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SOME PERTURBED OSTROWSKI TYPE INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS

[1]HÜSEYIN BUDAK, [1]MEHMET ZEKI SARIKAYA, AND [2,3]S. SEVER DRAGOMIR

Abstract. The main aim of this paper is to establish some perturbed Ostrowski type integral inequalities for functions whose second derivatives are either bounded or of bounded variation.

1. Introduction

In 1938, Ostrowski [27] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Definition 1. *Let $P:a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum\Delta f(P)$ denotes the sum $\sum_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b(f):=\sup\left\{\sum\Delta f(P):P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [15], Dragomir proved the following Ostrowski type inequalities for functions of bounded variation:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [12], authors obtained the following Ostroski type inequalities for functions whose second derivatives are bounded:

Theorem 3. *Letf $:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'':(a,b)\to\mathbb{R}$ is bounded on $(a;b)$. Then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\ \leq\ &\frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2\left\|f''\right\|_\infty\\ \leq\ &\frac{\left\|f''\right\|_\infty}{6}(b-a)^2\end{aligned}$$

for all $x\in[a,b]$.

Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many authors have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example ([1]-[17], [25],[26],[28]-[33]). Moreover, Dragomir proved some perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to [18]-[24]. In this study, we establish some perturbed Ostrowski type inequalities for twice differentiable functions whose second derivatives are either bounded or of bounded variation.

2. Some Identities

Before we start our main results, we state and prove the following lemma:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b). Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex number the following identity holds*

$$\begin{aligned}&\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt\\ &-\frac{1}{2(b-a)}\left[\frac{\lambda_1(x)(x-a)^3+\lambda_2(x)(b-x)^3}{3}\right]\\ =\ &\frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t)-\lambda_2(x)\right]dt\right],\end{aligned} \tag{2.1}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Using the integration by parts, we have

$$\begin{aligned}
(2.2)\quad & \int_a^x (t-a)^2 \left[f''(t)-\lambda_1(x)\right] dt \\
&= \int_a^x (t-a)^2 f''(t)dt - \lambda_1(x)\int_a^x (t-a)^2\, dt \\
&= (t-a)^2 f'(t)\Big|_a^x - 2\int_a^x (t-a) f'(t)dt - \frac{\lambda_1(x)}{3}(t-a)^3\Big|_a^x \\
&= (x-a)^2 f'(x) - 2\left[(t-a) f(t)|_a^x - \int_a^x f(t)dt\right] - \frac{\lambda_1(x)}{3}(x-a)^3 \\
&= (x-a)^2 f'(x) - 2(x-a) f(x) + 2\int_a^x f(t)dt - \frac{\lambda_1(x)}{3}(x-a)^3
\end{aligned}$$

and

$$\begin{aligned}
(2.3)\quad & \int_x^b (t-b)^2 \left[f''(t)-\lambda_2(x)\right] dt \\
&= \int_x^b (t-b)^2 f''(t)dt - \lambda_2(x)\int_x^b (t-b)^2\, dt \\
&= (t-b)^2 f'(t)\Big|_x^b - 2\int_x^b (t-b) f'(t)dt - \frac{\lambda_1(x)}{3}(t-b)^3\Big|_x^b \\
&= -(b-x)^2 f'(x) - 2\left[(t-b) f(t)|_x^b - \int_x^b f(t)dt\right] - \frac{\lambda_2(x)}{3}(b-x)^3 \\
&= (b-x)^2 f'(x) - 2(b-x) f(x) + 2\int_x^b f(t)dt - \frac{\lambda_1(x)}{3}(x-a)^3 .
\end{aligned}$$

If we add the equality (2.2) and (2.3) and devide by $2(b-a)$, we obtain required identity. □

Corollary 1. *Under assumption of Lemma 1 with $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\begin{aligned}
(2.4)\quad & \left(x-\frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda(x)}{6(b-a)}\left[(x-a)^3+(b-x)^3\right] \\
&= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t)-\lambda(x)\right] dt + \frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t)-\lambda(x)\right] dt\right].
\end{aligned}$$

Remark 1. *If we choose $\lambda(x) = 0$ in (2.4), then we have the following identity*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \tag{2.5}$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 f''(t)dt + \frac{1}{b-a}\int_x^b (t-b)^2 f''(t)dt\right]$$

for all $x \in [a, b]$.

Corollary 2. *Under assumption of Lemma 1 with $\lambda_1(x) = \lambda_1 \in \mathbb{C}$ and $\lambda_2(x) = \lambda_2 \in \mathbb{C}$, we get*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt \tag{2.6}$$

$$-\frac{1}{6(b-a)}\left[\lambda_1(x-a)^3 + \lambda_2(b-x)^3\right]$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 \left[f''(t) - \lambda_1\right] dt + \frac{1}{b-a}\int_x^b (t-b)^2 \left[f''(t) - \lambda_2\right] dt\right].$$

In particular, taking $\lambda_1 = \lambda_2 = \lambda$ we have

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{\lambda}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right] \tag{2.7}$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 \left[f''(t) - \lambda\right] dt + \frac{1}{b-a}\int_x^b (t-b)^2 \left[f''(t) - \lambda\right] dt\right].$$

Corollary 3. *Under assumption of Lemma 1 with $\lambda_1(x) = \lambda_2(x) = f''(x)$, $x \in (a, b)$, we have the equality*

$$\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a}\int_a^b f(t)dt - \frac{f''(x)}{6(b-a)}\left[(x-a)^3 + (b-x)^3\right] \tag{2.8}$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2 \left[f''(t) - f''(x)\right] dt + \frac{1}{b-a}\int_x^b (t-b)^2 \left[f''(t) - f''(x)\right] dt\right].$$

Corollary 4. *Under assumption of Lemma 1, we assume that the lateral derivatives $f''_+(a)$ and $f''_-(b)$ exist and finite. If we take $\lambda_1(x) = f''_+(a)$ and $\lambda_2(x) = f''_-(b)$ in*

(2.1), then we have

$$\begin{aligned}(2.9)\quad & \left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt \\ & -\frac{1}{2(b-a)}\left[\frac{f''_+(a)(x-a)^3+f''_-(b)(b-x)^3}{3}\right] \\ = & \ \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t)-f''_+(a)\right]dt+\frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t)-f''_-(b)\right]dt\right].\end{aligned}$$

In particular, we get

$$\begin{aligned}(2.10)\quad & \frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{48}\left[f''_+(a)+f''_-(b)\right] \\ = & \ \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}} (t-a)^2\left[f''(t)-f''_+(a)\right]dt+\int_{\frac{a+b}{2}}^b (t-b)^2\left[f''(t)-f''_-(b)\right]dt\right].\end{aligned}$$

Corollary 5. *Under assumption of Lemma 1, we assume that the derivatives* $f''_+(a)$, $f''_-(b)$ *and* $f''(x)$ *exist and finite. If we choose* $\lambda_1(x)=\frac{f''_+(a)+f''(x)}{2}$ *and* $\lambda_2(x)=\frac{f''(x)+f''_-(b)}{2}$ *in (2.1), then we have*

$$\begin{aligned}(2.11)\quad & \left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(x-a)^3+(b-x)^3}{6(b-a)}f''(x) \\ & -\frac{1}{6(b-a)}\left[(x-a)^3f''_+(a)+(b-x)^3f''_-(b)\right] \\ = & \ \frac{1}{2}\left[\frac{1}{b-a}\int_a^x (t-a)^2\left[f''(t)-\frac{f''_+(a)+f''(x)}{2}\right]dt\right. \\ & \left.+\frac{1}{b-a}\int_x^b (t-b)^2\left[f''(t)-\frac{f''(x)+f''_-(b)}{2}\right]dt\right].\end{aligned}$$

In particular,

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) - \frac{(b-a)}{48}\left[f''_+(a)+f''_-(b)\right] \tag{2.12}$$

$$= \frac{1}{2}\left[\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)^2 \left[f''(t) - \frac{f''_+(a)+f''\left(\frac{a+b}{2}\right)}{2}\right] dt \right.$$

$$\left. + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)^2 \left[f''(t) - \frac{f''\left(\frac{a+b}{2}\right)+f''_-(b)}{2}\right] dt\right].$$

3. Inequalities for Functions Whose Second Derivatives are Bounded

Recall the sets of complex-valued functions:

$$\overline{U}_{[a,b]}(\gamma,\Gamma)$$

$$:= \left\{ f:[a,b]\to\mathbb{C} \,\middle|\, \left[(\Gamma - f(t))\left(\overline{f(t)}\right) - \overline{\gamma}\right] \geq 0 \text{ for almast every } t\in[a,b]\right\}$$

and

$$\overline{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f:[a,b]\to\mathbb{C} \,\middle|\, \left|f(t) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

Proposition 1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty and closed sets and*

$$\overline{U}_{[a,b]}(\gamma,\Gamma) = \overline{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b) and $x\in(a,b)$. Suppose that $\gamma_i,\Gamma_i\in\mathbb{C}$, $\gamma_i\neq\Gamma_i$, $i=1,2$ and $f''\in\overline{U}_{[a,x]}(\gamma_1,\Gamma_2)\cap$*

$\overline{U}_{[x,b]}(\gamma_2,\Gamma_2)$, *then we have the inequalities*

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt - \frac{(\gamma_1+\Gamma_1)(x-a)^3+(\gamma_2+\Gamma_2)(b-x)^3}{12(b-a)}\right| \tag{3.1}$$
$$\leq \frac{(b-a)^2}{12}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^3+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^3\right]$$
$$\leq \begin{cases} \left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\max\{|\Gamma_1-\gamma_1|,|\Gamma_2-\gamma_2|,\} \\ \left[\left(\frac{x-a}{b-a}\right)^{3p}+\left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left(|\Gamma_1-\gamma_1|^q+|\Gamma_2-\gamma_2|^q\right)^{\frac{1}{q}}, \\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right]. \end{cases}$$

Proof. Taking the modulus identity (2.1) for $\lambda_1(x)=\frac{\gamma_1+\Gamma_1}{2}$ and $\lambda_2(x)=\frac{\gamma_2+\Gamma_2}{2}$, since $f''\in\overline{U}_{[a,x]}(\gamma_1,\Gamma_2)\cap\overline{U}_{[x,b]}(\gamma_2,\Gamma_2)$, we have

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt - \frac{(\gamma_1+\Gamma_1)(x-a)^3+(\gamma_2+\Gamma_2)(b-x)^3}{12(b-a)}\right|$$
$$\leq \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t)-\frac{\gamma_1+\Gamma_1}{2}\right|dt + \int_x^b (t-b)^2\left|f''(t)-\frac{\gamma_2+\Gamma_2}{2}\right|dt\right]$$
$$\leq \frac{1}{2(b-a)}\left[\frac{|\Gamma_1-\gamma_1|}{2}\int_a^x (t-a)^2\,dt+\frac{|\Gamma_2-\gamma_2|}{2}\int_x^b (t-b)^2\,dt\right]$$
$$= \frac{(b-a)^2}{12}\left[|\Gamma_1-\gamma_1|\left(\frac{x-a}{b-a}\right)^3+|\Gamma_2-\gamma_2|\left(\frac{b-x}{b-a}\right)^3\right]$$

which completes the first inequality in (3.1).

The proof of first and third branches of second inequality in (3.1) are obvious. Using the Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{\frac{1}{\alpha}} \left(n^\beta + q^\beta\right)^{\frac{1}{\beta}}, \; m, n, p, q \geq 0 \text{ and } \alpha > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1$$

we can easily obtain the second branch of second inequality in (3.1). □

Corollary 6. *Let $f : [a, b] \to \mathbb{C}$ be a twice differantiable function on (a, b) and $x \in (a, b)$. If $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ and $f'' \in \overline{U}_{[a,b]}(\gamma, \Gamma)$, then we have*

$$\left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt - \frac{\gamma + \Gamma}{12(b-a)} \left[(x-a)^3 + (b-x)^3\right] \right|$$
$$\leq \frac{|\Gamma - \gamma|}{12(b-a)} \left[(x-a)^3 + (b-x)^3\right].$$

Corollary 7. *Under assumption of Theorem 4 with $x = \frac{a+b}{2}$, we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f(x) + \frac{(b-a)^2}{48} \left[\frac{\gamma_1 + \Gamma_1}{2} + \frac{\gamma_2 + \Gamma_2}{2}\right] \right|$$
$$\leq \frac{1}{96} \left[|\Gamma_1 - \gamma_1| + |\Gamma_2 - \gamma_2|\right] (b-a)^2.$$

4. Inequalities for Functions Whose Second Derivatives are of Bounded Variation

Assume that $f : f : [a, b] \to \mathbb{C}$ be a twice differantiable function on I° (the interior of I) and $[a, b] \subset I^\circ$. Then, as in (2.11), we have the identity

$$\begin{aligned}
&\left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt - \frac{(x-a)^3 + (b-x)^3}{6(b-a)} f''(x) \\
&- \frac{1}{6(b-a)} \left[(x-a)^3 f''(a) + (b-x)^3 f''(b)\right] \\
= &\frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 \left[f''(t) - \frac{f''(a) + f''(x)}{2}\right] dt \right. \\
&\left. + \frac{1}{b-a} \int_x^b (t-b)^2 \left[f''(t) - \frac{f''(x) + f''(b)}{2}\right] dt \right],
\end{aligned} \tag{4.1}$$

for any $x \in [a, b]$.

Theorem 5. *Let* : $f:[a,b]\to\mathbb{C}$ *be a twice differantiable function on* I° *and* $[a,b]\subset I^{\circ}$. *If the second derivative* f'' *is of bounded variation on* $[a,b]$, *then*

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(x-a)^3+(b-x)^3}{6(b-a)}f''(x)\right. \tag{4.2}$$
$$\left.-\frac{1}{6(b-a)}\left[(x-a)^3f''(a)+(b-x)^3f''(b)\right]\right|$$
$$=\frac{(b-a)^2}{12}\left[\left(\frac{x-a}{b-a}\right)^3\bigvee_a^x(f'')+\left(\frac{b-x}{b-a}\right)^3\bigvee_x^b(f'')\right]$$
$$\le\frac{(b-a)^2}{12}\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right]\left[\frac{1}{2}\bigvee_a^b(f'')+\frac{1}{2}\left|\bigvee_a^x(f'')-\bigvee_x^b(f'')\right|\right],\\ \left[\left(\frac{x-a}{b-a}\right)^{3p}+\left(\frac{b-x}{b-a}\right)^{3p}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x(f'')\right)^q+\left(\bigvee_x^b(f'')\right)^q\right]^{\frac{1}{q}}\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^3\bigvee_a^b(f''),\end{cases}$$

for any $x\in[a,b]$.

Proof. Taking modulus (4.1), we get

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(x-a)^3+(b-x)^3}{6(b-a)}f''(x)\right. \tag{4.3}$$
$$\left.-\frac{1}{6(b-a)}\left[(x-a)^3f''(a)+(b-x)^3f''(b)\right]\right|$$
$$\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^x(t-a)^2\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right|dt\right.$$
$$\left.+\frac{1}{b-a}\int_x^b(t-b)^2\left|f''(t)-\frac{f''(x)+f''(b)}{2}\right|dt\right].$$

Since f'' is of bounded variation on $[a,x]$, we get

$$\begin{aligned}\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right| &\le \frac{|2f''(t)-f''(a)-f''(x)|}{2}\\ &\le \frac{|f''(t)-f''(a)|+|f''(x)-f''(t)|}{2}\\ &\le \frac{1}{2}\bigvee_a^x(f'').\end{aligned}$$

Thus,

$$\int_a^x (t-a)^2 \left| f''(t) - \frac{f''(a)+f''(x)}{2} \right| dt \leq \frac{1}{2} \bigvee_a^x (f'') \int_a^x (t-a)^2 \, dt \leq \frac{(x-a)^3}{6} \bigvee_a^x (f''). \tag{4.4}$$

Similarly, Since f'' is of bounded variation on $[x,b]$, we have

$$\left| f''(t) - \frac{f''(x)+f''(b)}{2} \right| \leq \frac{1}{2} \bigvee_x^b (f'')$$

and thus,

$$\int_x^b (t-b)^2 \left| f''(t) - \frac{f''(x)+f''(b)}{2} \right| dt \leq \frac{(b-x)^3}{6} \bigvee_x^b (f''). \tag{4.5}$$

If we substitute the inequalities (4.4) and (4.5) in (4.3), we obtain the first inequality in (4.2). The second inequality follows by Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{\frac{1}{\alpha}} \left(n^\beta + q^\beta\right)^{\frac{1}{\beta}}, \ m,n,p,q \geq 0 \text{ and } \alpha > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

□

Corollary 8. *Under assumptions of Theorem 5 with $x = \frac{a+b}{2}$, we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) - \frac{(b-a)}{48} \left[f''(a) + f''(b)\right] \right| \leq \frac{(b-a)^2}{96} \bigvee_a^b (f'').$$

5. Inequalities for Functions Whose Second Derivatives are Lipschitzian

Theorem 6. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° and $[a,b] \subset I^\circ$. If the second derivative f'' is Lipschitzian with the costant $L_1(x)$ on $[a,x]$ and*

$L_2(x)$ on $[x,b]$, then we have

$$\left|\left(x-\frac{a+b}{2}\right)f'(x)-f(x)+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(x-a)^3+(b-x)^3}{6(b-a)}f''(x)\right. \tag{5.1}$$

$$\left.-\frac{1}{6(b-a)}\left[(x-a)^3f''(a)+(b-x)^3f''(b)\right]\right|$$

$$=\frac{(b-a)^3}{12}\left[\left(\frac{x-a}{b-a}\right)^4L_1(x)+\left(\frac{b-x}{b-a}\right)^4L_2(x)\right]$$

$$\leq\frac{(b-a)^3}{12}\begin{cases}\left[\left(\frac{x-a}{b-a}\right)^4+\left(\frac{b-x}{b-a}\right)^4\right]\max\{L_1(x),L_2(x)\},\\ \\ \left[\left(\frac{x-a}{b-a}\right)^{4p}+\left(\frac{b-x}{b-a}\right)^{4p}\right]^{\frac{1}{p}}\left[(L_1(x))^q+(L_1(x))^q\right]^{\frac{1}{q}}\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \max\left\{\left(\frac{x-a}{b-a}\right)^4,\left(\frac{b-x}{b-a}\right)^4\right\}\left[L_1(x)+L_2(x)\right],\end{cases}$$

for any $x\in[a,b]$.

Proof. Since f'' is Lipschitzian with the costant $L_1(x)$ on $[a,x]$, we get

$$\begin{aligned}\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right| &\leq \frac{|2f''(t)-f''(a)-f''(x)|}{2}\\ &\leq \frac{|f''(t)-f''(a)|+|f''(x)-f''(t)|}{2}\\ &\leq \frac{1}{2}L_1(x)\left[|t-a|\right]+|x-t|\\ &= \frac{1}{2}L_1(x)(x-a).\end{aligned}$$

Thus,

$$\begin{aligned}\int_a^x(t-a)^2\left|f''(t)-\frac{f''(a)+f''(x)}{2}\right|dt &\leq \frac{1}{2}L_1(x)(x-a)\int_a^x(t-a)^2\,dt\\ &\leq \frac{1}{6}(x-a)^4L_1(x).\end{aligned} \tag{5.2}$$

Similarly,f'' is Lipschitzian with the costant $L_2(x)$ on $[x,b]$, we get

$$\left|f''(t)-\frac{f''(x)+f''(b)}{2}\right|\leq\frac{1}{2}L_2(x)(b-x)$$

and thus,

$$\int_x^b(t-b)^2\left|f''(t)-\frac{f''(x)+f''(b)}{2}\right|dt\leq\frac{1}{6}(b-x)^4L_2(x). \tag{5.3}$$

If we substitute the inequalities (5.2) and (5.3) in (4.3), we obtain the first inequality in (5.1). The second inequalities con be proved as in Theorem 4 and Theorem 5. □

Corollary 9. *Under assumption of Theorem 6 with $L_1(x) = L_2(x) = L$, we have*

$$\left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt - \frac{(x-a)^3 + (b-x)^3}{6(b-a)} f''(x) \right. \tag{5.4}$$

$$\left. - \frac{1}{6(b-a)} \left[(x-a)^3 f''(a) + (b-x)^3 f''(b) \right] \right|$$

$$= \frac{1}{12} \left[\left(\frac{x-a}{b-a} \right)^4 + \left(\frac{b-x}{b-a} \right)^4 \right] L(b-a)^3.$$

Corollary 10. *If we choose $x = \frac{a+b}{2}$ in (5.4), we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2} \right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2} \right) - \frac{(b-a)}{48} \left[f''(a) + f''(b) \right] \right|$$

$$\leq \frac{1}{192} L(b-a)^3.$$

References

[1] M. W. Alomari, *A Generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] M.W. Alomari and S.S. Dragomir, *Mercer–Trapezoid rule for the Riemann–Stieltjes integral with applications*, Journal of Advances in Mathematics, 2 (2)(2013), 67–85.

[4] H. Budak and M.Z. Sarıkaya, *On generalization of Dragomir's inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.

[5] H. Budak and M.Z. Sarıkaya, *New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation,* RGMIA Research Report Collection, 18(2015), Article 43, 8 pp.

[6] H. Budak and M.Z. Sarikaya, *A new generalization of Ostrowski type inequality for mappings of bounded variation*, RGMIA Research Report Collection, 18(2015), Article 47, 9 pp.

[7] H. Budak and M.Z. Sarikaya, *On generalization of weighted Ostrowski type inequalities for functions of bounded variation,* RGMIA Research Report Collection, 18(2015), Article 51, 11 pp.

[8] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan J. Pure Appl. Anal., 2(1)(2016), 1–11.

[9] H. Budak and M.Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, RGMIA Research Report Collection, 19(2016), Article 24, 10 pp.

[10] H. Budak, M.Z. Sarikaya and A. Qayyum, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, RGMIA Research Report Collection, 19(2016), Article 25, 11 pp.

[11] P. Cerone, S. S. Dragomir and E. Kikianty, *Jensen-Ostrowski Type Inequalities and Applications for f-Divergence Measures,* RGMIA Research Report Collection, 18(2015), Article 47, 20 pp.

[12] S. S. Dragomir and N.S. Barnett, *An Ostrowski type inequality for mappings whose second derivatives are bounded and applications*, RGMIA Research Report Collection, 1(2)(1998) .

[13] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495-508.

[14] S. S. Dragomir, and A. Sofo, *An integral inequality for twice differentiable mappings and application*, Tamkang J. Math., 31(4) 2000.

[15] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4 (2001), no. 1, 59–66.

[16] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5 (2014) No. 1, 89-97 pp.

[17] S. S. Dragomir, *Approximating real functions which possess nth derivatives of bounded variation and applications*, Computers and Mathematics with Applications 56 (2008) 2268–2278.

[18] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation,* Asian-European Journal of Mathematics, 8(4)(2015,),14 pages. DOI:10.1142/S1793557115500692.

[19] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (I)*, Acta Universitatis Matthiae Belii, series Mathematics 23(2015), 71–86.

[20] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (II)*, RGMIA Research Report Collection, 16(2013), Article 93, 16 pp.

[21] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for absolutely continuous functions (III)*, TJMM, 7(1)(2015),31-43.

[22] S. S. Dragomir, *Perturbed companions of Ostrowski's inequality for functions of bounded variation,* RGMIA Research Report Collection, 17(2014), Article 1, 16 pp.

[23] S. S. Dragomir, *Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I)*, RGMIA Research Report Collection, 17(2014), Article 7, 15 pp.

[24] S. S. Dragomir, *Perturbed companions of Ostrowski's inequality for absolutely continuous functions (II)*, GMIA Research Report Collection, 17(2014), Article 19, 11 pp.

[25] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).

[26] Z. Liu, *Some Ostrowski type inequalities*, Mathematical and Computer Modelling 48 (2008) 949–960.

[27] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[28] A. Rafiq, N.A. Mir and F. Zafar, *A generalized Ostrowski-Grüss Type inequality for twice differentiable mappings and application*, JIPAM, 7(4)(2006), article 124.

[29] M. Z. Sarikaya, *On the Ostrowski type integral inequality,* Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 1(2010),129-134.

[30] M. Z. Sarikaya and E. Set, *On New Ostrowski Type Integral Inequalities*, Thai Journal of Mathematics, 12(1)(2014) 145-154.

[31] E. Set and M. Z. Sarikaya, *On a new Ostroski-type inequality and related results*, Kyungpook Mathematical Journal, 54(2014), 545-554.

[32] A. Qayyum, M. Shoaib and I. Faye, *Companion of Ostrowski-type inequality based on 5-step quadratic kernel and applications*, Journal of Nonlinear Science and Applications, 9 (2016), 537–552.

[33] A. Qayyum, M. Shoaib and I. Faye, *On new refinements and applications of efficient quadrature rules using n-times differentiable mappings*, RGMIA Research Report Collection, 19(2016), Article 9, 22 pp.

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey

E-mail address: hsyn.budak@gmail.com

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey

E-mail address: sarikayamz@gmail.com

[2]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

[3]School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail address: sever.dragomir@vu.edu.au