

SOME PERTURBED OSTROWSKI TYPE INEQUALITY FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE OF BOUNDED VARIATION

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ABSTRACT. The main aim of this paper is to establish some new perturbed Ostrowski type integral inequalities for functions whose first derivatives are of bounded variation. Some perturbed Ostrowski type inequalities for Lipschitzian and monotonic mappings are also obtained.

1. INTRODUCTION

In 1938, Ostrowski [21] established a following useful inequality:

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, we have the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible.

The following definitions will be frequently used to prove our results.

Definition 1. *Let $P : a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^m |\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^n |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number*

$$\bigvee_a^b(f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

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In [15], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$(1.2) \quad \left| \int_a^b f(t)dt - (b-a)f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

In the past, many authors have worked on Ostrowski type inequalities for function of bounded variation, see for example ([1]-[4],[6],[9],[12]-[17],[20]).

For a function of bounded variation $v : [a, b] \rightarrow \mathbb{C}$, we define the *Cumulative Variation Function* (CVF) $V : [a, b] \rightarrow [0, \infty)$ by

$$V(t) := \bigvee_a^t(v),$$

the total variation of v on the interval $[a, t]$ with $t \in [a, b]$.

It is known that the CVF is monotonic nondecreasing on $[a, b]$ and is continuous in a point $c \in [a, b]$ if and only if the generating function v is continuing in that point. If v is *Lipschitzian* with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \leq L |t - s|, \text{ for any } t, s \in [a, b],$$

then V is also Lipschitzian with the same constant.

A simple proof of the following Lemma was given in [16].

Lemma 1. *Let $f, u : [a, b] \rightarrow \mathbb{C}$. If f is continuous on $[a, b]$ and u is of bounded variation on $[a, b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ exist and*

$$(1.3) \quad \left| \int_a^b f(t)du(t) \right| \leq \int_a^b |f(t)| d \left(\bigvee_a^t(u) \right) \leq \max_{t \in [a, b]} |f(t)| \bigvee_a^b(u).$$

In [8], authors gave the following Ostrowski type inequality for mapping whose first derivatives are of bounded variation:

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\ & \quad \left. + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\ & \leq \frac{1}{16} \left[\frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4 \left| x - \frac{3a+b}{4} \right| \right] \bigvee_a^b(f') \end{aligned}$$

for any $x \in [a, \frac{a+b}{2}]$.

For recent related results, see [5],[7] and [9]. Moreover, Dragomir proved some perturbed Ostrowski type inequalities for functions of bounded variation in [18, 19]. The aim of this paper is to obtain new perturbed Ostrowski type inequalities for mappings whose first derivatives are of bounded variation.

2. SOME IDENTITIES

Before we start our main results, we state and prove following lemma:

Lemma 2. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a twice differentiable function on (a, b) . Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex number the following identity holds*

$$\begin{aligned}
 (2.1) \quad & \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \\
 & - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \\
 = & \frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] + \frac{1}{b-a} \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \right],
 \end{aligned}$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. Using the integration by parts for Riemann-Stieltjes, we have

$$\begin{aligned}
 (2.2) \quad & \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] \\
 = & \int_a^x (t-a)^2 df'(t) - \lambda_1(x) \int_a^x (t-a)^2 dt \\
 = & (t-a)^2 f'(t) \Big|_a^x - 2 \int_a^x (t-a) f'(t) dt - \frac{\lambda_1(x)}{3} (t-a)^3 \Big|_a^x \\
 = & (x-a)^2 f'(x) - 2 \left[(t-a) f(t) \Big|_a^x - \int_a^x f(t) dt \right] - \frac{\lambda_1(x)}{3} (x-a)^3 \\
 = & (x-a)^2 f'(x) - 2(x-a) f(x) + 2 \int_a^x f(t) dt - \frac{\lambda_1(x)}{3} (x-a)^3
 \end{aligned}$$

and

$$\begin{aligned}
(2.3) \quad & \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \\
&= \int_x^b (t-b)^2 df'(t) - \lambda_2(x) \int_x^b (t-b)^2 dt \\
&= (t-b)^2 f'(t) \Big|_x^b - 2 \int_x^b (t-b) f'(t) dt - \frac{\lambda_1(x)}{3} (t-b)^3 \Big|_x^b \\
&= -(b-x)^2 f'(x) - 2 \left[(t-b) f(t) \Big|_x^b - \int_x^b f(t) dt \right] - \frac{\lambda_2(x)}{3} (b-x)^3 \\
&= (b-x)^2 f'(x) - 2(b-x) f(x) + 2 \int_x^b f(t) dt - \frac{\lambda_1(x)}{3} (x-a)^3.
\end{aligned}$$

If we add the equality (2.2) and (2.3) and divide by $2(b-a)$, we obtain required identity. \square

Corollary 1. *Under assumption of Lemma 2 with $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\begin{aligned}
(2.4) \quad & \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt - \frac{\lambda(x)}{6(b-a)} [(x-a)^3 + (b-x)^3] \\
&= \frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 d[f'(t) - \lambda(x)t] + \frac{1}{b-a} \int_x^b (t-b)^2 d[f'(t) - \lambda(x)t] \right]
\end{aligned}$$

for all $x \in [a, b]$.

Remark 1. *If we choose $\lambda(x) = 0$ in (2.4), then we have the following identity*

$$\begin{aligned}
(2.5) \quad & \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \\
&= \frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 df'(t) + \frac{1}{b-a} \int_x^b (t-b)^2 df'(t) \right]
\end{aligned}$$

for all $x \in [a, b]$.

Corollary 2. *Under assumption of Lemma 2 with $\lambda_1(x) = \lambda_1 \in \mathbb{C}$ and $\lambda_2(x) = \lambda_2 \in \mathbb{C}$, we get*

$$\begin{aligned}
 (2.6) \quad & \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \\
 & - \frac{1}{6(b-a)} [\lambda_1(x-a)^3 + \lambda_2(b-x)^3] \\
 = & \frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 d[f'(t) - \lambda_1 t] + \frac{1}{b-a} \int_x^b (t-b)^2 d[f'(t) - \lambda_2 t] \right].
 \end{aligned}$$

In particular, taking $\lambda_1 = \lambda_2 = \lambda$ we have

$$\begin{aligned}
 (2.7) \quad & \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt - \frac{\lambda}{6(b-a)} [(x-a)^3 + (b-x)^3] \\
 = & \frac{1}{2} \left[\frac{1}{b-a} \int_a^x (t-a)^2 d[f'(t) - \lambda t] + \frac{1}{b-a} \int_x^b (t-b)^2 d[f'(t) - \lambda t] \right].
 \end{aligned}$$

3. INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE OF BOUNDED VARIATION

We denote by $\ell : [a, b] \rightarrow [a, b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a, b]$.

Theorem 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a twice differentiable function on I° and $[a, b] \subset I^\circ$. If the first derivative f' is of bounded variation on $[a, b]$, then*

$$\begin{aligned}
 (3.1) \quad & \left| \left(x - \frac{a+b}{2}\right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\
 & \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\
 \leq & \frac{1}{(b-a)} \left[\int_a^x (t-a) \left(\bigvee_t^x (f' - \lambda_1(x)\ell) \right) dt + \int_x^b (b-t) \left(\bigvee_x^t (f' - \lambda_2(x)\ell) \right) dt \right] \\
 \leq & \frac{1}{2(b-a)} \left[(x-a)^2 \bigvee_a^x (f' - \lambda_1(x)\ell) + (b-x)^2 \bigvee_x^b (f' - \lambda_2(x)\ell) \right]
 \end{aligned}$$

$$\leq \frac{1}{2(b-a)} \times \begin{cases} \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \max \left\{ \bigvee_a^x (f' - \lambda_1(x)\ell), \bigvee_x^b (f' - \lambda_2(x)\ell) \right\} (b-a)^2, \\ \max \{ (x-a)^2, (b-x)^2 \} \left[\bigvee_a^x (f' - \lambda_1(x)\ell) + \bigvee_x^b (f' - \lambda_2(x)\ell) \right] \end{cases}$$

for any $x \in [a, b]$.

Proof. Taking modulus (2.1) and applying Lemma 1, we get

$$\begin{aligned} (3.2) & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\ & \quad \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\ & \leq \frac{1}{2} \left[\frac{1}{b-a} \left| \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] \right| + \frac{1}{b-a} \left| \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \right| \right] \\ & \leq \frac{1}{2(b-a)} \left[\int_a^x (t-a)^2 d \left(\bigvee_a^t (f' - \lambda_1(x)\ell) \right) + \int_x^b (t-b)^2 d \left(\bigvee_a^t (f' - \lambda_2(x)\ell) \right) \right]. \end{aligned}$$

Integrating by parts in the Riemann-Stieltjes integral, we get

$$\begin{aligned} (3.3) \quad & \int_a^x (t-a)^2 d \left(\bigvee_a^t (f' - \lambda_1(x)\ell) \right) \\ & = (t-a)^2 \bigvee_a^t (f' - \lambda_1(x)\ell) \Big|_a^x - 2 \int_a^x (t-a) \left(\bigvee_a^t (f' - \lambda_1(x)\ell) \right) dt \\ & = (x-a)^2 \bigvee_a^x (f' - \lambda_1(x)\ell) - 2 \int_a^x (t-a) \left(\bigvee_a^t (f' - \lambda_1(x)\ell) \right) dt \\ & = 2 \int_a^x (t-a) \left(\bigvee_a^x (f' - \lambda_1(x)\ell) \right) dt - 2 \int_a^x (t-a) \left(\bigvee_a^t (f' - \lambda_1(x)\ell) \right) dt \\ & = 2 \int_a^x (t-a) \left(\bigvee_t^x (f' - \lambda_1(x)\ell) \right) dt \end{aligned}$$

and

$$\begin{aligned}
(3.4) \quad & \int_x^b (t-b)^2 d \left(\bigvee_a^t (f' - \lambda_2(x)\ell) \right) \\
&= (t-b)^2 \bigvee_a^t (f' - \lambda_2(x)\ell) \Big|_x^b - 2 \int_x^b (t-b) \left(\bigvee_a^t (f' - \lambda_2(x)\ell) \right) dt \\
&= -(x-b)^2 \bigvee_a^x (f' - \lambda_2(x)\ell) - 2 \int_x^b (t-b) \left(\bigvee_a^t (f' - \lambda_2(x)\ell) \right) dt \\
&= -2 \int_x^b (b-t) \left(\bigvee_a^x (f' - \lambda_2(x)\ell) \right) dt + 2 \int_x^b (b-t) \left(\bigvee_a^t (f' - \lambda_2(x)\ell) \right) dt \\
&= 2 \int_x^b (b-t) \left(\bigvee_x^t (f' - \lambda_2(x)\ell) \right) dt.
\end{aligned}$$

If we put the identities (3.3) and (3.4) in (3.2), then we obtain the first inequality in (3.1). Moreover, we have,

$$(3.5) \quad \int_a^x (t-a) \left(\bigvee_t^x (f' - \lambda_1(x)\ell) \right) dt \leq \frac{1}{2} (x-a)^2 \bigvee_a^x (f' - \lambda_1(x)\ell)$$

and

$$(3.6) \quad \int_x^b (b-t) \left(\bigvee_x^t (f' - \lambda_2(x)\ell) \right) dt \leq \frac{1}{2} (b-x)^2 \bigvee_x^b (f' - \lambda_2(x)\ell).$$

With the inequalities (3.5) and (3.6), the proof of Theorem is completed. \square

Corollary 3. *If we choose $\lambda_1(x) = \lambda_2(x) = 0$, then we have the following inequality*

$$\begin{aligned}
& \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{1}{(b-a)} \left[\int_a^x (t-a) \left(\bigvee_t(f') \right) dt + \int_x^b (b-t) \left(\bigvee_x(f' \ell) \right) dt \right] \\
& \leq \frac{1}{2(b-a)} \left[(x-a)^2 \bigvee_a^x(f') + (b-x)^2 \bigvee_x^b(f') \right] \\
& \leq \frac{1}{2(b-a)} \begin{cases} \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] \left[\frac{1}{2} \bigvee_a^b(f') + \frac{1}{2} \left| \bigvee_a^x(f') - \bigvee_x^b(f') \right| \right] (b-a)^2, \\ \max \{ (x-a)^2, (b-x)^2 \} \bigvee_a^b(f') \end{cases}
\end{aligned}$$

for all $x \in [a, b]$.

Corollary 4. *Under assumption of Theorem 4 with $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\begin{aligned}
(3.7) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt - \frac{\lambda(x)}{6(b-a)} [(x-a)^3 + (b-x)^3] \right| \\
& \leq \frac{1}{(b-a)} \left[\int_a^x (t-a) \left(\bigvee_t(f' - \lambda(x)\ell) \right) dt + \int_x^b (b-t) \left(\bigvee_x(f' - \lambda(x)\ell) \right) dt \right] \\
& \leq \frac{1}{2(b-a)} \left[(x-a)^2 \bigvee_a^x(f' - \lambda(x)\ell) + (b-x)^2 \bigvee_x^b(f' - \lambda(x)\ell) \right] \\
& \leq \frac{1}{2(b-a)} \begin{cases} (b-a)^2 \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] \\ \times \left[\frac{1}{2} \bigvee_a^b(f' - \lambda(x)\ell) + \frac{1}{2} \left| \bigvee_a^x(f' - \lambda(x)\ell) - \bigvee_x^b(f' - \lambda(x)\ell) \right| \right], \\ \max \{ (x-a)^2, (b-x)^2 \} \bigvee_a^b(f' - \lambda(x)\ell) \end{cases}
\end{aligned}$$

for all $x \in [a, b]$.

Corollary 5. *If we choose $\lambda(x) = \lambda$ and $x = \frac{a+b}{2}$ in (3.7), then we have the following identity*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) - \frac{\lambda(b-a)^2}{24} \right| \\ & \leq \frac{1}{(b-a)} \left[\int_a^{\frac{a+b}{2}} (t-a) \left(\bigvee_t^{\frac{a+b}{2}} (f' - \lambda \ell) \right) dt + \int_{\frac{a+b}{2}}^b (b-t) \left(\bigvee_t^{\frac{a+b}{2}} (f' - \lambda \ell) \right) dt \right] \\ & \leq \frac{(b-a)}{8} \bigvee_a^b (f' - \lambda(x) \ell). \end{aligned}$$

4. INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE LIPSCHITZIAN

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a twice differentiable function on I° and $[a, b] \subset I^\circ$. If there exist the positive numbers $K_1(x)$ and $K_2(x)$ such that $f' - \lambda_1(x)\ell$ is Lipschitzian with the constant $K_1(x)$ on the interval $[a, x]$ and $f' - \lambda_2(x)\ell$ is Lipschitzian with the constant $K_2(x)$ on the interval $[x, b]$, then we have for any $x \in [a, b]$*

$$\begin{aligned} (4.1) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\ & \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\ & \leq \frac{(b-a)^2}{6} \left[K_1(x) \left(\frac{x-a}{b-a} \right)^3 + K_2(x) \left(\frac{b-x}{b-a} \right)^3 \right] \\ & \leq \frac{(b-a)^2}{6} \begin{cases} \left[\left(\frac{x-a}{b-a} \right)^3 + \left(\frac{b-x}{b-a} \right)^3 \right] \max \{ K_1(x), K_2(x) \}, \\ \left[\left(\frac{x-a}{b-a} \right)^{3p} + \left(\frac{b-x}{b-a} \right)^{3p} \right]^{\frac{1}{p}} [(K_1(x))^q + (K_2(x))^q]^{\frac{1}{q}}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^3 [K_1(x) + K_2(x)]. \end{cases} \end{aligned}$$

Proof. It is known that, if $g : [c, d] \rightarrow \mathbb{C}$ is Riemann integrable and $u : [c, d] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then the Riemann-Stieltje integral $\int_c^d g(t) du(t)$ exist and

$$\left| \int_c^d g(t) du(t) \right| \leq K \int_c^d |g(t)| dt.$$

Taking the madulus (2.1), we get

$$\begin{aligned}
& \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\
& \quad \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\
& \leq \frac{1}{2(b-a)} \left[\left| \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] \right| + \left| \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \right| \right] \\
& \leq \frac{1}{2(b-a)} \left[K_1(x) \int_a^x |(t-a)^2| dt + K_2(x) \int_x^b |(t-b)^2| dt \right] \\
& = \frac{(b-a)^2}{6} \left[K_1(x) (x-a)^3 + K_2(x) (b-x)^3 \right] \\
& = \frac{(b-a)^2}{6} \left[K_1(x) \left(\frac{x-a}{b-a} \right)^3 + K_2(x) \left(\frac{b-x}{b-a} \right)^3 \right].
\end{aligned}$$

This completes the proof of first inequality in (4.1).

Using the Hölder's inequality, we have

$$\begin{aligned}
& K_1(x) \left(\frac{x-a}{b-a} \right)^3 + K_2(x) \left(\frac{b-x}{b-a} \right)^3 \\
& \leq \begin{cases} \left[\left(\frac{x-a}{b-a} \right)^3 + \left(\frac{b-x}{b-a} \right)^3 \right] \max \{ K_1(x), K_2(x) \}, \\ \left[\left(\frac{x-a}{b-a} \right)^{3p} + \left(\frac{b-x}{b-a} \right)^{3p} \right]^{\frac{1}{p}} [(K_1(x))^q + (K_2(x))^q]^{\frac{1}{q}} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^3 [L_1(x) + L_2(x)] \end{cases}
\end{aligned}$$

which completes the proof. \square

Corollary 6. *Under assumption of Theorem 5 with $K_1(x) = K_2(x) = K$ and $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\begin{aligned}
(4.2) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt - \frac{\lambda(x)}{6(b-a)} [(x-a)^3 + (b-x)^3] \right| \\
& \leq \frac{1}{6} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^3 K(b-a)^2.
\end{aligned}$$

Corollary 7. *If we choose $x = \frac{a+b}{2}$ and $\lambda(x) = \lambda \in \mathbb{C}$ in (4.2), we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) - \frac{\lambda(b-a)^2}{48} \right| \leq \frac{1}{48} K(b-a)^2.$$

5. INEQUALITIES FOR MAPPINGS WHOSE FIRST DERIVATIVES ARE MONOTONIC FUNCTION

Theorem 6. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a twice differentiable function on I° and $[a, b] \subset I^\circ$. If $\lambda_1(x)$ and $\lambda_2(x)$ are real numbers such that $f' - \lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a, x]$ and $f' - \lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x, b]$, then for any $x \in [a, b]$ the following inequalities hold:*

$$\begin{aligned} (5.1) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\ & \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\ & \leq \frac{1}{2(b-a)} \left[(x-a)^2 [f'(x) - f'(a) - \lambda_1(x)(x-a)] \right. \\ & \quad \left. + (b-x)^2 [f'(b) - f'(x) - \lambda_2(x)(b-x)] \right] \\ & \leq \frac{1}{2(b-a)} \left\{ \begin{aligned} & \left[\frac{1}{2} [f'(b) - f'(a) - \lambda_1(x)(x-a) - \lambda_2(x)(b-x)] \right. \\ & \quad \left. + \left| f'(x) - \frac{f'(a)+f'(b)}{2} - \frac{1}{2}\lambda_1(x)(x-a) + \frac{1}{2}\lambda_2(x)(b-x) \right| \right] \\ & \quad \times \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)^2, \\ & \max \left\{ (x-a)^2, (b-x)^2 \right\} \\ & \quad \times [f'(b) - f'(a) - \lambda_1(x)(x-a) - \lambda_2(x)(b-x)]. \end{aligned} \right. \end{aligned}$$

Proof. Taking the madulus (2.1), we have

$$\begin{aligned} (5.2) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t) dt \right. \\ & \left. - \frac{1}{2(b-a)} \left[\frac{\lambda_1(x)(x-a)^3 + \lambda_2(x)(b-x)^3}{3} \right] \right| \\ & \leq \frac{1}{2(b-a)} \left[\left| \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] \right| + \left| \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \right| \right] \end{aligned}$$

Since $f' - \lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a, x]$, we have

$$\begin{aligned}
 (5.3) \quad & \int_a^x (t-a)^2 d[f'(t) - \lambda_1(x)t] \\
 & \leq (x-a)^2 [f'(x) - \lambda_1(x)x - f'(a) + \lambda_1(x)a] \\
 & = (x-a)^2 [f'(x) - f'(a) - \lambda_1(x)(x-a)]
 \end{aligned}$$

and similarly, since $f' - \lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x, b]$, we have

$$\begin{aligned}
 (5.4) \quad & \int_x^b (t-b)^2 d[f'(t) - \lambda_2(x)t] \\
 & \leq (b-x)^2 [f'(b) - \lambda_2(x)b - f'(x) + \lambda_2(x)x] \\
 & = (b-x)^2 [f'(b) - f'(x) - \lambda_2(x)(b-x)].
 \end{aligned}$$

If we put (5.3) and (5.4) in (5.2), we obtain the first inequality in (5.1).

The proofs of last inequalities are obvious, they are omitted. \square

Corollary 8. *Under assumption of Theorem 6 with $\lambda_1(x) = \lambda_2(x) = \lambda(x)$, we have*

$$\begin{aligned}
 (5.5) \quad & \left| \left(x - \frac{a+b}{2} \right) f'(x) - f(x) + \frac{1}{b-a} \int_a^b f(t)dt - \frac{\lambda(x)}{6(b-a)} [(x-a)^3 + (b-x)^3] \right| \\
 & \leq \frac{1}{2(b-a)} \left[(x-a)^2 [f'(x) - f'(a) - \lambda(x)(x-a)] \right. \\
 & \quad \left. + (b-x)^2 [f'(b) - f'(x) - \lambda(x)(b-x)] \right] \\
 & \leq \frac{1}{2(b-a)} \\
 & \quad \times \begin{cases} \left[\frac{f'(b)-f'(a)}{2} - \frac{1}{2}\lambda(x)(b-a) \right] \left| f'(x) - \frac{f'(a)+f'(b)}{2} - \lambda(x)\left(x - \frac{a+b}{2}\right) \right| \\ \times \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)^2, \\ \max \left\{ (x-a)^2, (b-x)^2 \right\} [f'(b) - f'(a) - \lambda(x)(b-a)]. \end{cases}
 \end{aligned}$$

Corollary 9. *If we choose $x = \frac{a+b}{2}$ and $\lambda(x) = \lambda$ in (5.5), we get the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{\lambda(b-a)^2}{48} \right| \leq \frac{(b-a)}{8} [f'(b) - f'(a) - \lambda(b-a)].$$

REFERENCES

- [1] M. W. Alomari, *A Generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.
- [2] M. W. Alomari and M.A. Latif, *Weighted companion for the Ostrowski and the generalized trapezoid inequalities for mappings of vounded variation*, RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.
- [3] M.W. Alomari and S.S. Dragomir, *Mercer–Trapezoid rule for the Riemann–Stieltjes integral with applications*, Journal of Advances in Mathematics, 2 (2)(2013), 67–85.
- [4] H. Budak and M.Z. Sarikaya, *On generalization of Dragomir’s inequalities*, RGMIA Research Report Collection, 17(2014), Article 155, 10 pp.
- [5] H. Budak and M.Z. Sarikaya, *New weighted Ostrowski type inequalities for mappings with first derivatives of bounded variation*, RGMIA Research Report Collection, 18(2015), Article 43, 8 pp.
- [6] H. Budak and M.Z. Sarikaya, *A new generalization of Ostrowski type inequality for mappings of bounded variation*, RGMIA Research Report Collection, 18(2015), Article 47, 9 pp.
- [7] H. Budak and M.Z. Sarikaya, *On generalization of weighted Ostrowski type inequalities for functions of bounded variation*, RGMIA Research Report Collection, 18(2015), Article
- [8] H. Budak and M. Z. Sarikaya, *A new Ostrowski type inequality for functions whose first derivatives are of bounded variation*, Moroccan Journal of Pure and Applied Analysis 2(1)(2016), 1–11.
- [9] H. Budak and M.Z. Sarikaya, *A companion of Ostrowski type inequalities for mappings of bounded variation and some applications*, RGMIA Research Report Collection, 19(2016), Article 24, 10 pp.
- [10] H. Budak, M.Z. Sarikaya and A. Qayyum, *Improvement in companion of Ostrowski type inequalities for mappings whose first derivatives are of bounded variation and application*, RGMIA Research Report Collection, 19(2016), Article 25, 11 pp.
- [11] P. Cerone, S. S. Dragomir and E. Kikianty, *Jensen–Ostrowski Type Inequalities and Applications for f -Divergence Measures*, Applied Mathematics and Computation 266 (2015) 304–315.
- [12] S. S. Dragomir, *Approximating real functions which possess n th derivatives of bounded variation and applications*, Computers and Mathematics with Applications 56(2008) 2268–2278.
- [13] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bulletin of the Australian Mathematical Society, 60(1) (1999), 495–508.
- [14] S. S. Dragomir, *On the Ostrowski’s integral inequality for mappings with bounded variation and applications*, Mathematical Inequalities & Applications, 4 (2001), no. 1, 59–66.
- [15] S. S. Dragomir, *A companion of Ostrowski’s inequality for functions of bounded variation and applications*, International Journal of Nonlinear Analysis and Applications, 5 (2014) No. 1, 89–97 pp.
- [16] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*. Arch. Math. (Basel) 91 (2008), no. 5, 450–460.
- [17] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Research Report Collection, 16 (2013), Art. 93.
- [18] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, RGMIA Research Report Collection, 16(2013), Article 93, 14 pp.
- [19] S. S. Dragomir, *Perturbed companions of Ostrowski’s inequality for functions of bounded variation*, RGMIA Research Report Collection, 17(2014), Article 1, 16 pp.
- [20] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, (2012).
- [21] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*, Comment. Math. Helv. 10(1938), 226–227.

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