Received 24/03/16

SOME WEIGHT FUNCTIONALS ASSOCIATED TO CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. The superadditivity and monotonicity of some weight functionals in the general setting of Lebesgue integral associated to convex functions are established. Applications for discrete inequalities and for arithmetic and geometric means are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. (almost every) $x\in\Omega$, consider the *Lebesgue space*

$$L_w(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\,|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ -a.e. $x\in\Omega$ and $\int_\Omega wd\mu>0$, we consider the functional

$$J(w;\Phi,f) := \int_\Omega w(\Phi\circ f)\,d\mu - \Phi\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)\int_\Omega wd\mu \geq 0, \tag{1.1}$$

where $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in L_w(\Omega,\mu)$.

In [7] we proved the following result:

Theorem 1. *Let $w_i:\Omega\to\mathbb{R}$, with $w_i(x)\geq 0$ for μ-a.e. (almost every) $x\in\Omega$ and $\int_\Omega w_i d\mu>0$, $i\in\{1,2\}$. If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in L_{w_1}(\Omega,\mu)\cap L_{w_2}(\Omega,\mu)$, then*

$$J(w_1+w_2;\Phi,f)\geq J(w_1;\Phi,f)+J(w_2;\Phi,f)\geq 0, \tag{1.2}$$

i.e. J is a superadditive functional of weights.

Moreover, if $w_2\geq w_1\geq 0$ μ-a.e. on Ω, then

$$J(w_2;\Phi,f)\geq J(w_1;\Phi,f)\geq 0, \tag{1.3}$$

i.e. J is a monotonic nondecreasing functional of weights.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20.

Key words and phrases. Jensen's inequality, Measurable functions, Lebesgue integral, Young's inequality, Arithmetic mean-geometric mean inequality.

The above theorem has a simple however interesting consequence that provides both a refinement and a reverse for the Jensen's integral inequality:

Corollary 1. *Let $w_i : \Omega \to \mathbb{R}$, with $w_i(x) \geq 0$ for μ-a.e. $x \in \Omega$, $\int_\Omega w_i d\mu > 0$, $i \in \{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$0 \leq \gamma \leq \frac{w_2}{w_1} \leq \Gamma < \infty \ \mu\text{-a.e. on } \Omega. \tag{1.4}$$

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi \circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \gamma \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1(\Phi\circ f)\,d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right] \\ &\leq \frac{\int_\Omega w_2(\Phi\circ f)\,d\mu}{\int_\Omega w_2 d\mu} - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right) \\ &\leq \Gamma \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1(\Phi\circ f)\,d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right]. \end{aligned} \tag{1.5}$$

Remark 1. *Assume that $\mu(\Omega) < \infty$ and let $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. $x \in \Omega$, $\int_\Omega w d\mu > 0$ and w is essentially bounded, i.e.* $\operatorname{essinf}_{x\in\Omega} w(x)$ *and* $\operatorname{essup}_{x\in\Omega} w(x)$ *are finite.*

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi \circ f \in L_w(\Omega,\mu) \cap L(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \frac{\operatorname{essinf}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi\circ f)\,d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right] \\ &\leq \frac{\int_\Omega w(\Phi\circ f)\,d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \\ &\leq \frac{\operatorname{essup}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi\circ f)\,d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right]. \end{aligned} \tag{1.6}$$

This result can be used to provide the following result related to the *Hermite-Hadamard inequality* for convex functions that states that

$$\frac{1}{b-a}\int_a^b \Phi(t)\,dt \geq \Phi\left(\frac{a+b}{2}\right)$$

for any convex function $\Phi : [a,b] \to \mathbb{R}$.

Indeed , if $w : [a,b] \to [0,\infty)$ is Lebesgue integrable, then we have by (1.6) that

$$\begin{aligned} 0 &\leq \frac{\operatorname{essinf}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\,dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{\int_a^b w(t)\,\Phi(t)\,dt}{\int_a^b w(t)\,dt} - \Phi\left(\frac{\int_\Omega w(t)\,tdt}{\int_a^b w(t)\,dt}\right) \\ &\leq \frac{\operatorname{essup}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\,dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{1.7}$$

Now we consider another functional depending on the weights

$$K(w;\Phi,f) := \frac{J(w;\Phi,f)}{\int_\Omega w d\mu} = \frac{\int_\Omega w(\Phi\circ f)\,d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \geq 0$$

and the composite functional

$$L(w;\Phi,f) := \left(\int_\Omega w d\mu\right) \ln\left[K(w;\Phi,f)+1\right] \geq 0,$$

where $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi \circ f \in L_w(\Omega,\mu)$.

Theorem 2. *With the assumptions of Theorem 1, the functional $L(\cdot;\Phi,f)$ is a superadditive and monotonic nondecreasing functional of weights.*

The following result provides another refinement and reverse of the Jensen inequality:

Corollary 2. *Let $w_i : \Omega \to \mathbb{R}$ with $w_i(x) \geq 0$ for μ-a.e. $x \in \Omega$, $\int_\Omega w_i d\mu > 0$, $i \in \{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$0 \leq \gamma \leq \frac{w_2}{w_1} \leq \Gamma < \infty \ \mu\text{-a.e. on } \Omega.$$

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi \circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$, then

$$\begin{aligned}(1.8)\qquad 0 &\leq \left[\frac{\int_\Omega w_1(\Phi\circ f)d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\gamma\frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}} - 1 \\ &\leq \frac{\int_\Omega w_2(\Phi\circ f)d\mu}{\int_\Omega w_2 d\mu} - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right) \\ &\leq \left[\frac{\int_\Omega w_1(\Phi\circ f)d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\Gamma\frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}} - 1.\end{aligned}$$

Remark 2. *Assume that $\mu(\Omega) < \infty$ and let $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ -a.e. $x \in \Omega$, $\int_\Omega w d\mu > 0$ and w is essentially bounded, i.e.* $\operatorname{essinf}_{x\in\Omega} w(x)$ *and* $\operatorname{essup}_{x\in\Omega} w(x)$ *are finite.*

If $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi \circ f \in L_w(\Omega,\mu) \cap L(\Omega,\mu)$, then

$$\begin{aligned}(1.9)\qquad 0 &\leq \left[\frac{\int_\Omega (\Phi\circ f)d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\inf_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega w d\mu\right)}} - 1 \\ &\leq \frac{\int_\Omega w(\Phi\circ f)d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \\ &\leq \left[\frac{\int_\Omega (\Phi\circ f)d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\sup_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega w d\mu\right)}} - 1.\end{aligned}$$

In particular, if $w : [a,b] \to [0,\infty)$ is Lebesgue integrable, then we have the following result related to the Hermite-Hadamard inequality for the convex function

$\Phi : [a, b] \to \mathbb{R}$

$$0 \le \left[\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) + 1\right]^{\frac{\operatorname{essinf}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}} - 1 \tag{1.10}$$
$$\le \frac{\int_a^b w(t)\,\Phi(t)\,dt}{\int_a^b w(t)\,dt} - \Phi\left(\frac{\int_\Omega w(t)\,tdt}{\int_a^b w(t)\,dt}\right)$$
$$\le \left[\frac{1}{b-a}\int_a^b \Phi(t)\,dt - \Phi\left(\frac{a+b}{2}\right) + 1\right]^{\frac{\operatorname{essup}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}} - 1.$$

For other related results, see [3]-[8] and [11].

Motivated by the above results, we establish in this paper the superadditivity and monotonicity of some weight functionals in the general setting of Lebesgue integral associated to convex functions defined on real intervals. Applications for discrete inequalities and for arithmetic and geometric means are also given.

2. Main Results

Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $f : \Omega \to I$ be μ-measurable on Ω. We consider the *cone* of positive weights

$$\mathcal{W}_{+,f}(\Omega) := \left\{w : \Omega \to [0,\infty),\ \int_\Omega w d\mu > 0 \text{ and } f \in L_w(\Omega,\mu)\right\}$$

and the functional $C(\cdot;\Phi,f) : \mathcal{W}_{+,f}(\Omega) \to \mathbb{R}$ given by

$$C(w;\Phi,f) := \Phi\left(\frac{\int_\Omega wf d\mu}{\int_\Omega w d\mu}\right)\int_\Omega w d\mu. \tag{2.1}$$

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $f : \Omega \to I$ be μ-measurable on Ω. Then the functional $C(\cdot;\Phi,f)$ is convex on $\mathcal{W}_{+,f}(\Omega)$.*

Proof. Let w_1, $w_2 \in \mathcal{W}_{+,f}(\Omega)$ and $\lambda \in [0,1]$. We have

$$C(\lambda w_1 + (1-\lambda) w_2;\Phi,f) \tag{2.2}$$
$$= \Phi\left(\frac{\int_\Omega [\lambda w_1 + (1-\lambda) w_2] f d\mu}{\int_\Omega [\lambda w_1 + (1-\lambda) w_2] d\mu}\right)\int_\Omega [\lambda w_1 + (1-\lambda) w_2] d\mu$$
$$= \Phi\left(\frac{\lambda\int_\Omega w_1 f d\mu + (1-\lambda)\int_\Omega w_2 f d\mu}{\lambda\int_\Omega w_1 d\mu + (1-\lambda)\int_\Omega w_2 d\mu}\right)\int_\Omega [\lambda w_1 + (1-\lambda) w_2] d\mu$$
$$= \Phi\left(\frac{\lambda\int_\Omega w_1 d\mu\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu} + (1-\lambda)\int_\Omega w_2 d\mu\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}}{\lambda\int_\Omega w_1 d\mu + (1-\lambda)\int_\Omega w_2 d\mu}\right)$$
$$\times \int_\Omega [\lambda w_1 + (1-\lambda) w_2] d\mu.$$

By the convexity of Φ we have

$$\begin{aligned}
&\Phi\left(\frac{\lambda\int_\Omega w_1 d\mu \frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu} + (1-\lambda)\int_\Omega w_2 d\mu \frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}}{\lambda\int_\Omega w_1 d\mu + (1-\lambda)\int_\Omega w_2 d\mu}\right) \\
&\quad\le \frac{\lambda\int_\Omega w_1 d\mu \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right) + (1-\lambda)\int_\Omega w_2 d\mu \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)}{\lambda\int_\Omega w_1 d\mu + (1-\lambda)\int_\Omega w_2 d\mu}.
\end{aligned} \tag{2.3}$$

Then by (2.2) and (2.3) we have

$$\begin{aligned}
&C(\lambda w_1 + (1-\lambda) w_2; \Phi, f) \\
&\le \frac{\lambda\int_\Omega w_1 d\mu \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right) + (1-\lambda)\int_\Omega w_2 d\mu \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)}{\lambda\int_\Omega w_1 d\mu + (1-\lambda)\int_\Omega w_2 d\mu} \\
&\times \int_\Omega [\lambda w_1 + (1-\lambda) w_2] d\mu \\
&= \lambda\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu + (1-\lambda)\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\int_\Omega w_2 d\mu \\
&= \lambda C(w_1; \Phi, f) + (1-\lambda) C(w_2; \Phi, f),
\end{aligned}$$

for any $w_1, w_2 \in \mathcal{W}_{+,f}(\Omega)$ and $\lambda \in [0,1]$, which proves that $C(\cdot; \Phi, f)$ is convex on $\mathcal{W}_{+,f}(\Omega)$. □

Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $f : \Omega \to I$ be μ-measurable on Ω. We consider the cone

$$\mathcal{W}_{+,f,\Phi}(\Omega) := \{w \in \mathcal{W}_{+,f}(\Omega),\ \Phi \circ f \in L_w(\Omega, \mu)\}.$$

As in the introduction, we can define the following weight functionals $J(\cdot; \Phi, f)$, $K(\cdot; \Phi, f)$ and $L(\cdot; \Phi, f)$ defined on $\mathcal{W}_{+,f,\Phi}(\Omega)$ and given by

$$\begin{aligned}
J(w; \Phi, f) &:= \int_\Omega w(\Phi \circ f) d\mu - C(w; \Phi, f) \\
&= \int_\Omega w(\Phi \circ f) d\mu - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)\int_\Omega w d\mu \ge 0,
\end{aligned} \tag{2.4}$$

$$K(w; \Phi, f) := \frac{J(w; \Phi, f)}{\int_\Omega w d\mu} = \frac{\int_\Omega w(\Phi \circ f) d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \ge 0 \tag{2.5}$$

and the composite functional

$$L(w; \Phi, f) := \left(\int_\Omega w d\mu\right)\ln[K(w; \Phi, f) + 1] \ge 0. \tag{2.6}$$

We have:

Theorem 4. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $f : \Omega \to I$ be μ-measurable on Ω. Then the functionals $J(\cdot; \Phi, f)$ and $L(\cdot; \Phi, f)$ are concave and positive homogeneous on $\mathcal{W}_{+,f,\Phi}(\Omega)$.*

Proof. Let $w_1,\ w_2 \in \mathcal{W}_{+,f,\Phi}(\Omega)$ and $\lambda \in [0,1]$. By the convexity of the functional $C(\cdot;\Phi,f)$ we have

$$
\begin{aligned}
& J(\lambda w_1+(1-\lambda)w_2;\Phi,f) \\
&= \int_\Omega (\lambda w_1+(1-\lambda)w_2)(\Phi\circ f)\,d\mu - C(\lambda w_1+(1-\lambda)w_2;\Phi,f) \\
&= \lambda\int_\Omega w_1(\Phi\circ f)\,d\mu + (1-\lambda)\int_\Omega w_2(\Phi\circ f)\,d\mu \\
&- C(\lambda w_1+(1-\lambda)w_2;\Phi,f) \\
&\geq \lambda\int_\Omega w_1(\Phi\circ f)\,d\mu + (1-\lambda)\int_\Omega w_2(\Phi\circ f)\,d\mu \\
&- \lambda C(w_1;\Phi,f) - (1-\lambda)C(w_2;\Phi,f) \\
&= \lambda\left[\int_\Omega w_1(\Phi\circ f)\,d\mu - C(w_1;\Phi,f)\right] \\
&+ (1-\lambda)\left[\int_\Omega w_2(\Phi\circ f)\,d\mu - C(w_2;\Phi,f)\right] \\
&= \lambda J(w_1;\Phi,f) + (1-\lambda)J(w_2;\Phi,f),
\end{aligned}
$$

which shows that $J(\cdot;\Phi,f)$ is concave on $\mathcal{W}_{+,f,\Phi}(\Omega)$.

Now, observe that by the concavity of $J(\cdot;\Phi,f)$, we have for any $w_1,\ w_2 \in \mathcal{W}_{+,f,\Phi}(\Omega)$ and $\lambda \in [0,1]$ that

$$
\begin{aligned}
& L(\lambda w_1+(1-\lambda)w_2;\Phi,f) \\
&= \int_\Omega (\lambda w_1+(1-\lambda)w_2)\,d\mu \ln\left[\frac{J(\lambda w_1+(1-\lambda)w_2;\Phi,f)}{\int_\Omega (\lambda w_1+(1-\lambda)w_2)\,d\mu}+1\right] \\
&\geq \int_\Omega (\lambda w_1+(1-\lambda)w_2)\,d\mu \\
&\times \ln\left[\frac{\lambda J(w_1;\Phi,f)+(1-\lambda)J(w_2;\Phi,f)}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}+1\right] \\
&= \int_\Omega (\lambda w_1+(1-\lambda)w_2)\,d\mu \\
&\times \ln\left[\frac{\lambda\int_\Omega w_1 d\mu\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+(1-\lambda)\int_\Omega w_2 d\mu\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}+1\right] \\
&= \int_\Omega (\lambda w_1+(1-\lambda)w_2)\,d\mu \\
&\times \ln\left[\frac{\lambda\int_\Omega w_1 d\mu\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+(1-\lambda)\int_\Omega w_2 d\mu\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}\right] \\
&=: A.
\end{aligned}
\tag{2.7}
$$

By the weighted arithmetic mean-geometric mean inequality we have

$$\begin{aligned}(2.8)\qquad & \frac{\lambda\int_\Omega w_1 d\mu\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+(1-\lambda)\int_\Omega w_2 d\mu\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}\\ & \geq\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)^{\frac{\lambda\int_\Omega w_1 d\mu}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}}\\ & \times\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)^{\frac{\lambda\int_\Omega w_2 d\mu}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}}\end{aligned}$$

for any $w_1,\ w_2\in\mathcal{W}_{+,f,\Phi}(\Omega)$ and $\lambda\in[0,1]$.

Taking the logarithm in (2.8) we have

$$\begin{aligned}(2.9)\qquad & \ln\left[\frac{\lambda\int_\Omega w_1 d\mu\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+(1-\lambda)\int_\Omega w_2 d\mu\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\lambda\int_\Omega w_1 d\mu+(1-\lambda)\int_\Omega w_2 d\mu}\right]\\ & \geq\frac{\lambda\int_\Omega w_1 d\mu}{\int_\Omega(\lambda w_1+(1-\lambda)w_2)\,d\mu}\ln\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)\\ & +\frac{(1-\lambda)\int_\Omega w_2 d\mu}{\int_\Omega(\lambda w_1+(1-\lambda)w_2)\,d\mu}\ln\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)\end{aligned}$$

for any $w_1,\ w_2\in\mathcal{W}_{+,f,\Phi}(\Omega)$ and $\lambda\in[0,1]$.

Now, if we multiply (2.9) by $\int_\Omega(\lambda w_1+(1-\lambda)w_2)\,d\mu>0$ we get

$$\begin{aligned}A & \geq\lambda\int_\Omega w_1 d\mu\ln\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)\\ & +(1-\lambda)\int_\Omega w_2 d\mu\ln\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)\\ & =\lambda L(w_1;\Phi,f)+(1-\lambda)L(w_2;\Phi,f)\end{aligned}$$

for any $w_1,\ w_2\in\mathcal{W}_{+,f,\Phi}(\Omega)$ and $\lambda\in[0,1]$, which, by (2.7) shows that $L(\cdot;\Phi,f)$ is concave on $\mathcal{W}_{+,f,\Phi}(\Omega)$.

We observe that

$$J(\alpha w;\Phi,f)=\alpha J(w;\Phi,f)\ \text{ and }\ L(\alpha w;\Phi,f)=\alpha L(w;\Phi,f)$$

for any $w\in\mathcal{W}_{+,f,\Phi}(\Omega)$ and $\alpha>0$ that proves that $J(\cdot;\Phi,f)$ and $L(\cdot;\Phi,f)$ are positive homogeneous on $\mathcal{W}_{+,f,\Phi}(\Omega)$. □

Remark 3. *We observe that by the concavity and positive homogeneity of* $J(\cdot;\Phi,f)$ *and* $L(\cdot;\Phi,f)$ *we have for any* $w_1,\ w_2\in\mathcal{W}_{+,f,\Phi}(\Omega)$ *that*

$$\begin{aligned}J(w_1+w_2;\Phi,f) & =2J\left(\frac{w_1+w_2}{2};\Phi,f\right)\geq 2\left[\frac{J(w_1;\Phi,f)+J(w_2;\Phi,f)}{2}\right]\\ & =J(w_1;\Phi,f)+J(w_2;\Phi,f)\end{aligned}$$

and a similar relation for $L(\cdot;\Phi,f)$*, that proves the superadditivity of the functionals* $J(\cdot;\Phi,f)$ *and* $L(\cdot;\Phi,f)$ *as pointed out in Theorem 1 and Theorem 2.*

Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$ and $f : \Omega \to [m, M]$ be μ-measurable on Ω. Then by convexity of Φ we have

$$\frac{(t-m)\Phi(M) + (M-t)\Phi(m)}{M-m} \geq \Phi\left[\frac{(t-m)M + (M-t)m}{M-m}\right] = \Phi(t) \tag{2.10}$$

for any $t \in [m, M]$.

We then have, by taking $t = \frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu} \in [m, M]$ in (2.10) that

$$\frac{\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu} - m\right)\Phi(M) + \left(M - \frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)\Phi(m)}{M-m} \geq \Phi\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)$$

that is equivalent to

$$\begin{aligned}\frac{1}{M-m}\bigg[\left(\int_\Omega wfd\mu - m\int_\Omega wd\mu\right)\Phi(M) \\ + \left(M\int_\Omega wd\mu - \int_\Omega wfd\mu\right)\Phi(m)\bigg] \\ \geq \Phi\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)\int_\Omega wfd\mu.\end{aligned} \tag{2.11}$$

We define the trapezoidal functional $T : \mathcal{W}_{+,f}(\Omega) \to [0, \infty)$ by

$$\begin{aligned}T(w; \Phi, f) := \frac{1}{M-m}\bigg[\left(\int_\Omega wfd\mu - m\int_\Omega wd\mu\right)\Phi(M) \\ + \left(M\int_\Omega wd\mu - \int_\Omega wfd\mu\right)\Phi(m)\bigg].\end{aligned} \tag{2.12}$$

We can consider the functionals P, $Q : \mathcal{W}_{+,f}(\Omega) \to [0, \infty)$ defined by

$$P(w; \Phi, f) := T(w; \Phi, f) - C(w; \Phi, f) \tag{2.13}$$

and

$$Q(w; \Phi, f) := \left(\int_\Omega wd\mu\right)\ln\left(\frac{P(w; \Phi, f)}{\int_\Omega wd\mu} + 1\right). \tag{2.14}$$

Theorem 5. *Let $\Phi : [m, M] \to \mathbb{R}$ be a continuous convex function on the interval of real numbers $[m, M]$ and $f : \Omega \to [m, M]$ be μ-measurable on Ω. Then the functionals $P(\cdot; \Phi, f)$ and $Q(\cdot; \Phi, f)$ are concave and positive homogeneous on $\mathcal{W}_{+,f}(\Omega)$.*

Proof. We observe that the functional $T(\cdot; \Phi, f)$ is additive and positive homogeneous on $\mathcal{W}_{+,f}(\Omega)$. Therefore, by the convexity and positive homogeneity of $C(\cdot; \Phi, f)$ we can conclude that $P(\cdot; \Phi, f)$ is concave and positive homogeneous on $\mathcal{W}_{+,f}(\Omega)$.

The proof of concavity of $Q(\cdot; \Phi, f)$ follows in a similar way to the one in the proof of Theorem 4 and we omit the details. □

Corollary 3. *With the assumptions of Theorem 4 we have that the functionals $P(\cdot; \Phi, f)$ and $Q(\cdot; \Phi, f)$ are superadditive and monotonic nondecreasing functionals of weights.*

We also have the following upper and lower bounds:

Corollary 4. *Let $\Phi:[m,M]\to\mathbb{R}$ be a continuous convex function on the interval of real numbers $[m,M]$ and $f:\Omega\to[m,M]$ be μ-measurable on Ω. If $w_1,\ w_2\in\mathcal{W}_{+,f}(\Omega)$ and there exists the nonnegative constants γ, Γ such that*

$$0\le\gamma\le\frac{w_2}{w_1}\le\Gamma<\infty\ \mu\text{-a.e. on }\Omega, \tag{2.15}$$

then we have

$$\begin{aligned}
&\gamma\frac{\int_\Omega w_1d\mu}{\int_\Omega w_2d\mu}\left[\frac{\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\Phi(m)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\right]\\
&\le\frac{\left(\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}\right)\Phi(m)}{M-m}\\
&-\Phi\left(\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}\right)\\
&\le\Gamma\frac{\int_\Omega w_1d\mu}{\int_\Omega w_2d\mu}\left[\frac{\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\Phi(m)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\right]
\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}
&\left[\frac{\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\Phi(m)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)+1\right]^{\gamma\frac{\int_\Omega w_1d\mu}{\int_\Omega w_2d\mu}}-1\\
&\le\frac{\left(\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}\right)\Phi(m)}{M-m}\\
&-\Phi\left(\frac{\int_\Omega w_2fd\mu}{\int_\Omega w_2d\mu}\right)\\
&\le\left[\frac{\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)\Phi(m)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega w_1fd\mu}{\int_\Omega w_1d\mu}\right)+1\right]^{\Gamma\frac{\int_\Omega w_1d\mu}{\int_\Omega w_2d\mu}}-1.
\end{aligned}\tag{2.17}$$

Proof. From (2.15) we have $\gamma w_1\le w_2\le\Gamma w_1<\infty$ μ-a.e. on Ω and by the monotonicity property of the functional $P(\cdot;\Phi,f)$ we get

$$P(\gamma w_1;\Phi,f)\le P(w_2;\Phi,f)\le P(\Gamma w_1;\Phi,f). \tag{2.18}$$

Since the the functional is positive homogeneous, namely $P(\alpha w;\Phi,f)=\alpha P(w;\Phi,f)$, then we get from (2.18) that

$$\begin{aligned}\gamma\left[T(w_1;\Phi,f)-C(w_1;\Phi,f)\right]&\le T(w_2;\Phi,f)-C(w_2;\Phi,f)\\&\le\Gamma\left[T(w_1;\Phi,f)-C(w_1;\Phi,f)\right],\end{aligned}$$

which is equivalent to (2.16).

The inequality (2.17) follows in a similar way from the monotonicity of the functional $Q(\cdot;\Phi,f)$. □

Assume that $\mu(\Omega)<\infty$ and let $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ-a.e. $x\in\Omega$, $\int_\Omega wd\mu>0$ and w is essentially bounded, i.e. $\operatorname{essinf}_{x\in\Omega}w(x)$ and $\operatorname{essup}_{x\in\Omega}w(x)$ are finite. If $\Phi:[m,M]\to\mathbb{R}$ is a continuous convex function on the interval of real numbers $[m,M]$ and $f:\Omega\to\mathbb{R}$ is μ-measurable, then by (2.16) we have

$$\begin{aligned}(2.19)\qquad&\frac{\operatorname{essinf}_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega wd\mu}\left[\frac{\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)\Phi(m)}{M-m}\right.\\&\left.-\Phi\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)\right]\\&\le\frac{\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)\Phi(m)}{M-m}\\&-\Phi\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)\\&\le\frac{\operatorname{essup}_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega wd\mu}\left[\frac{\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}-m\right)\Phi(M)+\left(M-\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)\Phi(m)}{M-m}\right.\\&\left.-\Phi\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)\right],\end{aligned}$$

while from (2.17) we have

$$\begin{aligned}
&\left[\frac{\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}-m\right)\Phi\left(M\right)+\left(M-\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\Phi\left(m\right)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega f d\mu}{\mu\left(\Omega\right)}\right)+1\right]^{\frac{\operatorname{essinf}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}}-1\\
&\leq \frac{\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}-m\right)\Phi\left(M\right)+\left(M-\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)\Phi\left(m\right)}{M-m}\\
&-\Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)\\
&\leq \left[\frac{\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}-m\right)\Phi\left(M\right)+\left(M-\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\Phi\left(m\right)}{M-m}\right.\\
&\left.-\Phi\left(\frac{\int_\Omega f d\mu}{\mu\left(\Omega\right)}\right)+1\right]^{\frac{\operatorname{essup}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}}-1.
\end{aligned} \tag{2.20}$$

This result can be used to provide the following result related to the *Jensen's difference* for convex functions

$$\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}\geq \Phi\left(\frac{a+b}{2}\right)$$

for any convex function $\Phi:[a,b]\to\mathbb{R}$.

Indeed, if $w:[a,b]\to[0,\infty)$ is Lebesgue integrable, then we have by (2.19) and (2.20) that

$$\begin{aligned}
&\frac{\operatorname{essinf}_{x\in[a,b]} w\left(x\right)}{\frac{1}{b-a}\int_a^b w\left(t\right)dt}\left[\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\\
&\leq \frac{\left(\frac{\int_a^b w(t)tdt}{\int_a^b w(t)dt}-a\right)\Phi\left(b\right)+\left(b-\frac{\int_a^b w(t)tdt}{\int_a^b w(t)dt}\right)\Phi\left(a\right)}{b-a}\\
&-\Phi\left(\frac{\int_a^b w\left(t\right)tdt}{\int_a^b w\left(t\right)dt}\right)\\
&\leq \frac{\operatorname{essup}_{x\in[a,b]} w\left(x\right)}{\frac{1}{b-a}\int_a^b w\left(t\right)dt}\left[\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]
\end{aligned} \tag{2.21}$$

and

$$\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\operatorname{essinf}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1 \tag{2.22}$$
$$\leq \frac{\left(\frac{\int_a^b w(t)tdt}{\int_a^b w(t)dt}-a\right)\Phi(b)+\left(b-\frac{\int_a^b w(t)tdt}{\int_a^b w(t)dt}\right)\Phi(a)}{b-a}$$
$$-\Phi\left(\frac{\int_a^b w(t)\,tdt}{\int_a^b w(t)\,dt}\right)$$
$$\leq \left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\operatorname{essup}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1.$$

3. Discrete Inequalities

Consider the convex function $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ defined on the interval I, $x = (x_1, ..., x_n) \in I^n$ an n-tuple of real numbers and $p = (p_1, ..., p_n)$ a probability distribution, i.e. $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$.

Let $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ be probability distributions such that there exists $0 < r \leq 1 \leq R < \infty$ with the property that

$$r \leq \frac{p_i}{q_i} \leq R \text{ for any } i \in \{1, ..., n\}. \tag{3.1}$$

By writing the inequalities (1.5) and (1.8) for the discrete measure we have

$$0 \leq r\left[\sum_{i=1}^n q_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n q_ix_i\right)\right] \tag{3.2}$$
$$\leq \sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_ix_i\right)$$
$$\leq R\left[\sum_{i=1}^n q_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n q_ix_i\right)\right]$$

and

$$0 \leq \left[\sum_{i=1}^n q_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n q_ix_i\right)+1\right]^r-1 \tag{3.3}$$
$$\leq \sum_{i=1}^n p_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n p_ix_i\right)$$
$$\leq \left[\sum_{i=1}^n q_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n q_ix_i\right)+1\right]^R-1.$$

For the corresponding inequality to (3.2) for functions defined on convex sets in linear spaces, see [6]. We notice also that the inequality (3.3) can be extended for general convex functions defined on linear spaces by utilizing a similar argument to the one pointed out above for integrals.

For x, y real numbers and $\lambda \in [0, 1]$ define the (generalized) *weighted arithmetic mean* by $A_\lambda(x, y) := (1-\lambda)x + \lambda y$.

If $p,q\in[0,1]$ then by (3.2) and (3.3) we get for $n=2$ that

$$\begin{aligned}0&\le\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_q\left(x,y\right)\right)\right]\\&\le A_p\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_p\left(x,y\right)\right)\\&\le\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\ \left[A_q\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_q\left(x,y\right)\right)\right]\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}0&\le\left[A_q\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_q\left(x,y\right)\right)+1\right]^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1\\&\le A_p\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_p\left(x,y\right)\right)\\&\le\left[A_q\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_q\left(x,y\right)\right)+1\right]^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1,\end{aligned}\tag{3.5}$$

for any $x,\ y\in I$ and $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ is convex on I.

If $A\left(x,y\right)$ denotes the *arithmetic mean* $\frac{x+y}{2}$ and $p\in[0,1]$ then by (3.4) and (3.5) we get

$$\begin{aligned}0&\le2\min\left\{p,1-p\right\}\left[A\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A\left(x,y\right)\right)\right]\\&\le A_p\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_p\left(x,y\right)\right)\\&\le2\max\left\{p,1-p\right\}\left[A\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A\left(x,y\right)\right)\right]\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}0&\le\left[A\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A\left(x,y\right)\right)+1\right]^{2\min\{p,1-p\}}-1\\&\le A_p\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A_p\left(x,y\right)\right)\\&\le\left[A\left(\Phi\left(x\right),\Phi\left(y\right)\right)-\Phi\left(A\left(x,y\right)\right)+1\right]^{2\max\{p,1-p\}}-1,\end{aligned}\tag{3.7}$$

for any $x,\ y\in I$ and $\Phi:I\subset\mathbb{R}\to\mathbb{R}$ is convex on I.

Consider the convex function $\Phi:[m,M]\to\mathbb{R}$ defined on the interval $[m,M]$, $x=(x_1,...,x_n)\in[m,M]^n$ an n-tuple of real numbers and $p=(p_1,...,p_n)$ a probability distribution, i.e. $p_i\ge0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

Let $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ be probability distributions such that there exists $0<r\le1\le R<\infty$ with the property that (3.1) is valid. By writing the inequalities (2.16) and (2.17) for the discrete measure we then have for any $(x_1,...,x_n)\in[m,M]^n$ that

$$\begin{aligned}&r\left[\frac{\left(\sum_{i=1}^n q_ix_i-m\right)\Phi\left(M\right)+\left(M-\sum_{i=1}^n q_ix_i\right)\Phi\left(m\right)}{M-m}-\Phi\left(\sum_{i=1}^n q_ix_i\right)\right]\\&\le\frac{\left(\sum_{i=1}^n p_ix_i-m\right)\Phi\left(M\right)+\left(M-\sum_{i=1}^n p_ix_i\right)\Phi\left(m\right)}{M-m}-\Phi\left(\sum_{i=1}^n p_ix_i\right)\\&\le R\left[\frac{\left(\sum_{i=1}^n q_ix_i-m\right)\Phi\left(M\right)+\left(M-\sum_{i=1}^n q_ix_i\right)\Phi\left(m\right)}{M-m}-\Phi\left(\sum_{i=1}^n q_ix_i\right)\right]\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}(3.9)\qquad &\left[\frac{\left(\sum_{i=1}^{n} q_i x_i - m\right)\Phi(M) + \left(M - \sum_{i=1}^{n} q_i x_i\right)\Phi(m)}{M-m}\right.\\ &\left.-\Phi\left(\sum_{i=1}^{n} q_i x_i\right) + 1\right]^{r} - 1\\ &\leq \frac{\left(\sum_{i=1}^{n} p_i x_i - m\right)\Phi(M) + \left(M - \sum_{i=1}^{n} p_i x_i\right)\Phi(m)}{M-m} - \Phi\left(\sum_{i=1}^{n} p_i x_i\right)\\ &\leq \left[\frac{\left(\sum_{i=1}^{n} q_i x_i - m\right)\Phi(M) + \left(M - \sum_{i=1}^{n} q_i x_i\right)\Phi(m)}{M-m}\right.\\ &\left.-\Phi\left(\sum_{i=1}^{n} q_i x_i\right) + 1\right]^{R} - 1.\end{aligned}$$

If p, $q \in (0,1)$ then by (3.8) and (3.9) we get for $n = 2$ that

$$\begin{aligned}(3.10)\qquad &\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\\ &\times\left[\frac{\left(A_q(x,y) - m\right)\Phi(M) + \left(M - A_q(x,y)\right)\Phi(m)}{M-m} - \Phi\left(A_q(x,y)\right)\right]\\ &\leq \frac{\left(A_p(x,y) - m\right)\Phi(M) + \left(M - A_p(x,y)\right)\Phi(m)}{M-m} - \Phi\left(A_p(x,y)\right)\\ &\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}\\ &\times\left[\frac{\left(A_q(x,y) - m\right)\Phi(M) + \left(M - A_q(x,y)\right)\Phi(m)}{M-m} - \Phi\left(A_q(x,y)\right)\right]\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad &\left[\frac{\left(A_q(x,y) - m\right)\Phi(M) + \left(M - A_q(x,y)\right)\Phi(m)}{M-m}\right.\\ &\left.-\Phi\left(A_q(x,y)\right) + 1\right]^{\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} - 1\\ &\leq \frac{\left(A_p(x,y) - m\right)\Phi(M) + \left(M - A_p(x,y)\right)\Phi(m)}{M-m} - \Phi\left(A_p(x,y)\right)\\ &\leq \left[\frac{\left(A_q(x,y) - m\right)\Phi(M) + \left(M - A_q(x,y)\right)\Phi(m)}{M-m}\right.\\ &\left.-\Phi\left(A_q(x,y)\right) + 1\right]^{\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} - 1\end{aligned}$$

for any x, $y \in [m, M]$ and the convex function $\Phi : [m, M] \to \mathbb{R}$.

By (3.10) and (3.11) we have

$$
\begin{aligned}
&2\min\{p,1-p\} \\
&\times\left[\frac{(A(x,y)-m)\Phi(M)+(M-A(x,y))\Phi(m)}{M-m}-\Phi(A(x,y))\right] \\
&\leq \frac{(A_p(x,y)-m)\Phi(M)+(M-A_p(x,y))\Phi(m)}{M-m}-\Phi(A_p(x,y)) \\
&\leq \max\{p,1-p\} \\
&\times\left[\frac{(A(x,y)-m)\Phi(M)+(M-A(x,y))\Phi(m)}{M-m}-\Phi(A(x,y))\right]
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\Bigg[\frac{(A(x,y)-m)\Phi(M)+(M-A(x,y))\Phi(m)}{M-m} \\
&-\Phi(A(x,y))+1\Bigg]^{2\min\{p,1-p\}}-1 \\
&\leq \frac{(A_p(x,y)-m)\Phi(M)+(M-A_p(x,y))\Phi(m)}{M-m}-\Phi(A_p(x,y)) \\
&\leq \Bigg[\frac{(A(x,y)-m)\Phi(M)+(M-A(x,y))\Phi(m)}{M-m} \\
&-\Phi(A(x,y))+1\Bigg]^{\max\{p,1-p\}}-1,
\end{aligned}
\tag{3.13}
$$

for any x, $y\in[m,M]$ and the convex function $\Phi:[m,M]\to\mathbb{R}$.

4. Applications for Arithmetic and Geometric Means

Consider the *weighted arithmetic* and *geometric means*

$$A_n(p,x):=\sum_{i=1}^n p_i x_i,\ \ G_n(p,x):=\prod_{i=1}^n x_i^{p_i}$$

of positive numbers $x=(x_1,...,x_n)$ with the positive weights $p=(p_1,...,p_n)$, a probability distribution, i.e. $p_i\geq 0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$.

If $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ satisfy the condition (3.1) then by the inequalities (3.2) and (3.3) for the convex function $\Phi(t)=-\ln t$, $t>0$ we have for positive numbers $x=(x_1,...,x_n)$ that [6]

$$\left(\frac{A_n(q,x)}{G_n(q,x)}\right)^r\leq\frac{A_n(p,x)}{G_n(p,x)}\leq\left(\frac{A_n(q,x)}{G_n(q,x)}\right)^R \tag{4.1}$$

and

$$
\begin{aligned}
0 &\leq \exp\left\{\left[\ln\left(\frac{A_n(q,x)}{G_n(q,x)}\right)+1\right]^r-1\right\} \\
&\leq \frac{A_n(p,x)}{G_n(p,x)} \\
&\leq \exp\left\{\left[\ln\left(\frac{A_n(q,x)}{G_n(q,x)}\right)+1\right]^R-1\right\}.
\end{aligned}
\tag{4.2}
$$

If $(x_1,...,x_n)\in[m,M]^n$ for $M>m>0$ then by the inequalities (3.8) and (3.9) we have

$$\left(\frac{A_n(q,x)}{M^{\frac{A_n(q,x)-m}{M-m}}m^{\frac{M-A_n(q,x)}{M-m}}}\right)^r \le \frac{A_n(p,x)}{M^{\frac{A_n(p,x)-m}{M-m}}m^{\frac{M-A_n(p,x)}{M-m}}} \le \left(\frac{A_n(q,x)}{M^{\frac{A_n(q,x)-m}{M-m}}m^{\frac{M-A_n(q,x)}{M-m}}}\right)^R \tag{4.3}$$

and

$$\begin{aligned}&\exp\left\{\left[\ln\left(\frac{A_n(q,x)}{M^{\frac{A_n(q,x)-m}{M-m}}m^{\frac{M-A_n(q,x)}{M-m}}}\right)+1\right]^r-1\right\}\\ &\le \frac{A_n(p,x)}{M^{\frac{A_n(p,x)-m}{M-m}}m^{\frac{M-A_n(p,x)}{M-m}}}\\ &\le \exp\left\{\left[\ln\left(\frac{A_n(q,x)}{M^{\frac{A_n(q,x)-m}{M-m}}m^{\frac{M-A_n(q,x)}{M-m}}}\right)+1\right]^R-1\right\},\end{aligned} \tag{4.4}$$

where $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ satisfy the condition (3.1).

If $p,\ q\in(0,1)$, then by the above inequalities we get

$$\left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \le \frac{A_p(x,y)}{G_p(x,y)} \le \left(\frac{A_q(x,y)}{G_q(x,y)}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} \tag{4.5}$$

and

$$\begin{aligned}0&\le \exp\left\{\left[\ln\left(\frac{A_q(x,y)}{G_q(x,y)}\right)+1\right]^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1\right\}\\ &\le \frac{A_p(x,y)}{G_p(x,y)}\\ &\le \exp\left\{\left[\ln\left(\frac{A_q(x,y)}{G_q(x,y)}\right)+1\right]^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1\right\},\end{aligned} \tag{4.6}$$

for any $x,y>0$.

If $x,\ y\in[m,M]\subset(0,\infty)$ then by (4.4) and (4.5) we have

$$\begin{aligned}\left(\frac{A_q(x,y)}{M^{\frac{A_q(x,y)-m}{M-m}}m^{\frac{M-A_q(x,y)}{M-m}}}\right)^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}} &\le \frac{A_p(x,y)}{M^{\frac{A_p(x,y)-m}{M-m}}m^{\frac{M-A_p(x,y)}{M-m}}}\\ &\le \left(\frac{A_q(x,y)}{M^{\frac{A_q(x,y)-m}{M-m}}m^{\frac{M-A_q(x,y)}{M-m}}}\right)^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}(4.8)\qquad &\exp\left\{\left[\ln\left(\frac{A_q(x,y)}{M^{\frac{A_q(x,y)-m}{M-m}}m^{\frac{M-A_q(x,y)}{M-m}}}\right)+1\right]^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1\right\}\\ &\le \frac{A_p(x,y)}{M^{\frac{A_p(x,y)-m}{M-m}}m^{\frac{M-A_p(x,y)}{M-m}}}\\ &\le \exp\left\{\left[\ln\left(\frac{A_q(x,y)}{M^{\frac{A_q(x,y)-m}{M-m}}m^{\frac{M-A_q(x,y)}{M-m}}}\right)+1\right]^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1\right\},\end{aligned}$$

for any $p,\ q\in(0,1)$.

If we take $q=\frac{1}{2}$ in (4.6) and (4.7) then we get

$$(4.9)\qquad \left(\frac{A(x,y)}{G(x,y)}\right)^{2\min\{p,1-p\}}\le\frac{A_p(x,y)}{G_p(x,y)}\le\left(\frac{A(x,y)}{G(x,y)}\right)^{2\max\{p,1-p\}}$$

and

$$\begin{aligned}(4.10)\qquad 0 &\le \exp\left\{\left[\ln\left(\frac{A(x,y)}{G(x,y)}\right)+1\right]^{2\min\{p,1-p\}}-1\right\}\\ &\le\frac{A_p(x,y)}{G_p(x,y)}\\ &\le \exp\left\{\left[\ln\left(\frac{A(x,y)}{G(x,y)}\right)+1\right]^{2\max\{p,1-p\}}-1\right\},\end{aligned}$$

for any $x,\ y>0$ and $p\in(0,1)$.

The first inequality in (4.9) was obtained by Zou et al. in [20] while the second by Liao et al. [17]. We also have shown in [10] that it can also be obtained from a more general result for convex functions from [6].

For other recent related results, see [1], [12]-[14] and [19] where further references are provided.

If we take $q=\frac{1}{2}$ in (4.7) and (4.8), then we get

$$\begin{aligned}(4.11)\qquad \left(\frac{A(x,y)}{M^{\frac{A(x,y)-m}{M-m}}m^{\frac{M-A(x,y)}{M-m}}}\right)^{2\min\{p,1-p\}} &\le \frac{A_p(x,y)}{M^{\frac{A_p(x,y)-m}{M-m}}m^{\frac{M-A_p(x,y)}{M-m}}}\\ &\le\left(\frac{A(x,y)}{M^{\frac{A(x,y)-m}{M-m}}m^{\frac{M-A(x,y)}{M-m}}}\right)^{2\max\{p,1-p\}}\end{aligned}$$

and

$$\begin{aligned}(4.12)\qquad &\exp\left\{\left[\ln\left(\frac{A(x,y)}{M^{\frac{A(x,y)-m}{M-m}}m^{\frac{M-A(x,y)}{M-m}}}\right)+1\right]^{2\min\{p,1-p\}}-1\right\}\\ &\le \frac{A_p(x,y)}{M^{\frac{A_p(x,y)-m}{M-m}}m^{\frac{M-A_p(x,y)}{M-m}}}\\ &\le \exp\left\{\left[\ln\left(\frac{A(x,y)}{M^{\frac{A(x,y)-m}{M-m}}m^{\frac{M-A(x,y)}{M-m}}}\right)+1\right]^{2\max\{p,1-p\}}-1\right\},\end{aligned}$$

for any $x,y\in[m,M]$.

Let $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ be probability distributions such that there exists $0<r\leq 1\leq R<\infty$ with the property (3.1). By writing the inequalities (3.2) and (3.3) for the convex function $f(t)=\exp t$ we have for $a=(a_1,...,a_n)\in\mathbb{R}^n$ that

$$
\begin{aligned}
0 &\leq r\left[\sum_{i=1}^{n} q_i \exp a_i - \exp\left(\sum_{i=1}^{n} q_i a_i\right)\right] \\
&\leq \sum_{i=1}^{n} p_i \exp a_i - \exp\left(\sum_{i=1}^{n} p_i a_i\right) \\
&\leq R\left[\sum_{i=1}^{n} q_i \exp a_i - \exp\left(\sum_{i=1}^{n} q_i a_i\right)\right]
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
0 &\leq \left[\sum_{i=1}^{n} q_i \exp a_i - \exp\left(\sum_{i=1}^{n} q_i a_i\right)+1\right]^{r}-1 \\
&\leq \sum_{i=1}^{n} p_i \exp a_i - \exp\left(\sum_{i=1}^{n} p_i a_i\right) \\
&\leq \left[\sum_{i=1}^{n} q_i \exp a_i - \exp\left(\sum_{i=1}^{n} q_i a_i\right)+1\right]^{R}-1.
\end{aligned} \tag{4.14}
$$

If we take $a_i=\ln x_i$ for positive numbers $x=(x_1,...,x_n)$, then we get

$$
\begin{aligned}
0 &\leq r\left[A_n(q,x)-G_n(q,x)\right] \leq A_n(p,x)-G_n(p,x) \\
&\leq R\left[A_n(q,x)-G_n(q,x)\right]
\end{aligned} \tag{4.15}
$$

and

$$
\begin{aligned}
0 &\leq \left[A_n(q,x)-G_n(q,x)+1\right]^{r}-1 \leq A_n(p,x)-G_n(p,x) \\
&\leq \left[A_n(q,x)-G_n(q,x)+1\right]^{R}-1.
\end{aligned} \tag{4.16}
$$

If $p,q\in(0,1)$ then by the above inequalities we get

$$
\begin{aligned}
0 &\leq \min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q(x,y)-G_q(x,y)\right] \\
&\leq A_p(x,y)-G_p(x,y) \\
&\leq \max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}\left[A_q(x,y)-G_q(x,y)\right]
\end{aligned} \tag{4.17}
$$

and

$$
\begin{aligned}
0 &\leq \left[A_q(x,y)-G_q(x,y)+1\right]^{\min\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1 \\
&\leq A_p(x,y)-G_p(x,y) \\
&\leq \left[A_q(x,y)-G_q(x,y)+1\right]^{\max\left\{\frac{p}{q},\frac{1-p}{1-q}\right\}}-1,
\end{aligned} \tag{4.18}
$$

for any $x,y>0$.

Moreover, if we take $q=\frac{1}{2}$ in (4.17) and (4.18) we get

$$\begin{aligned}(4.19)\qquad 0 &\le 2\min\{p,1-p\}\left(\sqrt{x}-\sqrt{y}\right)^2 \le A_p(x,y)-G_p(x,y)\\ &\le 2\max\{p,1-p\}\left(\sqrt{x}-\sqrt{y}\right)^2\end{aligned}$$

and

$$\begin{aligned}(4.20)\qquad 0 &\le \left[\left(\sqrt{x}-\sqrt{y}\right)^2+1\right]^{2\min\{p,1-p\}}-1 \le A_p(x,y)-G_p(x,y)\\ &\le \left[\left(\sqrt{x}-\sqrt{y}\right)^2+1\right]^{2\max\{p,1-p\}}-1,\end{aligned}$$

for any x, $y>0$ and $p\in[0,1]$.

The inequality (4.19) has been obtained by Kittaneh and Manasrah [15], [16]. We have shown in [9] that it can also be obtained from a more general result for convex functions from [6].

If $(a_1,...,a_n)\in[k,M]^n$ and $p=(p_1,...,p_n)$ and $q=(q_1,...,q_n)$ are probability distributions such that there exist $0<r\le 1\le R<\infty$ with the property (3.1), then by the inequalities (3.8) and (3.9) for the convex function $\Phi(t)=\exp t$, we have that

$$\begin{aligned}(4.21)\qquad & r\left[\frac{\left(\sum_{i=1}^n q_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n q_ia_i\right)\exp k}{K-k}-\exp\left(\sum_{i=1}^n q_ia_i\right)\right]\\ &\le \frac{\left(\sum_{i=1}^n p_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n p_ia_i\right)\exp k}{K-k}-\exp\left(\sum_{i=1}^n p_ia_i\right)\\ &\le R\left[\frac{\left(\sum_{i=1}^n q_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n q_ia_i\right)\exp k}{K-k}-\exp\left(\sum_{i=1}^n q_ia_i\right)\right]\end{aligned}$$

and

$$\begin{aligned}(4.22)\qquad & \left[\frac{\left(\sum_{i=1}^n q_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n q_ia_i\right)\exp k}{K-k}\right.\\ &\left.-\exp\left(\sum_{i=1}^n q_ia_i\right)+1\right]^r-1\\ &\le \frac{\left(\sum_{i=1}^n p_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n p_ia_i\right)\exp k}{K-k}-\exp\left(\sum_{i=1}^n p_ia_i\right)\\ &\le \left[\frac{\left(\sum_{i=1}^n q_ia_i-k\right)\exp K+\left(K-\sum_{i=1}^n q_ia_i\right)\exp k}{K-k}\right.\\ &\left.-\exp\left(\sum_{i=1}^n q_ia_i\right)+1\right]^R-1\end{aligned}$$

If $(x_1,...,x_n) \in [m,M]^n$ for $M > m > 0$, then by taking in (4.21) and (4.22) $a_i = \ln x_i$ for positive numbers $x = (x_1,...,x_n)$ and $k = \ln m$, $K = \ln M$, we get

$$
\begin{aligned}
(4.23)\qquad & r\left[\frac{(\ln G_n(q,x) - \ln m) M + (\ln M - \ln G_n(q,x)) m}{\ln M - \ln m} - G_n(q,x)\right] \\
&\leq \frac{(\ln G_n(p,x) - \ln m) M + (\ln M - \ln G_n(p,x)) m}{\ln M - \ln m} - G_n(p,x) \\
&\leq R\left[\frac{(\ln G_n(q,x) - \ln m) M + (\ln M - \ln G_n(q,x)) m}{\ln M - \ln m} - G_n(q,x)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(4.24)\qquad & \left[\frac{(\ln G_n(q,x) - \ln m) M + (\ln M - \ln G_n(q,x)) m}{\ln M - \ln m} - G_n(q,x) + 1\right]^r \\
& - 1 \\
&\leq \frac{(\ln G_n(p,x) - \ln m) M + (\ln M - \ln G_n(p,x)) m}{\ln M - \ln m} - G_n(p,x) \\
&\leq \left[\frac{(\ln G_n(q,x) - \ln m) M + (\ln M - \ln G_n(q,x)) m}{\ln M - \ln m} - G_n(q,x) + 1\right]^R \\
& - 1.
\end{aligned}
$$

If p, $q \in (0,1)$ then by the above inequalities we get for x, $y \in [m,M] \subset (0,\infty)$ that

$$
\begin{aligned}
(4.25)\qquad & \min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} \\
& \times\left[\frac{(\ln G_q(x,y) - \ln m) M + (\ln M - \ln G_q(x,y)) m}{\ln M - \ln m} - G_q(x,y)\right] \\
&\leq \frac{(\ln G_p(x,y) - \ln m) M + (\ln M - \ln G_p(x,y)) m}{\ln M - \ln m} - G_p(x,y) \\
&\leq \max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\} \\
& \times\left[\frac{(\ln G_q(x,y) - \ln m) M + (\ln M - \ln G_q(x,y)) m}{\ln M - \ln m} - G_q(x,y)\right]
\end{aligned}
$$

and

(4.26)

$$
\begin{aligned}
& \left[\frac{(\ln G_q(x,y) - \ln m) M + (\ln M - \ln G_q(x,y)) m}{\ln M - \ln m} - G_q(x,y) + 1\right]^{\min\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} \\
& - 1 \\
&\leq \frac{(\ln G_p(x,y) - \ln m) M + (\ln M - \ln G_p(x,y)) m}{\ln M - \ln m} - G_p(x,y) \\
&\leq \left[\frac{(\ln G_q(x,y) - \ln m) M + (\ln M - \ln G_q(x,y)) m}{\ln M - \ln m} - G_q(x,y) + 1\right]^{\max\left\{\frac{p}{q}, \frac{1-p}{1-q}\right\}} \\
& - 1.
\end{aligned}
$$

In particular, for $q=\frac{1}{2}$ we get

$$\begin{aligned}
&2\min\{p,1-p\} \\
&\times\left[\frac{(\ln G(x,y)-\ln m)M+(\ln M-\ln G(x,y))m}{\ln M-\ln m}-G(x,y)\right] \\
&\leq\frac{(\ln G_p(x,y)-\ln m)M+(\ln M-\ln G_p(x,y))m}{\ln M-\ln m}-G_p(x,y) \\
&\leq 2\max\{p,1-p\} \\
&\times\left[\frac{(\ln G(x,y)-\ln m)M+(\ln M-\ln G(x,y))m}{\ln M-\ln m}-G(x,y)\right]
\end{aligned}\tag{4.27}$$

and

$$\begin{aligned}
&\left[\frac{(\ln G(x,y)-\ln m)M+(\ln M-\ln G(x,y))m}{\ln M-\ln m}-G(x,y)+1\right]^{2\min\{p,1-p\}} \\
&-1 \\
&\leq\frac{(\ln G_p(x,y)-\ln m)M+(\ln M-\ln G_p(x,y))m}{\ln M-\ln m}-G_p(x,y) \\
&\leq\left[\frac{(\ln G(x,y)-\ln m)M+(\ln M-\ln G(x,y))m}{\ln M-\ln m}-G(x,y)+1\right]^{2\max\{p,1-p\}} \\
&-1.
\end{aligned}\tag{4.28}$$

for any $p\in[0,1]$ and $x,\,y\in[m,M]$.

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