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# OPERATOR SUPERADDITIVITY AND MONOTONICITY OF NONCOMMUTATIVE PERSPECTIVES

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Abstract. In this paper we establish some operator superadditivity and monotonicity properties for mappings associated to noncommutative perspectives of operator concave or operator convex functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

## 1. Introduction

Let $\Phi$ be a continuous function defined on the interval $I$ of real numbers, $B$ a selfadjoint operator on the Hilbert space $H$ and $A$ a positive invertible operator on $H$. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_{\Phi}(B,A)$ by setting

$$\mathcal{P}_{\Phi}(B,A):=A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If $A$ and $B$ are commutative, then

$$\mathcal{P}_{\Phi}(B,A)=A\Phi\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset \mathring{I}$.

It is well known that (see [7] and [6] or [8]), if $\Phi$ is an *operator convex (concave) function* defined in the positive half-line $(0,\infty)$, then the mapping

$$(B,A)\mapsto \mathcal{P}_{\Phi}(B,A)$$

*defined in pairs of positive definite operators, is operator convex (concave),* namely we have

$$\mathcal{P}_{\Phi}(\lambda B+(1-\lambda)D,\lambda A+(1-\lambda)C)\le(\ge)\lambda\mathcal{P}_{\Phi}(B,A)+(1-\lambda)\mathcal{P}_{\Phi}(D,C) \tag{1.1}$$

in the operator order for any positive invertible operators $A$, $B$, $C$, $D$ and $\lambda\in[0,1]$.

In the recent paper [2] we established the following reverse inequality for the perspective $\mathcal{P}_{\Phi}(B,A)$.

Let $\Phi:[m,M]\to\mathbb{R}$ be a *convex function* on the real interval $[m,M]$, $A$ a positive invertible operator and $B$ a selfadjoint operator such that

$$mA\le B\le MA, \tag{1.2}$$

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then we have

$$0 \le \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A) \le \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right) \le \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]A. \tag{1.3}$$

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of $J$. Suppose that there exists the constants $d$, $D$ such that

$$d \le \Phi''(t) \le D \text{ for any } t \in \mathring{J}. \tag{1.4}$$

If $A$ is a positive invertible operator and $B$ a selfadjoint operator such that the condition (1.2) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well [3]

$$\frac{1}{2}d\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right) \le \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A) \le \frac{1}{2}D\left(MA^{1/2}-BA^{-1/2}\right)\left(A^{-1/2}B-mA^{1/2}\right). \tag{1.5}$$

If $d > 0$, then the first inequality in (1.5) is better than the same inequality in (1.3).

Motivated by the above results, we establish in this paper some operator superadditivity and monotonicity properties for mappings associated to noncommutative perspectives of operator concave or operator convex functions. Applications for weighted operator geometric mean and relative operator entropy are also provided.

## 2. Superadditivity and Monotonicity Properties

The following result holds:

**Theorem 1.** *Let $\Phi$ be an operator concave (convex) function defined in the positive half-line then for any positive invertible operators $A$, $B$, $C$, $D$ we have*

$$\mathcal{P}_{\Phi}(B+D, A+C) \ge (\le)\, \mathcal{P}_{\Phi}(B,A)+\mathcal{P}_{\Phi}(D,C), \tag{2.1}$$

*i.e. $\mathcal{P}_{\Phi}$ is operator superadditive (subadditive) as a function of pairs of positive invertible operators.*

*In addition, if $\Phi$ is operator concave and nonnegative in the positive half-line and $A > C$ and $B > D$, then*

$$\mathcal{P}_{\Phi}(B,A) \ge \mathcal{P}_{\Phi}(D,C), \tag{2.2}$$

*i.e. $\mathcal{P}_{\Phi}$ is operator monotonic nondecreasing as a function of pairs of positive invertible operators.*

*Proof.* First, we observe that $\mathcal{P}_{\Phi}$ is positive homogeneous as a function of pairs of positive invertible operators, namely

$$\mathcal{P}_{\Phi}(\alpha B, \alpha A) := \alpha \mathcal{P}_{\Phi}(B,A)$$

for any $\alpha > 0$ and any pair of positive invertible operators $(B, A)$.

By property (1.1) we have for any positive invertible operators $A$, $B$, $C$, $D$ that

$$\begin{aligned}\mathcal{P}_{\Phi}(B+D,A+C) &= \mathcal{P}_{\Phi}\left(2\frac{B+D}{2},2\frac{A+C}{2}\right) \\ &= 2\mathcal{P}_{\Phi}\left(\frac{B+D}{2},\frac{A+C}{2}\right) \\ &\geq (\leq)\, 2\frac{\mathcal{P}_{\Phi}(B,A)+\mathcal{P}_{\Phi}(D,C)}{2} \\ &= \mathcal{P}_{\Phi}(B,A)+\mathcal{P}_{\Phi}(D,C)\end{aligned}$$

and the inequality (2.1) is proved.

If $A>C>0$ and $B>D>0$, then by (2.1) we have

$$\begin{aligned}\mathcal{P}_{\Phi}(B,A) &= \mathcal{P}_{\Phi}(B-D+D,A-C+C) \\ &\geq \mathcal{P}_{\Phi}(B-D,A-C)+\mathcal{P}_{\Phi}(D,C)\end{aligned}$$

giving that

$$\mathcal{P}_{\Phi}(B,A)-\mathcal{P}_{\Phi}(D,C)\geq \mathcal{P}_{\Phi}(B-D,A-C).$$

Since $\Phi$ is positive and $A>C$ and $B>D$ then

$$\begin{aligned}&\mathcal{P}_{\Phi}(B-D,A-C) \\ &= (A-C)^{1/2}\,\Phi\left((A-C)^{-1/2}(B-D)(A-C)^{-1/2}\right)(A-C)^{1/2}\geq 0\end{aligned}$$

and the inequality (2.2) is proved. □

**Corollary 1.** *Let $\Phi$ be a nonnegative and operator concave function defined in the positive half-line then for any positive invertible operators $A$, $B$, $C$, $D$ such that*

$$K(C,D)\geq (A,B)\geq k(C,D) \tag{2.3}$$

*namely, $KC\geq A\geq kC$ and $KD\geq B\geq kD$ for some positive constants $k$, $K$. Then we have*

$$K\mathcal{P}_{\Phi}(D,C)\geq \mathcal{P}_{\Phi}(B,A)\geq k\mathcal{P}_{\Phi}(D,C). \tag{2.4}$$

*Proof.* We have by (2.2) that

$$K\mathcal{P}_{\Phi}(D,C)=\mathcal{P}_{\Phi}(KD,KC)\geq \mathcal{P}_{\Phi}(B,A),$$

which proves the first inequality in (2.4).

The second inequality goes likewise and the corollary is proved. □

Let $\Phi:[m,M]\to\mathbb{R}$ be a continuous *convex function* on the real interval $[m,M]$. Then for any $t\in[m,M]$ we have by the convexity of $\Phi$ that

$$\frac{1}{M-m}\left[(M-t)\,\Phi(m)+(t-m)\,\Phi(M)\right]\geq \Phi(t).$$

If $A$ is positive invertible and $B$ is selfadjoint and $mA\leq B\leq MA$, holds, then by multiplying both sides of this inequality with $A^{-1/2}$ we have $mI\leq A^{-1/2}BA^{-1/2}\leq MI$. If we use the continuous functional calculus, then we have

$$\begin{aligned}&\frac{1}{M-m}\left[\Phi(m)\left(MI-A^{-1/2}BA^{-1/2}\right)+\Phi(M)\left(A^{-1/2}BA^{-1/2}-mI\right)\right] \\ &\geq \Phi\left(A^{-1/2}BA^{-1/2}\right).\end{aligned} \tag{2.5}$$

If we multiply both sides of (2.5) by $A^{1/2}$ we get

$$\frac{1}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]\geq\mathcal{P}_{\Phi}\left(B,A\right). \tag{2.6}$$

For $m$, $M$ with $M>m$ we define the following set of pairs of operators

$$\mathcal{S}\left(m,M\right):=\left\{\left(B,A\right),\ A\text{ is positive invertible, }B\text{ is selfadjoint and satisfy (1.2)}\right\}.$$

We observe that, if $\left(B,A\right),\left(D,C\right)\in\mathcal{S}\left(m,M\right)$ then also $\left(B,A\right)+\left(D,C\right)$, $\alpha\left(B,A\right)$ and $\lambda\left(B,A\right)+\left(1-\lambda\right)\left(D,C\right)\in\mathcal{S}\left(m,M\right)$ for any $\alpha>0$ and $\lambda\in[0,1]$ meaning that $\mathcal{S}\left(m,M\right)$ is a *cone* and, *a fortiori* a convex set.

We define the mapping $\mathcal{D}_{\Phi}$ on $\mathcal{S}\left(m,M\right)$ by

$$\mathcal{D}_{\Phi}\left(B,A\right):=\frac{1}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]-\mathcal{P}_{\Phi}\left(B,A\right).$$

From the above considerations, we see that, if $\Phi:[m,M]\to\mathbb{R}$ is a continuous *convex function* on the real interval $[m,M]$, then

$$\mathcal{D}_{\Phi}\left(B,A\right)\geq0 \tag{2.7}$$

for any $\left(B,A\right)\in\mathcal{S}\left(m,M\right)$.

We have:

**Theorem 2.** *Let $\Phi$ be an operator convex function defined on the interval $[m,M]$. Then the mapping $\mathcal{D}_{\Phi}$ is nonnegative, positive homogeneous, operator concave, operator superadditive and operator monotonic nondecreasing on $\mathcal{S}\left(m,M\right)$.*

*Proof.* If $\Phi$ is an operator convex function defined on the interval $[m,M]$, then it is convex and by (2.7) we have that $\mathcal{D}_{\Phi}$ is nonnegative on $\mathcal{S}\left(m,M\right)$ in the operator order.

If $\alpha>0$ and $\left(B,A\right)\in\mathcal{S}\left(m,M\right)$ then

$$\begin{aligned}&\mathcal{D}_{\Phi}\left(\alpha B,\alpha A\right)\\&=\frac{1}{M-m}\left[\Phi\left(m\right)\left(M\alpha A-\alpha B\right)+\Phi\left(M\right)\left(\alpha B-m\alpha A\right)\right]-\mathcal{P}_{\Phi}\left(\alpha B,\alpha A\right)\\&=\frac{\alpha}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]-\alpha\mathcal{P}_{\Phi}\left(B,A\right)\\&=\alpha\mathcal{D}_{\Phi}\left(B,A\right)\end{aligned}$$

that proves the positive homogeneity of $\mathcal{D}_{\Phi}$.

Let $\left(B,A\right),\left(D,C\right)\in\mathcal{S}\left(m,M\right)$ and $\lambda\in[0,1]$. Then we have

$$\begin{aligned}&\mathcal{D}_{\Phi}\left(\left(\lambda B+\left(1-\lambda\right)D,\lambda A+\left(1-\lambda\right)C\right)\right)\\&=\frac{1}{M-m}\left[\Phi\left(m\right)\left(M\left(\lambda A+\left(1-\lambda\right)C\right)-\left(\lambda B+\left(1-\lambda\right)D\right)\right)\right.\\&\left.+\Phi\left(M\right)\left(\lambda B+\left(1-\lambda\right)D-m\left(\lambda A+\left(1-\lambda\right)C\right)\right)\right]\\&-\mathcal{P}_{\Phi}\left(\left(\lambda B+\left(1-\lambda\right)D,\lambda A+\left(1-\lambda\right)C\right)\right)\\&=\frac{\lambda}{M-m}\left[\Phi\left(m\right)\left(MA-B\right)+\Phi\left(M\right)\left(B-mA\right)\right]\\&+\frac{1-\lambda}{M-m}\left[\Phi\left(m\right)\left(MC-D\right)+\Phi\left(M\right)\left(D-mC\right)\right]\\&-\mathcal{P}_{\Phi}\left(\left(\lambda B+\left(1-\lambda\right)D,\lambda A+\left(1-\lambda\right)C\right)\right)\\&=:U.\end{aligned}$$

By (1.1) we also have that

$$-\mathcal{P}_{\Phi}\left(\left(\lambda B+(1-\lambda) D, \lambda A+(1-\lambda) C\right)\right) \geq-\lambda \mathcal{P}_{\Phi}(B, A)-(1-\lambda) \mathcal{P}_{\Phi}(D, C)$$

for any $(B, A),\ (D, C) \in \mathcal{S}(m, M)$ and $\lambda \in[0,1]$.

Therefore

$$\begin{aligned} U \geq & \frac{\lambda}{M-m}[\Phi(m)(M A-B)+\Phi(M)(B-m A)]-\lambda \mathcal{P}_{\Phi}(B, A) \\ & +\frac{1-\lambda}{M-m}[\Phi(m)(M C-D)+\Phi(M)(D-m C)]-(1-\lambda) \mathcal{P}_{\Phi}(D, C) \\ & =\lambda \mathcal{D}_{\Phi}(B, A)+(1-\lambda) \mathcal{D}_{\Phi}(D, C) \end{aligned}$$

for any $(B, A),\ (D, C) \in \mathcal{S}(m, M)$ and $\lambda \in[0,1]$, showing that $\mathcal{D}_{\Phi}$ is operator concave on $\mathcal{S}(m, M)$.

If $(B, A),\ (D, C) \in \mathcal{S}(m, M)$, then by the above properties we have

$$\mathcal{D}_{\Phi}(B+D, A+C) \geq \mathcal{D}_{\Phi}(B, A)+\mathcal{D}_{\Phi}(D, C),$$

which proves the operator superadditivity of $\mathcal{D}_{\Phi}$ on $\mathcal{S}(m, M)$.

The operator monotonicity of $\mathcal{D}_{\Phi}$ follows in a similar way as in the proof of Theorem 1 and the details are omitted. □

**Corollary 2.** *Let $\Phi$ be an operator convex function defined on the interval $[m, M]$ and $(B, A),\ (D, C) \in \mathcal{S}(m, M)$. If there exists some positive constants $k$, $K$ such that $KC \geq A \geq kC$ and $KD \geq B \geq kD$, then we have the inequalities*

$$\begin{aligned} & K\left\{\frac{1}{M-m}[\Phi(m)(M C-D)+\Phi(M)(D-m C)]-\mathcal{P}_{\Phi}(D, C)\right\} \\ & \geq \frac{1}{M-m}[\Phi(m)(M A-B)+\Phi(M)(B-m A)]-\mathcal{P}_{\Phi}(B, A) \\ & \geq k\left\{\frac{1}{M-m}[\Phi(m)(M C-D)+\Phi(M)(D-m C)]-\mathcal{P}_{\Phi}(D, C)\right\} \geq 0 . \end{aligned} \tag{2.8}$$

## 3. Applications for Operator Geometric Mean

Assume that $A$, $B$ are positive invertible operators on a complex Hilbert space $(H,\langle\cdot, \cdot\rangle)$. We use the following notations for operators [18]

$$A \nabla_{\nu} B:=(1-\nu) A+\nu B,$$

the *weighted operator arithmetic mean* and

$$A \sharp_{\nu} B:=A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{\nu} A^{1/2},$$

the *weighted operator geometric mean,* where $\nu \in[0,1]$. When $\nu=\frac{1}{2}$ we write $A \nabla B$ and $A \sharp B$ for brevity, respectively.

The definition $A \sharp_{\nu} B$ can be extended accordingly for any real number $\nu$.

The following inequality is well known as the operator *Young inequality* or operator *$\nu$-weighted arithmetic-geometric mean inequality*:

$$A \sharp_{\nu} B \leq A \nabla_{\nu} B \text{ for all } \nu \in[0,1]. \tag{3.1}$$

For recent results on operator Young inequality see [11]-[14], [15] and [24]-[25].

If we consider the continuous function $\Phi_\nu : [0,\infty) \to [0,\infty)$, $\Phi_\nu(x) = x^\nu$ then the operator *$\nu$-weighted geometric mean* can be interpreted as the perspective $\mathcal{P}_{\Phi_\nu}(B,A)$, namely

$$\mathcal{P}_{\Phi_\nu}(B,A) = A\sharp_\nu B.$$

Since, for $\nu \in (0,1)$, $\Phi_\nu : [0,\infty) \to [0,\infty)$, $\Phi_\nu(x) = x^\nu$ is operator concave and positive on $[0,\infty)$, then by (1.1) we have that (see also [23, p. 146])

$$(tA + (1-t)C)\sharp_\nu(tB + (1-t)D) \geq tA\sharp_\nu B + (1-t)C\sharp_\nu D \tag{3.2}$$

and by (2.1) we have (see also [23, p. 146])

$$(A+C)\sharp_\nu(B+D) \geq A\sharp_\nu B + C\sharp_\nu D \tag{3.3}$$

for any positive invertible operators $A$, $B$, $C$, $D$ and $\nu \in [0,1]$.

For positive invertible operators $A$, $B$, $C$, $D$ such that $A > C$ and $B > D$, then by (2.2) we have (see also [23, p. 139])

$$A\sharp_\nu B \geq C\sharp_\nu D. \tag{3.4}$$

Moreover, if $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants $k$, $K$ then by (2.4) we have

$$KC\sharp_\nu D \geq A\sharp_\nu B \geq kC\sharp_\nu D. \tag{3.5}$$

For $\nu \in (0,1)$ we consider the mapping $\mathcal{D}_\nu$ on $\mathcal{S}(m,M)$ defined by

$$\mathcal{D}_\nu(B,A) := A\sharp_\nu B - \frac{1}{M-m}\left[m^\nu(MA-B) + M^\nu(B-mA)\right].$$

Using Theorem 2 we have that $\mathcal{D}_\nu$ is *nonnegative, positive homogeneous, operator concave and operator superadditive.*

If $(B,A)$, $(D,C) \in \mathcal{S}(m,M)$ with $A > C$ and $B > D$, then by the operator monotonicity of $\mathcal{D}_\nu$ we have

$$\begin{aligned} &A\sharp_\nu B - \frac{1}{M-m}\left[m^\nu(MA-B) + M^\nu(B-mA)\right] \\ &\geq C\sharp_\nu D - \frac{1}{M-m}\left[m^\nu(MC-D) + M^\nu(D-mC)\right] \geq 0. \end{aligned} \tag{3.6}$$

If $(B,A)$, $(D,C) \in \mathcal{S}(m,M)$ and $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants $k$, $K$ then by (2.8) we have

$$\begin{aligned} &K\left\{C\sharp_\nu D - \frac{1}{M-m}\left[m^\nu(MC-D) + M^\nu(D-mC)\right]\right\} \\ &\geq A\sharp_\nu B - \frac{1}{M-m}\left[m^\nu(MA-B) + M^\nu(B-mA)\right] \\ &\geq k\left\{C\sharp_\nu D - \frac{1}{M-m}\left[m^\nu(MC-D) + M^\nu(D-mC)\right]\right\} \geq 0. \end{aligned} \tag{3.7}$$

It is known that the function $\Phi_p(t) = t^p$ is operator convex on $(0,\infty)$ if either $1 \leq p \leq 2$ or $-1 \leq p \leq 0$. Consider the mapping

$$\mathcal{P}_{\Phi_p}(B,A) = A\sharp_p B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^p A^{1/2}.$$

In particular, we have

$$\mathcal{P}_2(B,A) = \mathcal{P}_{\Phi_2}(B,A) = A\sharp_2 B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^2 A^{1/2} = BA^{-1}B$$

and, symmetrically,

$$\mathcal{P}_{-1}(B,A)=\mathcal{P}_{\Phi_{-1}}(B,A)=A\sharp_{-1}B=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{-1}A^{1/2}=AB^{-1}A.$$

By utilizing Theorem 1 for operator convex functions, we conclude that $\mathcal{P}_{\Phi_p}(\cdot,\cdot)$ is subadditive as a function of positive invertible pairs. This implies that the functional

$$\rho_p(B,A):=\left\|A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^pA^{1/2}\right\|$$

is also subadditive as a function of positive invertible pairs for $p\in[-1,0]\cup[1,2]$.

In particular, we have the operator inequality

$$(B+D)(A+C)^{-1}(B+D)\le BA^{-1}B+DC^{-1}D$$

and the norm inequality

$$\left\|(B+D)(A+C)^{-1}(B+D)\right\|\le\left\|BA^{-1}B+DC^{-1}D\right\|\le\left\|BA^{-1}B\right\|+\left\|DC^{-1}D\right\|$$

for any positive invertible operators $A$, $B$, $C$, $D>0$.

Consider the mapping $\mathcal{D}_p$ on $\mathcal{S}(m,M)$ defined by

$$\mathcal{D}_p(B,A):=\frac{1}{M-m}\left[m^p(MA-B)+M^p(B-mA)\right]-A\sharp_pB$$

where

$$A\sharp_pB:=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^pA^{1/2},\ p\in[-1,0]\cup[1,2].$$

By Theorem 2 we have that the mapping $\mathcal{D}_p$ is *nonnegative, positive homogeneous, operator concave and operator superadditive on* $\mathcal{S}(m,M)$.

If $(B,A),\ (D,C)\in\mathcal{S}(m,M)$ with $A>C$ and $B>D$, then by the operator monotonicity of $\mathcal{D}_p$ we have

$$\begin{aligned}\frac{1}{M-m}&\left[m^p(MA-B)+M^p(B-mA)\right]-A\sharp_pB\\&\ge\frac{1}{M-m}\left[m^p(MC-D)+M^p(D-mC)\right]-C\sharp_pD\ge0,\end{aligned}\tag{3.8}$$

for any $p\in[-1,0]\cup[1,2]$.

In particular, we have

$$\begin{aligned}\frac{1}{M-m}&\left[\frac{1}{m}(MA-B)+\frac{1}{M}(B-mA)\right]-AB^{-1}A\\&\ge\frac{1}{M-m}\left[\frac{1}{m}(MC-D)+\frac{1}{M}(D-mC)\right]-CD^{-1}C\ge0\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}\frac{1}{M-m}&\left[m^2(MA-B)+M^2(B-mA)\right]-BA^{-1}B\\&\ge\frac{1}{M-m}\left[m^2(MC-D)+M^2(D-mC)\right]-DC^{-1}D\ge0.\end{aligned}\tag{3.10}$$

If $(B,A)$, $(D,C)\in\mathcal{S}(m,M)$ and $KC\geq A\geq kC$ and $KD\geq B\geq kD$ for some positive constants $k$, $K$ then by (2.8) we have

$$\begin{aligned}&K\left\{\frac{1}{M-m}\left[m^{p}(MC-D)+M^{p}(D-mC)\right]-C\sharp_{p}D\right\}\\&\geq\frac{1}{M-m}\left[m^{p}(MA-B)+M^{p}(B-mA)\right]-A\sharp_{p}B\\&\geq k\left\{\frac{1}{M-m}\left[m^{p}(MC-D)+M^{p}(D-mC)\right]-C\sharp_{p}D\right\}\geq 0\end{aligned}\tag{3.11}$$

for any $p\in[-1,0]\cup[1,2]$.

In particular, we have

$$\begin{aligned}&K\left\{\frac{1}{M-m}\left[\frac{1}{m}(MC-D)+\frac{1}{M}(D-mC)\right]-CD^{-1}C\right\}\\&\geq\frac{1}{M-m}\left[\frac{1}{m}(MA-B)+\frac{1}{M}(B-mA)\right]-AB^{-1}A\\&\geq k\left\{\frac{1}{M-m}\left[\frac{1}{m}(MC-D)+\frac{1}{M}(D-mC)\right]-CD^{-1}C\right\}\geq 0\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}&K\left\{\frac{1}{M-m}\left[m^{2}(MC-D)+M^{2}(D-mC)\right]-DC^{-1}D\right\}\\&\geq\frac{1}{M-m}\left[m^{2}(MA-B)+M^{2}(B-mA)\right]-BA^{-1}B\\&\geq k\left\{\frac{1}{M-m}\left[m^{2}(MC-D)+M^{2}(D-mC)\right]-DC^{-1}D\right\}\geq 0.\end{aligned}\tag{3.13}$$

## 4. Applications for Relative Operator Entropy

Kamei and Fujii [9], [10] defined the *relative operator entropy* $S(A|B)$, for positive invertible operators $A$ and $B$, by

$$S(A|B):=A^{1/2}\left(\ln A^{-1/2}BA^{-1/2}\right)A^{1/2},\tag{4.1}$$

which is a relative version of the operator entropy considered by Nakamura-Umegaki [22].

For some recent results on relative operator entropy see [4]-[5], [16]-[17] and [19]-[20].

Consider the logarithmic function $\ln$. Then the relative operator entropy can be interpreted as the permanent of ln, namely

$$\mathcal{P}_{\ln}(B,A)=S(A|B).$$

If we consider the entropy function $\eta(t)=-t\ln t$, then it is well known that for any positive invertible operators $A$, $B$ we have

$$S(A|B)=B^{1/2}\eta\left(B^{-1/2}AB^{-1/2}\right)B^{1/2}.\tag{4.2}$$

The function $\Phi(t)=t\ln t=-\eta(t)$, $t>0$, is convex, then the perspective of this function is

$$\mathcal{P}_{(\cdot)\ln(\cdot)}(B,A)=-A^{1/2}\eta\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=-S(B|A),$$

where for the last equality we used (4.2) for $A$ replacing $B$.

If $B \geq A$ and $A$ is positive and invertible, then $A^{-1/2}BA^{-1/2} \geq I$ and by the continuous functional calculus we have $\ln A^{-1/2}BA^{-1/2} \geq 0$, which implies by multiplying both sides with $A^{1/2}$ that $S(A|B) \geq 0$.

Since $\Phi(t) = \ln t$ is operator concave on $(0, \infty)$ then by (1.1) we have (see also [23, p. 153])

$$S(\nu A + (1-\nu) C|\nu B + (1-\nu) D) \geq \nu S(A|B) + (1-\nu) S(C|D) \tag{4.3}$$

for any positive invertible operators $A$, $B$, $C$, $D$ and $\nu \in [0,1]$, while by (2.1) we have (see also [23, p. 153])

$$S(A + C|B + D) \geq S(A|B) + S(C|D) \tag{4.4}$$

for any positive invertible operators $A$, $B$, $C$, $D$.

Moreover, if $A > C > 0$ and $B > D > 0$, then by (2.1) we have

$$S(A|B) - S(C|D) \geq S(A - C|B - D). \tag{4.5}$$

In addition, if $B - D \geq A - C$, or, equivalently, $B + C \geq A + D$, then we have

$$S(A|B) \geq S(C|D). \tag{4.6}$$

We consider the mapping $\mathcal{D}_{\ln}$ on $\mathcal{S}(m, M)$ defined by

$$\mathcal{D}_{\ln}(B, A) := S(A|B) - \frac{1}{M-m}[\ln m(MA - B) + \ln M(B - mA)].$$

Using Theorem 2 we have that $\mathcal{D}_{\ln}$ is *nonnegative, positive homogeneous, operator concave and operator superadditive.*

If $(B, A), (D, C) \in \mathcal{S}(m, M)$ with $A > C$ and $B > D$, then by the operator monotonicity of $\mathcal{D}_{\ln}$ we have

$$\begin{aligned} &S(A|B) - \frac{1}{M-m}[\ln m(MA - B) + \ln M(B - mA)] \\ &\geq S(C|D) - \frac{1}{M-m}[\ln m(MC - D) + \ln M(D - mC)] \geq 0. \end{aligned} \tag{4.7}$$

If $(B, A), (D, C) \in \mathcal{S}(m, M)$ and $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants $k$, $K$ then by (2.8) we have

$$\begin{aligned} &K\left\{S(C|D) - \frac{1}{M-m}[\ln m(MC - D) + \ln M(D - mC)]\right\} \\ &\geq S(A|B) - \frac{1}{M-m}[\ln m(MA - B) + \ln M(B - mA)] \\ &\geq k\left\{S(C|D) - \frac{1}{M-m}[\ln m(MC - D) + \ln M(D - mC)]\right\} \geq 0. \end{aligned} \tag{4.8}$$

The function $\Phi(t) = t\ln t$ is operator convex on $(0, \infty)$. We can consider the mapping $\mathcal{D}_{(\cdot)\ln(\cdot)}$ on $\mathcal{S}(m, M)$ defined by

$$\begin{aligned} \mathcal{D}_{(\cdot)\ln(\cdot)}(B, A) &:= \frac{1}{M-m}[m\ln m(MA - B) + M\ln M(B - mA)] - \mathcal{P}_{(\cdot)\ln(\cdot)}(B, A) \\ &= \frac{1}{M-m}[m\ln m(MA - B) + M\ln M(B - mA)] + S(B|A). \end{aligned}$$

Using Theorem 2 we have that $\mathcal{D}_{(\cdot)\ln(\cdot)}$ is *nonnegative, positive homogeneous, operator concave and operator superadditive.*

If $(B,A)$, $(D,C) \in \mathcal{S}(m,M)$ with $A > C$ and $B > D$, then by the operator monotonicity of $\mathcal{D}_{(\cdot)\ln(\cdot)}$ we have

$$\begin{aligned}(4.9)\qquad & \frac{1}{M-m}\left[m\ln m\,(MA-B)+M\ln M\,(B-mA)\right]+S\left(B|A\right) \\ & \geq \frac{1}{M-m}\left[m\ln m\,(MC-D)+M\ln M\,(D-mC)\right]+S\left(D|C\right)\geq 0.\end{aligned}$$

If $(B,A)$, $(D,C) \in \mathcal{S}(m,M)$ and $KC \geq A \geq kC$ and $KD \geq B \geq kD$ for some positive constants $k$, $K$ then by (2.8) we have

$$\begin{aligned}(4.10)\quad & K\left\{\frac{1}{M-m}\left[m\ln m\,(MC-D)+M\ln M\,(D-mC)\right]+S\left(D|C\right)\right\} \\ & \geq \frac{1}{M-m}\left[m\ln m\,(MA-B)+M\ln M\,(B-mA)\right]+S\left(B|A\right) \\ & \geq k\left\{\frac{1}{M-m}\left[m\ln m\,(MC-D)+M\ln M\,(D-mC)\right]+S\left(D|C\right)\right\}\geq 0.\end{aligned}$$

## References


[1] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[2] S. S. Dragomir, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [Online http://rgmia.org/papers/v18/v18a130.pdf].

[3] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135. [Online http://rgmia.org/papers/v18/v18a135.pdf].

[4] S. S. Dragomir, Some inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 145. [Online http://rgmia.org/papers/v18/v18a145.pdf].

[5] S. S. Dragomir, Further inequalities for relative operator entropy, *RGMIA Res. Rep. Coll.* **18** (2015), Art. 160.[Online http://rgmia.org/papers/v18/v18a160.pdf].

[6] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, *Proc. Natl. Acad. Sci. USA*, **108** (2011), no. 18, 7313–7314.

[7] E. G. Effros, A matrix convexity approach to some celebrated quantum inequalities, *Proc. Natl. Acad. Sci. USA* **106** (2009), 1006–1008.

[8] E. G. Effros and F. Hansen, Noncomutative perspectives, *Ann. Funct. Anal.* **5** (2014), no. 2, 74–79.

[9] J. I. Fujii and E. Kamei, Uhlmann's interpolational method for operator means. *Math. Japon.* **34** (1989), no. 4, 541–547.

[10] J. I. Fujii and E. Kamei, Relative operator entropy in noncommutative information theory. *Math. Japon.* **34** (1989), no. 3, 341–348.

[11] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[12] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.

[13] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.

[14] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra,* **59** (2011), 1031-1037.

[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.

[16] I. H. Kim, Operator extension of strong subadditivity of entropy, *J. Math. Phys.* 53(2012), 122204

[17] P. Kluza and M. Niezgoda, Inequalities for relative operator entropies, *Elec. J. Lin. Alg.* **27** (2014), Art. 1066.

[18] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.

[19] M. S. Moslehian, F. Mirzapour, and A. Morassaei, Operator entropy inequalities. *Colloq. Math.*, **130** (2013), 159–168.

[20] I. Nikoufar, On operator inequalities of some relative operator entropies, *Adv. Math.* **259** (2014), 376-383.
[21] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.
[22] M. Nakamura and H. Umegaki, A note on the entropy for operator algebras. *Proc. Japan Acad.* **37** (1961) 149–154.
[23] J. Pečarić, T. Furuta, J. Mićić Hot and Y. Seo, *Mond-Pečarić method in operator inequalities.* Inequalities for bounded selfadjoint operators on a Hilbert space. Monographs in Inequalities, 1. ELEMENT, Zagreb, 2005. xiv+262 pp.+loose errata. ISBN: 953-197-572-8
[24] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.
[25] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.

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