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SOME APPLICATIONS OF YOUNG-TYPE INEQUALITIES FOR OPERATORS

LOREDANA CIURDARIU

ABSTRACT. The aim of this paper is to present some applications of several new Young-type and Holder's inequalities given by Alzer, H., Fonseca, C. M. and Kovacec, A. for operators.

1. Introduction

The famous Young's inequality, as a classical result, state that:

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

when a and b are positive numbers, $a \neq b$ and $\nu \in (0,1)$.

In these years, there are many interesting generalizations of this well-known inequality and its reverse, see for example [9, 10, 8, 7, 1] many others and references therein.

As in [1], we consider $A_{\nu}(a,b) = \nu a + (1-\nu)b$, and $G_{\nu}(a,b) = a^{\nu}b^{1-\nu}$. The following result, given in [7] is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah, [9], [10].

Proposition 1. *For all $a, b > 0$ we have*

$$3\nu\left(A_{\frac{1}{3}}(a,b) - G_{\frac{1}{3}}(a,b)\right) \leq A_{\nu}(a,b) - G_{\nu}(a,b)$$

if $0 < \nu \leq \frac{1}{3}$, and

$$3\nu(1-\nu)\left(A_{\frac{2}{3}}(a,b) - G_{\frac{2}{3}}(a,b)\right) \leq A_{\nu}(a,b) - G_{\nu}(a,b)$$

if $\frac{1}{3} \leq \nu < 1$.

More recently, in [1] are given new results which extend many generalizations of Young's inequality given before. We recall these results below in order to use them in the next sections.

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Theorem 1. *Let* λ, ν *and* τ *be real numbers with* $\lambda \geq 1$ *and* $0 < \nu < \tau < 1$. *Then*

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_\nu(a,b)^\lambda - G_\nu(a,b)^\lambda}{A_\tau(a,b)^\lambda - G_\tau(a,b)^\lambda} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a *and* b. *Moreover, both bounds are sharp.*

Theorem 2. *Let* $\nu \in (0,1)$. *For all real numbers* a, b *with* $0 < a < b$ *we have*

$$\frac{\nu(1-\nu)}{2} a \log^2\left(\frac{b}{a}\right) < A_\nu(a,b) - G_\nu(a,b) < \frac{\nu(1-\nu)}{2} b \log^2\left(\frac{b}{a}\right)$$

and

$$\exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{a}{b}\right)^2\right) < \frac{A_\nu(a,b)}{G_\nu(a,b)} < \exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{b}{a}\right)^2\right).$$

In each inequality, the factor $\frac{\nu(1-\nu)}{2}$ *is the best possible.*

2. **The Young-type inequalities for operators**

As in [4], it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space $(H; < .,. >)$. The *Gelfand map* establishes a $*$- isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t\in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [4] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{1}$$

in the operator order of $B(H)$.

We consider $A,\ B$ two positive operators on a complex Hilbert space $(H, < .,. >)$ and the following notations for operators:

$$A\nabla_\nu B = (1-\nu)A + \nu B,\ \nu \in [0,1],$$

the weighted operator arithmetic mean and

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}},\ \nu \in [0,1],$$

the weighted operator geometric mean.

The following inequalities operators will use the results given in previous section.

Theorem 3. *For any A, B positive invertible operators on H we have*

$$\left(\frac{\nu}{\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(\tau B+(1-\tau)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right] <$$
$$< \left(A^{-\frac{1}{2}}(\nu B+(1-\nu)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu\lambda} <$$
$$< \left(\frac{1-\nu}{1-\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(\tau B+(1-\tau)A)A^{-\frac{1}{2}}\right)^\lambda - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

which can be also written,

$$\left(\frac{\nu}{\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(A\nabla_\tau B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\sharp_\tau B)A^{-\frac{1}{2}}\right] <$$
$$< \left(A^{-\frac{1}{2}}(A\nabla_\nu B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\sharp_\nu B)A^{-\frac{1}{2}} <$$
$$< \left(\frac{1-\nu}{1-\tau}\right)^\lambda \left[\left(A^{-\frac{1}{2}}(A\nabla_\tau B)A^{-\frac{1}{2}}\right)^\lambda - A^{-\frac{1}{2}}(A\sharp_\tau B)A^{-\frac{1}{2}}\right]$$

or

$$\left(\frac{\nu}{\tau}\right)^\lambda [A\sharp_\lambda(A\nabla_\tau B) - A\sharp_{\tau\lambda}B] <$$
$$< A\sharp_\tau(A\nabla_\nu B) - A\sharp_{\nu\lambda}B <$$
$$< \left(\frac{1-\nu}{1-\tau}\right)^\lambda [A\sharp_\lambda(A\nabla_\tau B) - A\sharp_{\tau\lambda}B],$$

for any real numbers λ, ν and τ with $\lambda \geq 1$ and $0 < \nu < \tau < 1$.

Proof. In Theorem 1 if we divide the inequality

$$\left(\frac{\nu}{\tau}\right)^\lambda < \frac{A_\nu(a,b)^\lambda - G_\nu(a,b)^\lambda}{A_\tau(a,b)^\lambda - G_\tau(a,b)^\lambda} < \left(\frac{1-\nu}{1-\tau}\right)^\lambda,$$

by b^λ we get the following:

$$\left(\frac{\nu}{\tau}\right)^\lambda < \frac{(\nu\frac{a}{b}+(1-\nu))^\lambda - (\frac{a}{b})^{\nu\lambda}}{(\tau\frac{a}{b}+(1-\tau))^\lambda - (\frac{a}{b})^{\tau\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^\lambda.$$

Now we use the continuous functional calculus as in [5] and we have for an operator $C > 0$ that

$$\left(\frac{\nu}{\tau}\right)^\lambda \left[(\tau C+(1-\tau)1_H)^\lambda - C^{\tau\lambda}\right] <$$
$$< (\nu C+(1-\nu)1_H)^\lambda - C^{\nu\lambda} <$$
$$< \left(\frac{1-\nu}{1-\tau}\right)^\lambda \left[(\tau C+(1-\tau)1_H)^\lambda - C^{\tau\lambda}\right].$$

We take $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and we have

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left[\left(\tau A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+(1-\tau)1_H\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]<$$

$$<\left(\nu A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+(1-\nu)1_H\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu\lambda}<$$

$$<\left(\frac{1-\nu}{1-\tau}\right)^{\lambda}\left[\left(\tau A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+(1-\tau)1_H\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]$$

or

$$\left(\frac{\nu}{\tau}\right)^{\lambda}\left[\left(A^{-\frac{1}{2}}(\tau B+(1-\tau A)A^{-\frac{1}{2}})\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right]<$$

$$<\left(A^{-\frac{1}{2}}(\nu B+(1-\nu A)A^{-\frac{1}{2}})\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu\lambda}<$$

$$<\left(\frac{1-\nu}{1-\tau}\right)^{\lambda}\left[\left(A^{-\frac{1}{2}}(\tau B+(1-\tau A)A^{-\frac{1}{2}})\right)^{\lambda}-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\tau\lambda}\right].$$

Now if we multiply both sides of previous inequality with $A^{\frac{1}{2}}$ we deduce last inequality of this theorem.
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Proposition 2. *For $\lambda = n \in \mathbf{N}$ and A, B positive invertible operators on H we have:*

$$\frac{\nu^n}{\tau^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^k A^{\frac{1}{2}}-A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau n}A^{\frac{1}{2}}\right]<$$

$$<\sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^k A^{\frac{1}{2}}-A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\nu n}A^{\frac{1}{2}}<$$

$$<\frac{(1-\nu)^n}{(1-\tau)^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^k A^{\frac{1}{2}}-A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau n}A^{\frac{1}{2}}\right],$$

or

$$\frac{\nu^n}{\tau^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A\sharp_k B-A\sharp_{\tau n}B\right]<$$

$$<\sum_{k=0}^{n}\binom{n}{k}(1-\nu)^{n-k}\nu^k A\sharp_k B-A\sharp_{\nu n}B<$$

$$<\frac{(1-\nu)^n}{(1-\tau)^n}\left[\sum_{k=0}^{n}\binom{n}{k}(1-\tau)^{n-k}\tau^k A\sharp_k B-A\sharp_{\tau n}B\right]$$

for any real numbers ν and τ with $0<\nu<\tau<1$.

Proof. We use the same method as in Tkeorem 3. ∎

Theorem 4. *(i) If A, B are positive invertible operators on H with $A < B$ and $\nu \in (0,1)$ then we have:*

$$\frac{\nu(1-\nu)}{2}A^{\frac{1}{2}}\left(\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}} <$$

$$< \nu A + (1-\nu)B - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu}A^{\frac{1}{2}} <$$

$$< \frac{\nu(1-\nu)}{2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}$$

(ii) (a) If A, B are positive invertible operators on H with $A < B$ then

$$\nu A + (1-\nu)B < A^{\frac{1}{2}}\left[(A^{-\frac{1}{2}}BA^{\frac{1}{2}})^{1-\nu}\exp\frac{\nu(1-\nu)}{2}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^2\right]A^{\frac{1}{2}},$$

where $\nu \in (0,1)$.

(b) If A, B are positive invertible operators on H with $A > B$ then

$$\nu A + (1-\nu)B > A^{\frac{1}{2}}\left[(A^{-\frac{1}{2}}BA^{\frac{1}{2}})^{1-\nu}\exp\frac{\nu(1-\nu)}{2}(I - A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^2\right]A^{\frac{1}{2}},$$

where $\nu \in (0,1)$.

Proof. (i) We divide by $a \neq 0$ in inequality

$$\frac{\nu(1-\nu)}{2}a\log^2\left(\frac{b}{a}\right) < A_\nu(a,b) - G_\nu(a,b) < \frac{\nu(1-\nu)}{2}b\log^2\left(\frac{b}{a}\right)$$

from Theorem 2 and we get the scalar inequality

$$\frac{\nu(1-\nu)}{2}\log^2\left(\frac{b}{a}\right) < \nu + (1-\nu)\frac{b}{a} - \frac{b^{1-\nu}}{a^{1-\nu}} <$$

$$< \frac{\nu(1-\nu)}{2}\frac{b}{a}\log^2\left(\frac{b}{a}\right),$$

when $\frac{b}{a} > 1$. Therefore we can write:

$$\frac{\nu(1-\nu)}{2}\log^2(c) < \nu + (1-\nu)c - c^{1-\nu} < \frac{\nu(1-\nu)}{2}c\log^2 c,$$

with $c > 1$. Now using the continous functional calculus as in [5] we find that

$$\frac{\nu(1-\nu)}{2}\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) < \nu 1_H + (1-\nu)A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu} <$$

$$< \frac{\nu(1-\nu)}{2}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\log^2\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right),$$

where $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} > 1_H$ because $B > A$. Then we multiply both sides of previous inequality with $A^{\frac{1}{2}}$ and we get:

$$\frac{\nu(1-\nu)}{2}A^{\frac{1}{2}}\left(\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}} <$$

$$< \nu A + (1-\nu)B - A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{1-\nu}A^{\frac{1}{2}} <$$

$$< \frac{\nu(1-\nu)}{2}A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\log^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)A^{\frac{1}{2}}.$$

(ii) (b) This time we divide the inequality

$$\exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{a}{b}\right)^2\right) < \frac{A_\nu(a,b)}{G_\nu(a,b)} < \exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{b}{a}\right)^2\right).$$

from Theorem 2 by $b \neq 0$ and we get

$$\exp\left(\frac{\nu(1-\nu)}{2}\left(1-\frac{a}{b}\right)^2\right) < \frac{\nu\frac{a}{b}+(1-\nu)}{\left(\frac{a}{b}\right)^\nu}$$

with $c = \frac{a}{b} < 1$. By continous functional calculus we get

$$C^\nu \exp\left(\frac{\nu(1-\nu)}{2}(1_H - C)^2\right) < \nu C + (1-\nu)1_H$$

when $C < 1_H$ Now we put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} < 1_H$ and we have,

$$(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu \exp\left(\frac{\nu(1-\nu)}{2}\left(1_H - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)^2\right) < \nu(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + (1-\nu)1_H.$$

We multiply both sides of previous inequality with $A^{\frac{1}{2}}$ and we obtain the desired inequality.

For (a) we take into account the second part of inequality from Theorem 2 and we use the same method.

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Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara