left(1-\left(1+q\right)t\right)\ {}_aD_qf\left(\left(1-t\right)a+tb\right)\ {}_0d_qt \\
&= \int_a^b \left(1-\left(1+q\right)\left(\frac{x-a}{b-a}\right)\right)\ {}_aD_qf\left(x\right)\ {}_ad_q\left(\frac{x-a}{b-a}\right) \\
&= \frac{1}{b-a}\int_a^b \left(1-\left(1+q\right)\left(\frac{x-a}{b-a}\right)\right)\ {}_aD_qf\left(x\right)_a d_qx \\
&= \frac{1}{b-a}\left[\left(1-\left(1+q\right)\left(\frac{x-a}{b-a}\right)\right)f\left(x\right)\Big|_a^b\right. \\
&\left.-\int_a^b f\left(qx+\left(1-q\right)a\right)_a D_q\left(1-\left(1+q\right)\left(\frac{x-a}{b-a}\right)\right)\ {}_ad_qx\right] \\
&= -\frac{qf\left(b\right)+f\left(a\right)}{b-a} + \frac{1+q}{\left(b-a\right)^2}\int_a^b f\left(qx+\left(1-q\right)a\right)_a d_qx.
\end{aligned} \tag{3.5}$$

Now by Definition 4, we have

$$
\begin{aligned}
(3.6)\quad & \int_a^b f\left(qx+(1-q)a\right)_a d_q x \\
&= (b-a)(1-q)\sum_{n=0}^{\infty} q^n f\left(q\left(q^n b+(1-q^n)a\right)+(1-q)a\right) \\
&= (b-a)(1-q)\sum_{n=0}^{\infty} q^n f\left(q^{n+1}b+\left(1-q^{n+1}\right)a\right) \\
&= \frac{(b-a)(1-q)}{q}\sum_{n=1}^{\infty} q^n f\left(q^n b+(1-q^n)a\right) \\
&= \frac{(b-a)(1-q)}{q}\sum_{n=0}^{\infty} q^n f\left(q^n b+(1-q^n)a\right) - \frac{(b-a)(1-q)f(b)}{q} \\
&= \frac{1}{q}\int_a^b f(x)_a d_q x - \frac{(b-a)(1-q)f(b)}{q}.
\end{aligned}
$$

Using (3.6) in (3.5), we get

$$
\begin{aligned}
(3.7)\quad & \int_0^1 (1-(1+q)t)\ {}_aD_q f\left((1-t)a+tb\right)\ {}_0d_q t \\
&= -\frac{qf(b)+f(a)}{b-a} + \frac{1+q}{q(b-a)^2}\int_a^b f(x)_a d_q x - \frac{\left(1-q^2\right)f(b)}{q(b-a)} \\
&= -\frac{qf(a)+f(b)}{b-a} + \frac{1+q}{q(b-a)^2}\int_a^b f(x)_a d_q x.
\end{aligned}
$$

Multiplying both sides of (3.7) by $\frac{q(b-a)}{1+q}$, we get (3.4). □

We are now able to present some new estimates for (1.1) in q-calculus by using s-convexity of functions.

Theorem 5. *Let $f : I \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $[0,\infty) \subset I^\circ$ and let ${}_aD_q$ be continuous and q-integrable on $[a,b]$, where a, $b \in [0,\infty)$, $a < b$ and $0 < q < 1$. If $\left|{}_aD_q f\right|^r$ is s-convex function for some fixed $s \in (0,1)$ and $r \geq 1$, then*

$$
\begin{aligned}
& \left|\Upsilon_q(a,b)(f)\right| \\
& \leq \frac{q(b-a)}{1+q}\left[\frac{2q}{(1+q)^2}\right]^{1-\frac{1}{r}} \left(\rho_1(q,s)\left|{}_aD_q f(a)\right|^r + \rho_2(q,s)\left|{}_aD_q f(b)\right|^r\right)^{\frac{1}{r}},
\end{aligned}
$$

where

$$
\rho_1(q,s) = \frac{q}{[s+1]}\left[2\left(\frac{q}{1+q}\right)^{s+1}-1\right] + \frac{1+q}{[s+2]}\left[1-2\left(\frac{q}{1+q}\right)^{s+2}\right]
$$

and

$$
\rho_2(q,s) = \frac{1}{[s+1]}\left[2\left(\frac{1}{1+q}\right)^{s+1}-1\right] + \frac{1+q}{[s+2]}\left[1-2\left(\frac{1}{1+q}\right)^{s+2}\right].
$$

Proof. Taking absolute value on both sides of (3.4) and using the Hölder's inequality, we have

$$|\Upsilon_q(a,b)(f)| \leq \frac{q(b-a)}{1+q}\left(\int_0^1 |1-(1+q)t|_0\, d_qt\right)^{1-\frac{1}{r}} \times \left(\int_0^1 |1-(1+q)t|\, |{}_aD_qf((1-t)a+tb)|^r\ {}_0d_qt\right)^{\frac{1}{r}}. \tag{3.8}$$

Since $|{}_aD_qf|^r$ is s-convex function for some fixed $s\in(0,1)$, we have

$$\int_0^1 |1-(1+q)t|\, |{}_aD_qf((1-t)a+tb)|^r\ {}_0d_qt \tag{3.9}$$
$$\leq |{}_aD_qf(a)|^r \int_0^1 |1-(1+q)t|\,(1-t)^s\ {}_0d_qt + |{}_aD_qf(b)|^r \int_0^1 |1-(1+q)t|\, t^s\ {}_0d_qt.$$

We also have

$$\begin{aligned} &\int_0^1 |1-(1+q)t|_0\, d_qt \\ &= \int_0^{\frac{1}{1+q}} (1-(1+q)t)_0\, d_qt + \int_{\frac{1}{1+q}}^1 ((1+q)t-1)_0\, d_qt \\ &= \int_0^{\frac{1}{1+q}} (1-(1+q)t)_0\, d_qt + \int_{\frac{1}{1+q}}^1 ((1+q)t-1)_0\, d_qt = \frac{2q}{(1+q)^2}. \end{aligned} \tag{3.10}$$

Now we calculate the other q-integrals involved in (3.9) as follows

$$\begin{aligned} &\int_0^1 |1-(1+q)t|\,(1-t)^s\ {}_0d_qt \\ &= \int_0^{\frac{1}{1+q}} (1-(1+q)t)(1-t)^s\ {}_0d_qt + \int_{\frac{1}{1+q}}^1 ((1+q)t-1)(1-t)^s\ {}_0d_qt \\ &= \int_1^{\frac{q}{1+q}} (1-(1+q)(1-x))\,x^s\ {}_0d_q(1-x) \\ &\quad + \int_{\frac{q}{1+q}}^0 ((1+q)(1-x)-1)\,x^s\ {}_0d_q(1-x) \\ &= -\int_1^{\frac{q}{1+q}} (1-(1+q)(1-x))\,x^s\ {}_0d_qx - \int_{\frac{q}{1+q}}^0 ((1+q)(1-x)-1)\,x^s\ {}_0d_qx \\ &= \int_{\frac{q}{1+q}}^1 (1-(1+q)(1-x))\,x^s\ {}_0d_qx + \int_0^{\frac{q}{1+q}} ((1+q)(1-x)-1)\,x^s\ {}_0d_qx \\ &= \int_0^1 (1-(1+q)(1-x))\,x^s\ {}_0d_qx - \int_0^{\frac{q}{1+q}} (1-(1+q)(1-x))\,x^s\ {}_0d_qx \end{aligned} \tag{3.11}$$

$$+\int_0^{\frac{q}{1+q}}((1+q)(1-x)-1)x^s\ {}_0d_qx$$

$$=\int_0^1\left(-qx^s+(1+q)x^{s+1}\right)\ {}_0d_qx-2\int_0^{\frac{q}{1+q}}\left(-qx^s+(1+q)x^{s+1}\right)\ {}_0d_qx$$

$$=q\left(\frac{1-q}{1-q^{s+1}}\right)\left[2\left(\frac{q}{1+q}\right)^{s+1}-1\right]+(1+q)\left(\frac{1-q}{1-q^{s+2}}\right)\left[1-2\left(\frac{q}{1+q}\right)^{s+2}\right].$$

$$\int_0^1|1-(1+q)t|t^s\ {}_0d_qt \tag{3.12}$$

$$=\int_0^{\frac{1}{1+q}}(1-(1+q)t)t^s\ {}_0d_qt+\int_{\frac{1}{1+q}}^1((1+q)t-1)t^s\ {}_0d_qt$$

$$=\int_0^{\frac{1}{1+q}}(1-(1+q)t)t^s\ {}_0d_qt+\int_0^1((1+q)t-1)t^s\ {}_0d_qt$$

$$-\int_0^{\frac{1}{1+q}}((1+q)t-1)t^s\ {}_0d_qt$$

$$=2\int_0^{\frac{1}{1+q}}(1-(1+q)t)t^s\ {}_0d_qt-\int_0^1(1-(1+q)t)t^s\ {}_0d_qt$$

$$=\left(\frac{1-q}{1-q^{s+1}}\right)\left[2\left(\frac{1}{1+q}\right)^{s+1}-1\right]+(1+q)\left(\frac{1-q}{1-q^{s+2}}\right)\left[1-2\left(\frac{1}{1+q}\right)^{s+2}\right].$$

Applying (3.9)-(3.12) in (3.8), we get the required inequality. □

Corollary 1. *As $q\to1^-$ in Theorem 5, the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(x)_a\,d_qx-\frac{f(a)+f(b)}{2}\right|$$
$$\le\frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{r}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{r}}\left(\left|f^{'}(a)\right|^r+\left|f^{'}(b)\right|^r\right)^{\frac{1}{r}}.$$

This inequality was proved in [9, Theroem 1, page 28].

Proof. As $q\to1^-$, we have by using L'Hospital rule that

$$\rho_1(q,s),\rho_2(q,s)\to\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}$$

and

$$|{}_aD_qf(a)|^r\to\left|f^{'}(a)\right|^r,|{}_aD_qf(b)|^r\to\left|f^{'}(b)\right|^r.$$

□

Theorem 6. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $[0,\infty)\subset I^\circ$ and let ${}_aD_q$ be continuous and q-integrable on on $[a,b]$, where a, $b\in[0,\infty)$, $a<b$ and $0<q<1$. If $|{}_aD_qf|^r$ is s-convex function for* *some fixed $s \in (0,1)$, where p and r are Hölder conjugates of each other, then*

$$
\begin{aligned}
(3.13)\qquad \left|\Upsilon_q(a,b)(f)\right| \le \frac{q(b-a)}{1+q}\left[\frac{1+q^{p+1}}{(1+q)[p+1]}\right]^{1-\frac{1}{p}} \\
\times\left(\frac{1}{[s+1]}\right)^{\frac{1}{r}}\left(\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf(b)\right|^r\right)^{\frac{1}{r}}.
\end{aligned}
$$

Proof. Taking absolute value on both sides of (3.4) and using Hölder inequality, we have

$$
\begin{aligned}
(3.14)\qquad & \left|\Upsilon_q(a,b)(f)\right| \\
\le & \frac{q(b-a)}{1+q}\left(\int_0^1\left|1-(1+q)t\right|^p\ {}_0d_qt\right)^{1-\frac{1}{p}}\left(\int_0^1\left|{}_aD_qf((1-t)a+tb)\right|^r\ {}_0d_qt\right)^{\frac{1}{r}}.
\end{aligned}
$$

We now evaluate the integrals involved in (3.14). We note that the suitable substitutions, yield

$$
\begin{aligned}
(3.15)\qquad & \int_0^1\left|1-(1+q)t\right|^p\ {}_0d_qt \\
&= \int_0^{\frac{1}{1+q}}(1-(1+q)t)^p\ {}_0d_qt+\int_{\frac{1}{1+q}}^1((1+q)t-1)^p\ {}_0d_qt \\
&= \int_1^0 x^p\ {}_0d_q\left(\frac{1-x}{1+q}\right)+\int_0^q x^p\ {}_0d_q\left(\frac{1+x}{1+q}\right) \\
&= \frac{1}{1+q}\int_0^1 x^p\ {}_0d_qx+\frac{1}{1+q}\int_0^q x^p\ {}_0d_qx \\
&= \frac{1}{1+q}\left(\frac{1-q}{1-q^{1+p}}\right)+\frac{1}{1+q}\left(\frac{1-q}{1-q^{1+p}}\right)q^{1+p}=\frac{(1-q)\left(1+q^{p+1}\right)}{(1+q)\left(1-q^{1+p}\right)}.
\end{aligned}
$$

Using the s-convexity of $\left|{}_aD_qf\right|^r$ for some fixed $s\in(0,1)$, we have

$$
\begin{aligned}
(3.16)\qquad & \int_0^1\left|{}_aD_qf((1-t)a+tb)\right|^r\ {}_0d_qt \\
&\le \left|{}_aD_qf(a)\right|^r\int_0^1(1-t)^s\ {}_0d_qt+\left|{}_aD_qf(b)\right|^r\int_0^1 t^s\ {}_0d_qt \\
&= \left|{}_aD_qf(a)\right|^r\int_1^0 x^s\ {}_0d_q(1-x)+\left|{}_aD_qf(b)\right|^r\int_0^1 t^s\ {}_0d_qt \\
&= \left|{}_aD_qf(a)\right|^r\int_0^1 t^s\ {}_0d_qt+\left|{}_aD_qf(b)\right|^r\int_0^1 t^s\ {}_0d_qt \\
&= \left(\frac{1-q}{1-q^{s+1}}\right)\left[\left|{}_aD_qf(a)\right|^r+\left|{}_aD_qf(b)\right|^r\right].
\end{aligned}
$$

Making use of (3.15) and (3.16) in (3.14), we get the required result. □

Corollary 2. *As $q \to 1^-$ in Theorem 6, the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(x)_a\, d_q x - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)}{2}\left(\frac{1}{p+1}\right)^{1-\frac{1}{p}}\left(\frac{\left|f'(a)\right|^r+\left|f'(b)\right|^r}{s+1}\right)^{\frac{1}{r}}. \tag{3.17}$$

Proof. Proof follows by using similar arguments as in proving Corollary 1. □

Theorem 7. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $[0,\infty) \subset I^\circ$ and let ${}_aD_q$ be continuous and q-integrable on on $[a,b]$, where a, $b \in [0,\infty)$, $a < b$ and $0 < q < 1$. If $|{}_aD_q f|^r$ is s-convex function for some fixed $s \in (0,1)$, where p and r are Hölder conjugates of each other, then*

$$|\Upsilon_q(a,b)(f)| \leq \frac{q(b-a)}{(1+q)^2}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}}\left(\frac{1}{[s+1]}\right)^{\frac{1}{r}} \times\left\{\left(\left|f\left(\frac{qa+b}{q+2}\right)\right|^r+|f(a)|^r\right)^{\frac{1}{r}}+q^{1+\frac{1}{p}}\left(\left|f\left(\frac{qa+b}{q+2}\right)\right|^r+|f(b)|^r\right)^{\frac{1}{r}}\right\}. \tag{3.18}$$

Proof. Taking absolute value on both sides of (3.4) and using Hölder inequality, we have

$$\begin{aligned}
|\Upsilon_q(a,b)(f)| &\leq \frac{q(b-a)}{1+q}\int_0^1 |1-(1+q)t|\ \ |{}_aD_q f((1-t)a+tb)|\ {}_0d_q t \\
&= \frac{q(b-a)}{1+q}\int_0^{\frac{1}{1+q}} (1-(1+q)t)\ \ |{}_aD_q f((1-t)a+tb)|\ {}_0d_q t \\
&\quad + \frac{q(b-a)}{1+q}\int_{\frac{1}{1+q}}^1 ((1+q)t-1)\ \ |{}_aD_q f((1-t)a+tb)|\ {}_0d_q t \\
&\leq \frac{q(b-a)}{1+q}\left(\int_0^{\frac{1}{1+q}} (1-(1+q)t)^p\ {}_0d_q t\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{1+q}} |{}_aD_q f((1-t)a+tb)|^r\ {}_0d_q t\right)^{\frac{1}{r}} \\
&\quad + \frac{q(b-a)}{1+q}\left(\int_{\frac{1}{1+q}}^1 ((1+q)t-1)^p\ {}_0d_q t\right)^{\frac{1}{p}}\left(\int_{\frac{1}{1+q}}^1 |{}_aD_q f((1-t)a+tb)|^r\ {}_0d_q t\right)^{\frac{1}{r}}.
\end{aligned} \tag{3.19}$$

By using (3.1), we obtain

$$\int_0^{\frac{1}{1+q}} |{}_aD_q f((1-t)a+tb)|^r\ {}_0d_q t \leq \frac{1-q}{(1+q)(1-q^{s+1})}\left(\left|{}_aD_q f\left(\frac{qa+b}{q+1}\right)\right|^r +_a D_q\,|f(a)|^r\right) \tag{3.20}$$

and

$$\int_{\frac{1}{1+q}}^{1} \left|{}_aD_qf\left((1-t)\,a+tb\right)\right|_0^r d_qt \tag{3.21}$$
$$\leq \frac{1-q}{(1+q)\left(1-q^{s+1}\right)}\left(\left|{}_aD_qf\left(\frac{qa+b}{q+1}\right)\right|^r +_a D_q\left|f(b)\right|^r\right).$$

Moreover,

$$\int_0^{\frac{1}{1+q}} \left(1-(1+q)\,t\right)^p \ {}_0d_qt = \frac{1}{1+q}\left(\frac{1-q}{1-q^{1+p}}\right) \tag{3.22}$$

and

$$\int_{\frac{1}{1+q}}^{1} \left((1+q)\,t-1\right)^p \ {}_0d_qt = \frac{1}{1+q}\left(\frac{1-q}{1-q^{1+p}}\right)q^{1+p}. \tag{3.23}$$

Using (3.20)-(3.23) in (3.19), we get the desired result. □

Corollary 3. *When $q \to 1^-$ in Theorem 7, we get the following corrected result proved in* [9, Theorem 3, page 31]

$$\left|\frac{1}{b-a}\int_a^b f(x)_a\, d_qx - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)}{4}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{r}} \tag{3.24}$$
$$\times\left\{\left(\left|f\left(\frac{a+b}{2}\right)\right|^r + \left|f(a)\right|^r\right)^{\frac{1}{r}} + \left(\left|f\left(\frac{a+b}{2}\right)\right|^r + \left|f(b)\right|^r\right)^{\frac{1}{r}}\right\}.$$

Proof. Proof follows by using L'Hospital's rule to evaluate the limits

$$\lim_{q\to 1^-}\frac{1-q}{1-q^{1+p}} \text{ and } \lim_{q\to 1^-}\frac{1-q}{1-q^{s+1}}.$$

□

Theorem 8. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a q-differentiable function on I° (the interior of I) with $[0,\infty) \subset I^\circ$ and let ${}_aD_q$ be continuous and q-integrable on on $[a,b]$, where a, $b \in [0,\infty)$, $a < b$ and $0 < q < 1$. If $\left|{}_aD_qf\right|^r$ is s-concave function for some fixed $s \in (0,1)$, where p and r are Hölder conjugates of each other, then*

$$\left|\Upsilon_q(a,b)(f)\right| \leq \frac{2^{\frac{s-1}{r}}q(b-a)}{(1+q)^2}\left(\frac{1}{[p+1]}\right)^{\frac{1}{p}} \tag{3.25}$$
$$\times\left\{\left|{}_aD_qf\left(\frac{(2q+1)\,a+b}{2\,(q+1)}\right)\right| + q^2\left|{}_aD_qf\left(\frac{(q+2)\,b+a}{2\,(q+1)}\right)\right|\right\}.$$

Proof. We continue from (3.19) and using (3.1), we have

$$\int_0^{\frac{1}{1+q}} \left|{}_aD_qf\left((1-t)\,a+tb\right)\right|^r \ {}_0d_qt \leq \frac{2^{s-1}}{1+q}\left|{}_aD_qf\left(\frac{(2q+1)\,a+b}{2\,(q+1)}\right)\right|^r$$

and

$$\int_{\frac{1}{1+q}}^{1} \left|{}_aD_qf\left((1-t)\,a+tb\right)\right|^r \ {}_0d_qt \leq \frac{q2^{s-1}}{1+q}\left|{}_aD_qf\left(\frac{(q+2)\,b+a}{2\,(q+1)}\right)\right|^r.$$

□

Corollary 4. *Letting $q \to 1^-$ in Theorem 8, we get the following corrected inequality proved in* [9, Theorem 4, page 32]

$$\left|\frac{1}{b-a}\int_a^b f(x)_a\, d_q x - \frac{f(a)+f(b)}{2}\right| \leq \frac{2^{\frac{s-1}{r}}(b-a)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left|f'\left(\frac{3a+b}{4}\right)\right| + \left|f'\left(\frac{3b+a}{4}\right)\right|\right\}. \tag{3.26}$$

4. Inequalities for products of two q-integrable functions

Theorem 9. *Let $f, g : [0,\infty) \to \mathbb{R}, a, b \in [0,\infty), a < b$, be functions such that f, g and fg are q-integrable over $[a,b]$, $0 < q < 1$. If f is non-negative and convex on $[a,b]$, and if g is s-convex for some fixed $s \in (0,1)$, then*

$$\frac{1}{b-a}\int_a^b f(x)\, g(x)\ {}_a d_q x \leq \frac{f(b)\, g(b) + f(a)\, g(a)}{[s+2]} + [f(a)\, g(b) + f(b)\, g(a)]\left(\frac{1}{[s+1]} - \frac{1}{[s+2]}\right). \tag{4.1}$$

Proof. By the convexity of f on $[a,b]$ and s-convexity of g, we have

$$f(tb+(1-t)a) \leq tf(b) + (1-t)f(a)$$

and

$$g(tb+(1-t)a) \leq t^s g(b) + (1-t)^s g(a)$$

for all $t \in [0,1]$. Since f and g are non-negative, we have

$$f(tb+(1-t)a)\, g(tb+(1-t)a) \leq t^{s+1} f(b)\, g(b) + t(1-t)^s f(b)\, g(a) + t^s(1-t) f(a)\, g(b) + (1-t)^{s+1} f(a)\, g(a).$$

By q-integration on both sides of the above inequality over the interval $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 & f(tb+(1-t)a)\, g(tb+(1-t)a)\ {}_0 d_q t \\ &\leq f(b)\, g(b)\int_0^1 t^{s+1}\ {}_0 d_q t + f(b)\, g(a)\int_0^1 t(1-t)^s\ {}_0 d_q t \\ &\quad + f(a)\, g(b)\int_0^1 t^s(1-t)\ {}_0 d_q t + f(a)\, g(a)\int_0^1 (1-t)^{s+1}\ {}_0 d_q t.\end{aligned} \tag{4.2}$$

By the change of variables technique, we have

$$\begin{aligned}\int_0^1 & f(tb+(1-t)a)\, g(tb+(1-t)a)\ {}_0 d_q t \\ &= \int_a^b f(x)\, g(x)\ {}_a d_q\left(\frac{x-a}{b-a}\right) = \frac{1}{b-a}\int_a^b f(x)\, g(x)\ {}_a d_q x.\end{aligned} \tag{4.3}$$

We also note that

$$\int_0^1 t^{s+1} = \int_0^1 (1-t)^{s+1}\ {}_0 d_q t = \frac{1-q}{1-q^{s+2}} \tag{4.4}$$

and

$$\int_0^1 t(1-t)^s \ {}_0d_qt = \int_0^1 t^s(1-t) \ {}_0d_qt = \frac{1-q}{1-q^{s+1}} - \frac{1-q}{1-q^{s+2}}. \tag{4.5}$$

Applying (4.3)-(4.5) in (4.2), we get the desired inequality. □

Remark 2. *Suppose $q \to 1^-$ in Theorem 9, we get the Theorem 5 proved* [9, page 32].

Remark 3. *If we choose $f(x) = 1$ for all $x \in [a,b]$ in Theorem 9, we get (3.1).*

Theorem 10. *Let $f, g : [0,\infty) \to \mathbb{R}, a, b \in [0,\infty), a < b$, be functions such that f, g and fg are q-integrable over $[a,b]$, $0 < q < 1$. If f is non-negative and convex on $[a,b]$, and if g is s-convex for some fixed $s \in (0,1)$, then*

$$\begin{aligned} 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,g(x) \ {}_ad_qx \\ \le \frac{1}{[s+2]}\left[f(a)\,g(b) + f(b)\,g(a)\right] \\ + \left(\frac{1}{[s+1]} - \frac{1}{[s+2]}\right)\left[f(b)\,g(b) + f(a)\,g(a)\right]. \end{aligned} \tag{4.6}$$

Proof. By using the convexity and non-negativity of f on $[a,b]$ and s-convexity of g, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ & \le f\left(\frac{tb+(1-t)a+(1-t)b+ta}{2}\right) g\left(\frac{tb+(1-t)a+(1-t)b+ta}{2}\right) \\ & \le \frac{1}{2^{s+1}}\left[f(tb+(1-t)a)\,g(tb+(1-t)a) + f(tb+(1-t)a)\,g((1-t)b+ta)\right. \\ & \left. f((1-t)b+ta)\,g(tb+(1-t)a) + f((1-t)b+ta)\,g((1-t)b+ta)\right] \\ & \le \frac{1}{2^{s+1}}\left[f(tb+(1-t)a)\,g(tb+(1-t)a) + f((1-t)b+ta)\,g((1-t)b+ta)\right] \\ & + \frac{1}{2^{s+1}}\left[f((1-t)b+ta)\,g(tb+(1-t)a) + f(tb+(1-t)a)\,g((1-t)b+ta)\right] \\ & = \frac{1}{2^{s+1}}\left[f(tb+(1-t)a)\,g(tb+(1-t)a) + f((1-t)b+ta)\,g((1-t)b+ta)\right] \\ & = \frac{1}{2^{s+1}}\left[t^s(1-t) + t(1-t)^s\right]\left[f(b)\,g(b) + f(a)\,g(a)\right] \\ & + \left[t^{s+1} + (1-t)^{s+1}\right]\left[f(a)\,g(b) + f(b)\,g(a)\right]. \end{aligned}$$

By q-integration on both sides of the above inequality and by using the change of variables technique give us the desired result. □

Remark 4. *When $q \to 1^-$, Theorem 10 reduces to the Theorem 7 in* [9, page 34].

For our next results, we need the q-beta and q-gamma functions which are defined as follows.

Definition 6. [8] *(1) For $\alpha>0$, the q-gamma function is defined as*

$$\Gamma_q(\alpha)=\int_0^{\frac{1}{1-q}} t^{\alpha-1}E_q^{-qt}\ {}_0d_qt,$$

where E_q^x is one of the following q-analogues of the exponential function

$$E_q^t=\sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}}\frac{t^n}{[n]!}=(1+(1-q)t)_q^{\infty}=\prod_{j=0}^{\infty}\left(1+q^j(1-q)t\right)$$

$$e_q^t=\sum_{n=0}^{\infty}\frac{t^n}{[n]!}=\frac{1}{(1-(1-q)t)_q^{\infty}}=\frac{1}{\prod_{j=0}^{\infty}(1-q^j(1-q)t)}.$$

(2) For $\alpha,\beta>0$, the q-beta function is defined as

$$B_q(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-qt)_q^{\beta-1}\ {}_0d_qt,$$

where

$$(1-qt)_q^{\beta-1}=\frac{(1-qt)_q^{\infty}}{(1-q^{\beta}t)_q^{\infty}}.$$

Some properties of q-beta and q-gamma functions are given in the following theorem.

Theorem 11. [8] *(a) $\Gamma_q(\alpha)$ can equivalently be expressed as*

$$\Gamma_q(\alpha)=\frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}$$

In particular one has

$$\Gamma_q(1+\alpha)=[\alpha]\Gamma_q(\alpha),\ \textit{for all } t>0,\ \Gamma_q(1)=1.$$

(b) The q-gamma and q-beta functions are related to each other by the following two equations

$$\Gamma_q(\alpha)=\frac{B_q(\alpha,\infty)}{(1-q)^{\alpha}},$$

$$B_q(\alpha,\beta)=\frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}.$$

Remark 5. *It is not difficult to observe that*

$$(1-t)^{\beta}\le(1-qt)^{\beta}\le(1-qt)_q^{\beta}$$

for $0\le t\le 1$, $\beta>0$ and $0<q<1$.

Theorem 12. *Let $f,g:[0,\infty)\to\mathbb{R}, a,b\in[0,\infty), a<b$, be functions such that f, g and fg are q-integrable over $[a,b]$, $0<q<1$. If f is s_1-convex and g is s_2-convex for some fixed $s_1,s_2\in(0,1)$, then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)\ {}_ad_qx &\le\frac{f(b)g(b)+f(a)g(a)}{[s_1+s_2+1]}\\ &\quad+[f(b)g(a)+f(a)g(b)]B_q(s_1+1,s_2+1),\end{aligned}\tag{4.7}$$

where $B_q(\alpha,\beta)$, $\alpha,\beta>0$ is the q-beta function.

Proof. Since f is s_1-convex and g is s_2-convex functions, we have

$$f(tb+(1-t)a) \leq t^{s_1} f(b) + (1-t)^{s_1} f(a)$$

and

$$g(tb+(1-t)a) \leq t^{s_2} g(b) + (1-t)^{s_2} g(a)$$

for all $t \in [0,1]$. The non-negativity of f and g gives

$$\begin{aligned}
& f(tb+(1-t)a)\, g(tb+(1-t)a) \\
& \leq t^{s_1+s_2} f(b)\, g(b) + t^{s_1}(1-t)^{s_2} f(b)\, g(a) \\
& + t^{s_2}(1-t)^{s_1} f(a)\, g(b) + (1-t)^{s_1}(1-t)^{s_2} f(a)\, g(a).
\end{aligned}$$

By q-integration on both sides of the above inequality over the interval $[0,1]$, we get

$$\begin{aligned}
(4.8) \qquad & \int_0^1 f(tb+(1-t)a)\, g(tb+(1-t)a)\ {}_0d_qt \\
& \leq f(b)\, g(b) \int_0^1 t^{s_1+s_2}\ {}_0d_qt + f(b)\, g(a) \int_0^1 t^{s_1}(1-t)^{s_2}\ {}_0d_qt \\
& + f(a)\, g(b) \int_0^1 t^{s_2}(1-t)^{s_1}\ {}_0d_qt + f(a)\, g(a) \int_0^1 (1-t)^{s_1+s_2}\ {}_0d_qt.
\end{aligned}$$

By using the change of variables technique, we have

$$\int_0^1 f(tb+(1-t)a)\, g(tb+(1-t)a)\ {}_0d_qt = \frac{1}{b-a}\int_a^b f(x)\, g(x)\ {}_ad_qx.$$

We also observe that

$$\int_0^1 t^{s_1+s_2}\ {}_0d_qt = \int_0^1 (1-t)^{s_1+s_2}\ {}_0d_qt = \frac{1-q}{1-q^{s_1+s_2+1}},$$

$$\int_0^1 t^{s_1}(1-t)^{s_2}\ {}_0d_qt \leq \int_0^1 t^{s_1}(1-qt)_q^{s_2}\ {}_0d_qt = B_q(s_1+1, s_2+1)$$

and

$$\begin{aligned}
\int_0^1 t^{s_2}(1-t)^{s_1}\ {}_0d_qt & \leq \int_0^1 t^{s_2}(1-qt)_q^{s_1}\ {}_0d_qt = B_q(s_2+1, s_1+1) \\
& = B_q(s_1+1, s_2+1).
\end{aligned}$$

Utilizing the above observations, we get from (4.8), the required result. □

Remark 6. *A similar result to that of Theorem 10 can be formulated when f is s_1-convex and g is s_2-convex for some fixed $s_1, s_2 \in (0,1)$.*

Remark 7. *Taking $f(x) = 1$ fro all $x \in [a,b]$ in Theorem 10, we obtain the following inequality*

$$(4.9) \qquad 2^s g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(x)\ {}_ad_qx \leq \frac{g(b)+g(a)}{[s+1]}.$$

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