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INEQUALITIES OF TURÁN TYPE FOR MITTAG–LEFFLER FUNCTIONS

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Abstract. In this paper some Turán type inequalities for Mittag–Leffler functions are considered. The method is based on proving monotonicity for special ratio of sections for series of Mittag–Leffler functions. Some applications are considered to Lazarević and Wilker–type inequalities for Mittag–Leffler functions.

keywords: Mittag–Leffler functions; Turán type inequalities; Lazarević–type inequalities; Wilker–type inequalities.

MSC (2010): 33E12, 26D07.

1. Introduction

We use a definition of Mittag–Leffler function by its series

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \ z\in\mathbb{C},\ \alpha,\beta\in\mathbb{C},\ Re(\alpha)>0,\ Re(\beta)>0. \tag{1}$$

This function was first introduced by G. Mittag–Leffler in 1903 for $\alpha=1$ and by A. Wiman in 1905 for the general case (1). For the mathematical theory and properties of Mittag–Leffler functions cf. [7], [10], [45].

First applications of the function (1) by Mittag–Leffler and Viman were in complex function theory (non–trivial examples of entire functions with fractional growth characteristics such as order and generalized summation methods). But its really important applications were found in 20–th century for fractional integral and differential equations. The most known result in this field is an explicit formula for the resolvent of Riemann–Liouville fractional integral proved by E. Hille and J. Tamarkin in 1930. On this and similar formulas many results are based still for solving fractional integral and differential equations. For numerous applications of the Mittag–Leffler function to fractional calculus cf. [35], [10], [6], [15], [33]. Due to many useful applications it was crowned by R. Gorenflo and F. Mainardi in [9] as a *"Queen function of Fractional Calculus"*! Besides fractional calculus the Mittag–Leffler function also plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, signal processing and others.

There are further related generalizations of the Mittag–Leffler function, namely Wright and Fox functions. Wright functions are defined in the same way as (1) but with more gamma–functions both in numerator and denominator, sometimes these functions are called "multi–indexed Mittag–Leffler functions", cf. [17],[18], [40]. Fox function is defined by the Mellin transform, cf. [16], [23], [24], [40], [41], it has also important applications e.g. in fractional diffusion theory, cf. [19], [20], [8].

An important result which initiated a new field of researches on inequalities for special functions was proved by Paul Turán, it is:

$$P_{n+1}(x)P_{n-1}(x)\leq[P_n(x)]^2, \tag{2}$$

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where $-1 < x < 1$, $n \in \mathbb{N}$ and $P_n(\cdot)$ stands for the classical Legendre polynomial. This inequality was published by Turán in 1950 in [42] but proved earlier in 1946 in a letter to Szegö. Since the publication of the above Turán inequality in 1948 by Szegö [44] many authors derived results of such type for classical orthogonal polynomials and different special functions. The Turán type inequalities now have an extensive literature and some of the results have been applied successfully to different problems in information theory, economic theory, biophysics, probability and statistics. For more details cf. [1], [2], [3], [22], [25], [43]. The results on Turán type inequalities are closely connected with log–convex and log–concave functions, cf. [11], [12], [13], [14].

Now Turán–type inequalities are proved for different classes of special functions: Kummer hypergeometric functions (cf. [3], [26], [27], [28], [36], [37], [38]), Gauss hypergeometric functions (cf. [13], [14], [26], [27], [28]), different types of Bessel functions (cf. [1], [2]), Dunkl kernel and q–Dunkl kernel (cf. [25]), $q-$Kummer hypergeometric functions (cf. [29], [30], [31]) and some others.

This paper is a continuation of some line of authors results. In 1990 one of the authors studied inequalities for sections of series for exponential function in [36]. Among other results in [36] a conjecture was proposed on monotonicity of ratios for Kummer hypergeometric function, cf. also [37]–[38]. This conjecture was proved recently by the authors in [26]–[27], cf. also [28]. After that q–versions of these results were proved in [29]–[30], cf. also [31].

The paper is organized as follows. In section 2 we collect some lemmas. In section 3 we give some Turán type inequalities for Mittag–Leffler functions. Moreover, we prove monotonicity of ratios for sections of series of Mittag–Leffler functions, the result is also closely connected with Turán-type inequalities. In section 4 we deduce new inequalities of Lazarević–type and Wilker–type for Mittag–Leffler functions.

2. **Useful lemmas**

We need the following two useful lemmas proved in [4], [34].

Lemma 1. *Let (a_n) and (b_n) $(n = 0, 1, 2...)$ be real numbers, such that $b_n > 0,\ n = 0, 1, 2, ...$ and $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is increasing (decreasing), then $\left(\frac{a_0+...+a_n}{b_0+...+b_n}\right)_n$ is also increasing (decreasing).*

Lemma 2. *Let (a_n) and (b_n) $(n = 0, 1, 2...)$ be real numbers and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < r$. If $b_n > 0,\ n = 0, 1, 2, ...$ and the sequence $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is (strictly) increasing (decreasing) , then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing on $[0, r)$.*

3. **Turán type inequalities for Mittag–Leffler functions**

Our first result is the next theorem.

Theorem 1. *Let $\alpha, \beta > 0$. Then the following assertions are true.*

a. *The function $\beta \mapsto \Gamma(\beta)E_{\alpha,\beta}(z)$ is log–convex on $(0, \infty)$.*

b. *The following Turán type inequality*

$$E^2_{\alpha,\beta+1}(z) \leq \frac{\beta+1}{\beta} E_{\alpha,\beta}(z)E_{\alpha,\beta+2}(z) \tag{3}$$

holds for all $z \in (0, \infty)$.

In particular, the following inequality

$$(e^z - 1)^2 \leq \frac{1}{3} e^z (6(e^z - 1) - z) \tag{4}$$

is valid for all $z>0$.

c. *For $n\in\mathbb{N}$ define the function $E^n_{\alpha,\beta}(z)$ by*

$$E^n_{\alpha,\beta}(z)=E_{\alpha,\beta}(z)-\sum_{k=0}^{n}\frac{z^k}{\Gamma(\alpha k+\beta)}=\sum_{k=n+1}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

Then the following Turán type inequality

$$E^n_{\alpha,\beta}(z)E^{n+2}_{\alpha,\beta}(z)\leq[E^{n+1}_{\alpha,\beta}(z)]^2. \tag{5}$$

is valid for all $n\in\mathbb{N}$, and $\alpha,\beta>0$ and $z>0$.

Proof. **a.** For log–convexity of $\beta\mapsto\Gamma(\beta)E_{\alpha,\beta}(z)$ we observe that it is enough to show the log–convexity of each individual term and to use the fact that the sum of log–convex functions is log–convex too. Thus, we just need to show that for each $k\geq 0$ we have

$$\frac{\partial^2}{\partial\beta^2}\log\left[\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right]=\psi^{'}(\beta)-\psi^{'}(\beta+\alpha k)\geq 0,$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the so–called digamma function. But ψ is known to be concave, and consequently the function $\beta\mapsto\frac{\Gamma(\beta)}{\Gamma(\beta+k\alpha)}$ is log–convex on $(0,\infty)$.

b. Since the function $\beta\mapsto\Gamma(\beta)E_{\alpha,\beta}(z)$ is log–convex, then for all $\beta_1,\beta_2>0,\ z>0$ and $t\in[0,1]$ we have

$$\Gamma(t\beta_1+(1-t)\beta_2)E_{\alpha,t\beta_1+(1-t)\beta_2}(z)\leq[\Gamma(\beta_1)E_{\alpha,\beta_1}(z)]^t[\Gamma(\beta_2)E_{\alpha,\beta_2}(z)]^{1-t}.$$

Now choosing $t=1/2,\ \beta_1=\beta,\ \beta_2=\beta+2$ we conclude that (3) holds. To prove the inequality (4) choose $\alpha=\beta=1$ in(3) and use a recurrence relation from ([24], Theorem 5.1)

$$E_{\alpha,\beta}(z)=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)}.$$

c. Let $n\in\mathbb{N}$, from the definition of the function $E_n(\alpha,\beta,z)$, we have

$$E^n_{\alpha,\beta}(z)=E^{n+1}_{\alpha,\beta}(z)+\frac{z^{n+1}}{\Gamma(\beta+(n+1)\alpha)}$$

and

$$E^{n+2}_{\alpha,\beta}(z)=E^{n+1}_{\alpha,\beta}(z)-\frac{z^{n+2}}{\Gamma(\beta+(n+2)\alpha)}.$$

It implies that

$$\begin{aligned}
&E^n_{\alpha,\beta}(z)E^{n+2}_{\alpha,\beta}(z)-[E^{n+1}_{\alpha,\beta}(z)]^2=\\
&=E^{n+1}_{\alpha,\beta}(z)\left(\frac{z^{n+1}}{\Gamma(\beta+(n+1)\alpha)}-\frac{z^{n+2}}{\Gamma(\beta+(n+2)\alpha)}\right)-\frac{z^{2n+3}}{\Gamma(\beta+(n+1)\alpha)\Gamma(\beta+(n+2)\alpha)}\\
&=\sum_{k=n+3}^{\infty}\left[\frac{1}{\Gamma(\beta+(n+1)\alpha)\Gamma(\beta+k\alpha)}-\frac{1}{\Gamma(\beta+(n+2)\alpha)\Gamma(\beta+(k-1)\alpha)}\right]z^{k+n+1}\\
&=\sum_{k=n+3}^{\infty}\frac{A_k(\alpha,\beta)}{\Gamma(\beta+(n+1)\alpha)\Gamma(\beta+k\alpha)\Gamma(\beta+(n+2)\alpha)\Gamma(\beta+(k-1)\alpha)}z^{k+n+1},
\end{aligned}$$

where

$$A_k(\alpha,\beta)=\Gamma(\beta+(n+2)\alpha)\Gamma(\beta+(k-1)\alpha)-\Gamma(\beta+(n+1)\alpha)\Gamma(\beta+k\alpha).$$

Due to log–convexity of $\Gamma(x)$, the ratio $x \mapsto \frac{\Gamma(x+a)}{\Gamma(x)}$ is increasing on $(0, \infty)$ when $a > 0$. It implies the following inequality

$$\frac{\Gamma(x+a)}{\Gamma(x)} \leq \frac{\Gamma(x+a+b)}{\Gamma(x+b)}. \tag{6}$$

holds for all $a, b > 0$. For $n \geq 0$ and $k \geq n+3$, let $x = \beta + (n+1)\alpha,\ a = \alpha\ , \beta = \alpha(k-(n+2))$ in (6) we conclude that $A_k(\alpha, \beta) \leq 0$, and so the Turán type inequality (5) is proved. ■

Corollary 1. *The following Turán type inequality*

$$E_{\alpha,\beta+(n+1)\alpha}(z)E_{\alpha,\beta+(n+3)\alpha}(z) \leq [E_{\alpha,\beta+(n+2)\alpha}(z)]^2, \tag{7}$$

holds for all $n \in \mathbb{N}$, and $\alpha, \beta \geq 0$ and $z > 0$.

Proof. In [24] the following formula for Mittag–Leffler functions was proved

$$z^n E_{\alpha,\beta+n\alpha}(z) = E_{\alpha,\beta}(z) - \sum_{k=0}^{n-1} \frac{z^k}{\Gamma(\beta+k\alpha)}, \tag{8}$$

■

it holds for all $\alpha > 0,\ \beta > 0$ and $n \in \mathbb{N}$. Using (5) and (8) we conclude that (7) holds.

Theorem 2. *Let $\alpha, \beta > 0$ and $n \in \mathbb{N}$. Then the function $h_n(\alpha, \beta, z)$ defined by*

$$x \mapsto h_n(\alpha, \beta, z) = \frac{E^n_{\alpha,\beta}(z)E^{n+2}_{\alpha,\beta}(z)}{[E^{n+1}_{\alpha,\beta}(z)]^2}$$

is increasing on $(0, \infty)$. So the following Turán type inequality

$$\frac{\Gamma^2(\beta+(n+1)\alpha)}{\Gamma(\beta+n\alpha)\Gamma(\beta+(n+2)\alpha)}[E^{n+1}_{\alpha,\beta}(z)]^2 \leq E^n_{\alpha,\beta}(z)E^{n+2}_{\alpha,\beta}(z), \tag{9}$$

holds for all $\alpha, \beta > 0,\ n \in \mathbb{N}$ and $z > 0$. The constant in LHS of inequality (9) is sharp.

Proof. From the Cauchy series product we get

$$h_n(\alpha, \beta, z) = \frac{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{\Gamma(\beta+(n+1+j)\alpha)\Gamma(\beta+(n+3+k-j)\alpha)}\right) z^{2n+2+k}}{\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{1}{\Gamma(\beta+(n+2+j)\alpha)\Gamma(\beta+(n+2+k-j)\alpha)}\right) z^{2n+2+k}} =$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j(\alpha, \beta) z^{2n+2+k}}{\sum_{k=0}^{\infty}\sum_{j=0}^{k} b_j(\alpha, \beta) z^{2n+2+k}}.$$

Now we consider the sequence $(U_j)_{j\geq 0}$ defined by

$$U_j = \frac{\Gamma(\beta+(n+2+j)\alpha)\Gamma(\beta+(n+2+k-j)\alpha)}{\Gamma(\beta+(n+1+j)\alpha)\Gamma(\beta+(n+3+k-j)\alpha)}.$$

Thus

$$\frac{U_{j+1}}{U_j} = \frac{\Gamma(\beta+(n+3+j)\alpha)\Gamma(\beta+(n+1+j)\alpha)}{\Gamma^2(\beta+(n+2+j)\alpha)} \cdot$$

$$\cdot \frac{\Gamma(\beta+(n+1+k-j)\alpha)\Gamma(\beta+(n+3+k-j)\alpha)}{\Gamma^2(\beta+(n+2+k-j)\alpha)} =$$

$$= \frac{\Gamma(\beta_1+(n+3)\alpha)\Gamma(\beta_1+(n+1)\alpha)}{\Gamma^2(\beta_1+(n+2)\alpha)} \cdot \frac{\Gamma(\beta_2+(n+1)\alpha)\Gamma(\beta_2+(n+3)\alpha)}{\Gamma^2(\beta_2+(n+2)\alpha)},$$

where $\beta_1 = \beta + j\alpha$ and $\beta_2 = \beta + (k-j)\alpha$. And again using the Turán type inequality (6) we conclude that $\frac{U_{j+1}}{U_j} \geq 1$ and consequently the sequence $(U_j)_{j\geq 0}$ is increasing. So from lemma 1 we conclude that $\frac{\sum_{j=0}^{k} a_j(\alpha,\beta)}{\sum_{j=0}^{k} b_j(\alpha,\beta)}$ is increasing. Therefore, the function $x \mapsto h_n(\alpha,\beta,z)$ is also increasing on $(0,\infty)$ by lemma 2. Finally,

$$\lim_{x\to 0} h_n(\alpha,\beta,z) = \frac{\Gamma^2(\beta+(n+1)\alpha)}{\Gamma(\beta+n\alpha)\Gamma(\beta+(n+2)\alpha)}.$$

And it follows that the next constant $\frac{\Gamma^2(\beta+(n+1)\alpha)}{\Gamma(\beta+n\alpha)\Gamma(\beta+(n+2)\alpha)}$ is the best possible for which the inequality (9) holds for all $\alpha,\beta > 0$, $n \in \mathbb{N}$ and $z > 0$. ■

4. Applications: Lazarević and Wilker–type inequalities for Mittag–Leffler functions

Theorem 3. *Let $\alpha,\beta_1,\beta_2 > 0$ be such that $\beta_1 \geq \beta_2 > 1$. Then the following inequality*

$$[E_{\alpha,\beta_1}(z)]^{\frac{\Gamma(\beta_1-1)}{\Gamma(\beta_1)}} \leq [E_{\alpha,\beta_2}(z)]^{\frac{\Gamma(\beta_2-1)}{\Gamma(\beta_2)}} \tag{10}$$

holds for all $z \in \mathbb{R}$.

Proof. From part (**a.**) of the theorem 1 the function $\beta \mapsto \log[\Gamma(\beta)E_{\alpha,\beta}(z)]$ is convex and hence it follows that $\beta \mapsto \log[\Gamma(\beta+a)E_{\alpha,\beta+a}(z)] - \log[\Gamma(\beta)E_{\alpha,\beta}(z)]$ is increasing for each $a > 0$. Thus, choosing $a = 1$ we obtain that indeed the function $\beta \mapsto \frac{\Gamma(\beta+1)E_{\alpha,\beta+1}(z)}{\Gamma(\beta)E_{\alpha,\beta}(z)}$ is increasing on $(0,\infty)$. Now, providing that $\beta_1 \geq \beta_2 > 1$ let define the function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\Phi(x) = \frac{\Gamma(\beta_2)\Gamma(\beta_1-1)}{\Gamma(\beta_2-1)\Gamma(\beta_1)} \log[\Gamma(\beta_1)E_{\alpha,\beta_1}(z)] - \log[\Gamma(\beta_2)E_{\alpha,\beta_2}(z)].$$

From the differentiation formula ([24], Theorem 5.1)

$$\frac{d}{dz}E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)}{\alpha z}$$

we get

$$\Phi'(x) =$$

$$= \frac{1}{\alpha z}\left[\frac{\Gamma(\beta_2)\Gamma(\beta_1-1)}{\Gamma(\beta_2-1)\Gamma(\beta_1)}\frac{E_{\alpha,\beta_1-1}(z)}{E_{\alpha,\beta_1}(z)} - \frac{E_{\alpha,\beta_2-1}(z)}{E_{\alpha,\beta_2}(z)} + (\beta_2-1) - \frac{\Gamma(\beta_2)\Gamma(\beta_1-1)}{\Gamma(\beta_2-1)\Gamma(\beta_1)}(\beta_1-1)\right] =$$

$$= \frac{\Gamma(\beta_2)}{\alpha z\Gamma(\beta_2-1)}\left[\frac{\Gamma(\beta_1-1)E_{\alpha,\beta_1-1}(z)}{\Gamma(\beta_1)E_{\alpha,\beta_1}(z)} - \frac{\Gamma(\beta_2-1)E(\alpha,\beta_2-1,z)}{\Gamma(\beta_2)E(\alpha,\beta_2,z)}\right].$$

Since the function $\beta \mapsto \frac{\Gamma(\beta+1)E_{\alpha,\beta+1}(z)}{\Gamma(\beta)E_{\alpha,\beta}(z)}$ is increasing on $(0,\infty)$ we derive for all $\beta_1 \geq \beta_2 > 1$ that

$$\frac{\Gamma(\beta_1-1)E_{\alpha,\beta_1-1}(z)}{\Gamma(\beta_1)E_{\alpha,\beta_1}(z)} \leq \frac{\Gamma(\beta_2-1)E_{\alpha,\beta_2-1}(z)}{\Gamma(\beta_2)E_{\alpha,\beta_2}(z)}.$$

From this we conclude that the function $x \mapsto \Phi(x)$ is decreasing on $[0,\infty)$ and increasing on $(-\infty,0]$. Consequently $\Phi(x) \leq \Phi(0) = 0$ for all $x \in \mathbb{R}$. So the proof of the theorem 3 is complete. ■

Remark 1. *Choosing $\beta_1 = \beta+1, \beta_2 = \beta$ in (10) we obtain*

$$E_{\alpha,\beta+1}(z) \leq [E_{\alpha,\beta}(z)]^{\frac{\beta}{\beta-1}},\ z \in \mathbb{R}. \tag{11}$$

If $\beta = 3/2$ we derive the Lazarević–type inequality [21] *for the Mittag–Leffler function*

$$E_{\alpha,\frac{5}{2}}(z) \leq [E_{\alpha,\frac{3}{2}}(z)]^3,\ z \in \mathbb{R}. \tag{12}$$

Corollary 2. *Let $\alpha, \beta_1, \beta_2 > 0$ be such that $\beta_1 \geq \beta_2 > 1$. Then the following inequality*

$$[E_{\alpha,\beta_2}(z)]^{\frac{\beta_1-\beta_2}{\beta_2-1}} + \frac{E_{\alpha,\beta_2}(z)}{E_{\alpha,\beta_1}(z)} \geq 2, \tag{13}$$

holds for all $z \in \mathbb{R}$.

Proof. By using the inequality (10) and the arithmetic–geometric mean inequality we conclude that

$$\frac{1}{2}\left([E_{\alpha,\beta_2}(z)]^{\frac{\beta_1-\beta_2}{\beta_2-1}} + \frac{E_{\alpha,\beta_2}(z)}{E_{\alpha,\beta_1}(z)}\right) \geq \sqrt{\frac{[E_{\alpha,\beta_2}(z)]^{\frac{\beta_1-1}{\beta_2-1}}}{E_{\alpha,\beta_1}(z)}} \geq 1.$$

∎

Note that with use of generalizations of AGM–inequality we may refine (13), on generalizations of the AGM–inequality cf. [5], [32] and related Cauchy–Bunyakovsky inequality cf. [39].

Remark 2. *For $\beta_1 = \beta + 1, \beta_2 = \beta$ in (13) we obtain*

$$[E_{\alpha,\beta}(z)]^{\frac{1}{\beta-1}} + \frac{E_{\alpha,\beta}(z)}{E_{\alpha,\beta+1}(z)} \geq 2, \; z \in \mathbb{R}. \tag{14}$$

In case $\beta = 3/2$ the next Wilker–type inequality [46] *for the Mittag–Leffler function follows*

$$[E_{\alpha,\frac{3}{2}}(z)]^2 + \frac{E_{\alpha,\frac{3}{2}}(z)}{E_{\alpha,\frac{5}{2}}(z)} \geq 2, \; z \in \mathbb{R}. \tag{15}$$

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