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PERTURBED COMPANION OF OSTROWSKI TYPE INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, some perturbed companion of Ostrowski type integral inequalities for functions whose second derivatives are either bounded or of bounded variation are established.

1. Introduction

In 1938, Ostrowski [29] established a following useful inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$.

The constant $\frac{1}{4}$ is the best possible.

Definition 1. *Let $P:a=x_0<x_1<...<x_n=b$ be any partition of $[a,b]$ and let $\Delta f(x_i)=f(x_{i+1})-f(x_i)$, then f is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m}|\Delta f(x_i)|$$

is bounded for all such partitions.

Definition 2. *Let f be of bounded variation on $[a,b]$, and $\sum\Delta f(P)$ denotes the sum $\sum\limits_{i=1}^{n}|\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number*

$$\bigvee_a^b(f):=\sup\left\{\sum\Delta f(P):P\in P([a,b])\right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [16], Dragomir proved the following Ostrowski type inequalitiesfor functions of bounded variation:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [13], Authors obtained the following Ostroski type inequalities for functions whose secand derivatives are bounded:

Theorem 3. *Let$f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'':(a,b)\to\mathbb{R}$ is bounded on $(a;b)$. Then we have the inequality*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right|\\
\leq\;&\frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2\left\|f''\right\|_\infty\\
\leq\;&\frac{\left\|f''\right\|_\infty}{6}(b-a)^2
\end{aligned}$$

for all $x\in[a,b]$.

Ostrowski inequality has potential applications in Mathematical Sciences. In the past, many authors have worked on Ostrowski type inequalities for functions (bounded, of bounded variation, etc.) see for example ([1]-[10], [13]-[19], [27],[28],[30]-[37]). Furthermore, several works were devoted to study of perturbed Ostrowski type inequalities for bounded functions and functions of bounded variation, please refer to ([11],[12], [20]-[26]). In this study, we establish some perturbed companion of Ostrowski type inequalities for twice differentiable functions whose second derivatives are either bounded or of bounded variation.

2. Some Identities

Before we start our main results, we state and prove the following lemma:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b). Then for any $\lambda_i(x)$, $i=1,2,3$ complex number and all $x\in\left[a,\frac{a+b}{2}\right]$ the following identity* *holds*

$$
\begin{aligned}
(2.1)\quad & \left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt \\
& -\frac{1}{6(b-a)}\left[(x-a)^3\left(\lambda_1(x)+\lambda_3(x)\right)+2\left(\frac{a+b}{2}-x\right)^3\lambda_2(x)\right] \\
= \ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left[f''(t)-\lambda_1(x)\right]dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)-\lambda_2(x)\right]dt\right. \\
& \left.+\int_{a+b-x}^b (t-b)^2\left[f''(t)-\lambda_3(x)\right]dt\right]
\end{aligned}
$$

where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Using the integration by parts, we have

$$
\begin{aligned}
(2.2)\qquad & \int_a^x (t-a)^2\left[f''(t)-\lambda_1(x)\right]dt \\
= \ & \int_a^x (t-a)^2 f''(t)dt-\lambda_1(x)\int_a^x (t-a)^2\,dt \\
= \ & (t-a)^2 f'(t)\Big|_a^x-2\int_a^x (t-a)f'(t)dt-\frac{\lambda_1(x)}{3}(t-a)^3\Big|_a^x \\
= \ & (x-a)^2 f'(x)-2\left[(t-a)f(t)|_a^x-\int_a^x f(t)dt\right]-\frac{\lambda_1(x)}{3}(x-a)^3 \\
= \ & (x-a)^2 f'(x)-2(x-a)f(x)+2\int_a^x f(t)dt-\frac{\lambda_1(x)}{3}(x-a)^3,
\end{aligned}
$$

$$
\begin{aligned}
(2.3) \qquad & \int_{x}^{a+b-x} \left(t-\frac{a+b}{2}\right)^{2} \left[f''(t)-\lambda_2(x)\right] dt \\
&= \int_{x}^{a+b-x} \left(t-\frac{a+b}{2}\right)^{2} f''(t)dt - \lambda_2(x) \int_{x}^{a+b-x} \left(t-\frac{a+b}{2}\right)^{2} dt \\
&= \left(t-\frac{a+b}{2}\right)^{2} f'(t)\Big|_{x}^{a+b-x} - 2\left(t-\frac{a+b}{2}\right) f(t)\Big|_{x}^{a+b-x} \\
&\quad +2\int_{x}^{a+b-x} f(t)dt - \frac{\lambda_2(x)}{3}\left(t-\frac{a+b}{2}\right)^{3}\Big|_{x}^{a+b-x} \\
&= \left(\frac{a+b}{2}-x\right)^{2} \left[f'(a+b-x)-f'(x)\right] \\
&\quad -2\left(\frac{a+b}{2}-x\right)\left[f(a+b-x)+f(x)\right] + 2\int_{x}^{a+b-x} f(t)dt \\
&\quad -\frac{2}{3}\lambda_2(x)\left(\frac{a+b}{2}-x\right)^{3}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.4) \qquad & \int_{a+b-x}^{b} (t-b)^{2}\left[f''(t)-\lambda_3(x)\right] dt \\
&= \int_{a+b-x}^{b} (t-b)^{2} f''(t)dt - \lambda_3(x)\int_{a+b-x}^{b} (t-b)^{2} dt \\
&= (t-b)^{2} f'(t)\Big|_{a+b-x}^{b} - 2\,(t-b)\,f(t)\Big|_{a+b-x}^{b} \\
&\quad +2\int_{a+b-x}^{b} f(t)dt - \frac{\lambda_3(x)}{3}(t-b)^{3}\Big|_{a+b-x}^{b} \\
&= -(x-a)^{2} f'(a+b-x) - 2\,(x-a)\,f(a+b-x) \\
&\quad +2\int_{a+b-x}^{b} f(t)dt - \frac{\lambda_3(x)}{3}(x-a)^{3}.
\end{aligned}
$$

If we add the equality (2.2)-(2.4) and divide by $2(b-a)$, we obtain required identity. □

Corollary 1. *Under assumption of Lemma 1 with $\lambda_i(x)=\lambda_i, i=1,2,3$*

i) if we choose $x = a$, we have

$$\frac{b-a}{8}\left[f'(b)-f'(a)\right]-\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(t)dt-\frac{(b-a)^2}{24}\lambda_2 \tag{2.5}$$

$$= \frac{1}{2(b-a)}\int_a^b \left(t-\frac{a+b}{2}\right)^2\left[f''(t)-\lambda_2\right]dt,$$

ii) if we choose $x = \frac{a+b}{2}$, we have

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{48}\left(\lambda_1+\lambda_3\right) \tag{2.6}$$

$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}(t-a)^2\left[f''(t)-\lambda_1\right]dt+\int_{\frac{a+b}{2}}^b (t-b)^2\left[f''(t)-\lambda_3\right]dt\right],$$

iii) if we choose $x = \frac{3a+b}{4}$, we have

$$\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{(b-a)^2}{384}\left(\lambda_1+2\lambda_2+\lambda_3\right) \tag{2.7}$$

$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{3a+b}{4}}(t-a)^2\left[f''(t)-\lambda_1\right]dt+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)-\lambda_2\right]dt\right.$$

$$\left.+\int_{\frac{a+3b}{4}}^b (t-b)^2\left[f''(t)-\lambda_3\right]dt\right].$$

Corollary 2. *If we take $\lambda_1 = -\lambda_3$ in (2.6), then we get*

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$

$$= \frac{1}{2(b-a)}\left[\int_a^{\frac{a+b}{2}}(t-a)^2\left[f''(t)-\lambda_1\right]dt+\int_{\frac{a+b}{2}}^b (t-b)^2\left[f''(t)+\lambda_1\right]dt\right],$$

and choosing $\lambda_1=\lambda_3=-\lambda_2$ in (2.7), we have the inequaity

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\
=\ & \frac{1}{2(b-a)}\left[\int_a^{\frac{3a+b}{4}}(t-a)^2\left[f''(t)-\lambda_1\right]dt+\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)^2\left[f''(t)+\lambda_1\right]dt\right. \\
&\left.+\int_{\frac{a+3b}{4}}^{b}(t-b)^2\left[f''(t)-\lambda_1\right]dt\right].
\end{aligned}
$$

3. Inequalities for Functions Whose Second Derivatives are Bounded

Recall the sets of complex-valued functions:

$$
\begin{aligned}
&\overline{U}_{[a,b]}(\gamma,\Gamma) \\
:\ &=\left\{f:[a,b]\to\mathbb{C}\,|\ \mathrm{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}\right)-\overline{\gamma}\right]\geq 0 \text{ for almast every } t\in[a,b]\right\}
\end{aligned}
$$

and

$$
\overline{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{f:[a,b]\to\mathbb{C}\,\left|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right| \text{ for a.e. } t\in[a,b]\right.\right\}.
$$

Proposition 1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\overline{U}_{[a,b]}(\gamma,\Gamma)$ and $\overline{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty and closed sets and*

$$
\overline{U}_{[a,b]}(\gamma,\Gamma)=\overline{\Delta}_{[a,b]}(\gamma,\Gamma).
$$

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a twice differantiable function on (a,b) and $x\in(a,b)$. Suppose that $\gamma_i(x),\Gamma_i(x)\in\mathbb{C}$, $\gamma_i(x)\neq\Gamma_i(x)$, $i=1,2,3$ and $f''\in\overline{U}_{[a,x]}(\gamma_1,\Gamma_1)\cap\overline{U}_{[x,a+b-x]}(\gamma_2,\Gamma_2)\cap\overline{U}_{[a+b-x,b]}(\gamma_3,\Gamma_3)$, then we have the inequalty*

$$
\begin{aligned}
&\left|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2}+\frac{1}{b-a}\int_a^b f(t)dt\right. \\
&-\frac{1}{12(b-a)}\left[(x-a)^3\left[\gamma_1(x)+\Gamma_1(x)+\gamma_3(x)+\Gamma_3(x)\right]\right. \\
&\left.\left.+2\left(\frac{a+b}{2}-x\right)^3\left[\gamma_2(x)+\Gamma_2(x)\right]\right]\right| \\
\leq\ & \frac{1}{12(b-a)}\left[(x-a)^3\left[\left|\Gamma_1(x)-\gamma_1(x)\right|+\left|\Gamma_3(x)-\gamma_3(x)\right|\right]\right. \\
&\left.+2\left(\frac{a+b}{2}-x\right)^3\left|\Gamma_2(x)-\gamma_2(x)\right|\right]
\end{aligned}
$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. Taking the modulus identity (2.1) for $\lambda_1(x) = \frac{\gamma_1(x)+\Gamma_1(x)}{2}$, $\lambda_2(x) = \frac{\gamma_2(x)+\Gamma_2(x)}{2}$ and $\lambda_3(x) = \frac{\gamma_3(x)+\Gamma_3(x)}{2}$, since $f'' \in \overline{U}_{[a,x]}(\gamma_1, \Gamma_1) \cap \overline{U}_{[x,a+b-x]}(\gamma_2, \Gamma_2) \cap \overline{U}_{[a+b-x,b]}(\gamma_3, \Gamma_3)$, we have

$$\begin{aligned}
&\Bigg|\left(x - \frac{3a+b}{4}\right)\frac{f'(x) - f'(a+b-x)}{2} - \frac{f(x)+f(a+b-x)}{2} + \frac{1}{b-a}\int_a^b f(t)dt \\
&-\frac{1}{12(b-a)}\left[(x-a)^3\left[\gamma_1(x)+\Gamma_1(x)+\gamma_3(x)+\Gamma_3(x)\right]\right. \\
&\left.+2\left(\frac{a+b}{2} - x\right)^3\left[\gamma_2(x)+\Gamma_2(x)\right]\right]\Bigg| \\
\leq\ & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left|f''(t) - \frac{\gamma_1(x)+\Gamma_1(x)}{2}\right|dt\right. \\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t) - \frac{\gamma_2(x)+\Gamma_2(x)}{2}\right|dt \\
&\left.+\int_{a+b-x}^b (t-b)^2\left|f''(t) - \frac{\gamma_3(x)+\Gamma_3(x)}{2}\right|dt\right] \\
\leq\ & \frac{1}{4(b-a)}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|\int_a^x (t-a)^2\,dt + \left|\Gamma_2(x)-\gamma_2(x)\right|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right. \\
&\left.+\left|\Gamma_3(x)-\gamma_3(x)\right|\int_{a+b-x}^b (t-b)^2\,dt\right] \\
=\ & \frac{1}{12(b-a)}\left[(x-a)^3\left|\Gamma_1(x)-\gamma_1(x)\right| + 2\left(\frac{a+b}{2}-x\right)^3\left|\Gamma_2(x)-\gamma_2(x)\right|\right. \\
&\left.+(x-a)^3\left|\Gamma_3(x)-\gamma_3(x)\right|\right]
\end{aligned}$$

which completes the proof. □

Corollary 3. *Under assumption of Theorem 4,*

i) if we choose $x = a$, then we have

$$\begin{aligned}
&\left|\frac{b-a}{8}\left[f'(b)-f'(a)\right] - \frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)^2}{48}\left[\gamma_2(x)+\Gamma_2(x)\right]\right| \quad (3.1)\\
\leq\ & \frac{(b-a)^2}{48}\left|\Gamma_2(x)-\gamma_2(x)\right|,
\end{aligned}$$

ii) if we choose $x = \frac{a+b}{2}$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{96}\left[\gamma_1(x)+\Gamma_1(x)+\gamma_3(x)+\Gamma_3(x)\right]\right| \le \frac{(b-a)^2}{96}\left[\left|\Gamma_1(x)-\gamma_1(x)\right| + \left|\Gamma_3(x)-\gamma_3(x)\right|\right], \tag{3.2}$$

iii) if we choose $x = \frac{3a+b}{4}$, then we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right. \\ &\left. - \frac{(b-a)^2}{768}\left(\gamma_1(x)+\Gamma_1(x)+2\gamma_2(x)+2\Gamma_2(x)+\gamma_3(x)+\Gamma_3(x)\right)\right| \\ \le\; & \frac{(b-a)^2}{768}\left[\left|\Gamma_1(x)-\gamma_1(x)\right| + 2\left|\Gamma_2(x)-\gamma_2(x)\right| + \left|\Gamma_3(x)-\gamma_3(x)\right|\right]. \end{aligned} \tag{3.3}$$

Corollary 4. *If we choose $\Gamma_1(x) = -\Gamma_3(x)$ and $\gamma_1(x) = -\gamma_3(x)$ in (3.2), then*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{48}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|\right],$$

and choosing $\Gamma_1(x) = \Gamma_3(x) = -\Gamma_2(x)$ and $\gamma_1(x) = \gamma_3(x) = -\gamma_2(x)$ in (3.3),we have

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \\ \le\; & \frac{(b-a)^2}{192}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|\right]. \end{aligned}$$

4. Inequalities for Functions Whose Second Derivatives are of Bounded Variation

Assume that $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° (the interior of I) and $[a,b] \subset I^\circ$. Then, from (2.1), we have for $\lambda_1(x) = f''(a)$, $\lambda_2(x) =$

$\frac{f''(x)+f''(a+b-x)}{2}$ and $\lambda_3(x) = f''(b)$

$$
\begin{aligned}
(4.1) & \left(x - \frac{3a+b}{4}\right) \frac{f'(x) - f'(a+b-x)}{2} - \frac{f(x)+f(a+b-x)}{2} + \frac{1}{b-a}\int_a^b f(t)dt \\
& -\frac{1}{6(b-a)}\left[(x-a)^3\left(f''(a)+f''(b)\right) + \left(\frac{a+b}{2} - x\right)^3\left[f''(x)+f''(a+b-x)\right]\right] \\
= & \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2\left[f''(t) - f''(a)\right]dt + \right. \\
& \int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)^2\left[f''(t) - \frac{f''(x)+f''(a+b-x)}{2}\right]dt \\
& \left. + \int_{a+b-x}^b (t-b)^2\left[f''(t) - f''(b)\right]dt\right]
\end{aligned}
$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Theorem 5. *Let : $f : [a,b] \to \mathbb{C}$ be a twice differantiable function on I° (the interior of I) and $[a,b] \subset I^\circ$. If the second derivative f'' is of bounded variation on $[a,b]$, then*

$$
\begin{aligned}
(4.2) & \left|\left(x - \frac{3a+b}{4}\right)\frac{f'(x) - f'(a+b-x)}{2} - \frac{f(x)+f(a+b-x)}{2}\right. \\
& + \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{6(b-a)}\left[(x-a)^3\left(f''(a)+f''(b)\right)\right. \\
& \left.\left. + \left(\frac{a+b}{2} - x\right)^3\left[f''(x)+f''(a+b-x)\right]\right]\right| \\
\leq & \frac{1}{6(b-a)}\left[(x-a)^3\bigvee_a^x(f'') + \left(\frac{a+b}{2} - x\right)^3\bigvee_x^{a+b-x}(f'') + (x-a)^3\bigvee_{a+b-x}^b(f'')\right] \\
\leq & \frac{1}{6(b-a)}\begin{cases} \max\left\{(x-a)^3, \left(\frac{a+b}{2} - x\right)^3\right\}\bigvee_a^b(f''), \\[2ex] \left[2(x-a)^3 + \left(\frac{a+b}{2} - x\right)^3\right]\max\left\{\bigvee_a^x(f''), \bigvee_x^{a+b-x}(f''), \bigvee_{a+b-x}^b(f'')\right\} \end{cases}
\end{aligned}
$$

for any $x \in [a,b]$.

Proof. Taking modulus (4.1), we get

$$
\begin{aligned}
&\Bigg|\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}-\frac{f(x)+f(a+b-x)}{2} \\
&+\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{6(b-a)}\Big[(x-a)^3\left(f''(a)+f''(b)\right) \\
&+\left(\frac{a+b}{2}-x\right)^3\left[f''(x)+f''(a+b-x)\right]\Big]\Bigg| \\
\leq\ & \frac{1}{2(b-a)}\Bigg[\int_a^x (t-a)^2\left|f''(t)-f''(a)\right|dt \\
&+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2\left|f''(t)-\frac{f''(x)+f''(a+b-x)}{2}\right|dt \\
&+\int_{a+b-x}^b (t-b)^2\left|f''(t)-f''(b)\right|dt\Bigg]:=T.
\end{aligned}
\tag{4.3}
$$

Since f'' is of bounded variation on $[a,b]$, we get

$$
\left|f''(t)-f''(a)\right|\leq\bigvee_a^t(f'')\text{ for }t\in[a,x]\,,
\tag{4.4}
$$

$$
\begin{aligned}
&\left|f''(t)-\frac{f''(x)+f''(a+b-x)}{2}\right| \\
\leq\ & \frac{1}{2}\left[\left|f''(t)-f''(x)\right|+\left|f''(a+b-x)-f''(t)\right|\right] \\
\leq\ & \frac{1}{2}\bigvee_x^{a+b-x}(f'')\text{ for }t\in[x,a+b-x]
\end{aligned}
\tag{4.5}
$$

and

$$
\left|f''(t)-f''(b)\right|\leq\bigvee_t^b(f'')\text{ for }t\in[a+b-x,b]\,.
\tag{4.6}
$$

If we put (4.4)-(4.6) in (4.3), we have

$$\begin{aligned} T &\leq \frac{1}{2(b-a)}\left[\int_a^x (t-a)^2 \left(\bigvee_a^t (f'')\right) dt + \int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)^2 \left(\frac{1}{2}\bigvee_x^{a+b-x}(f'')\right) dt \right. \\ &\quad \left. + \int_{a+b-x}^b (t-b)^2 \left(\bigvee_t^b (f'')\right) dt\right] \\ &\leq \frac{1}{2(b-a)}\left[\bigvee_a^x (f'') \int_a^x (t-a)^2\, dt + \frac{1}{2}\bigvee_x^{a+b-x}(f'') \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt \right. \\ &\quad \left. + \bigvee_{a+b-x}^b (f'') \int_{a+b-x}^b (t-b)^2\, dt\right] \\ &= \frac{1}{6(b-a)}\left[(x-a)^3 \bigvee_a^x (f'') + \left(\frac{a+b}{2}-x\right)^3 \bigvee_x^{a+b-x}(f'') + (x-a)^3 \bigvee_{a+b-x}^b (f'')\right] \end{aligned}$$

which completes the proof of the first inequality in (4.2).

The proof of second inequality in (4.2) is obvious from properties of maximum. □

Corollary 5. *Under assumptions of Theorem 5,*

i) if we take $x = a$, we have the inequality

$$\begin{aligned} &\left|\frac{b-a}{8}\left[f'(b)-f'(a)\right] - \frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int_a^b f(t)dt - \frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]\right| \\ &\leq \frac{(b-a)^2}{48}\bigvee_a^b (f'') \end{aligned}$$

ii) if we take $x = \frac{a+b}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{48}\left[f''(a)+f''(b)\right]\right| \leq \frac{(b-a)^2}{48}\bigvee_a^b (f'')$$

iii) if we take $x = \frac{3a+b}{4}$, we get

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]\right. \\ &\left. -\frac{(b-a)^2}{384}\left[f''(a)+f''(b)+f''\left(\frac{3a+b}{4}\right)+f''\left(\frac{a+3b}{4}\right)\right]\right| \\ &\leq \frac{(b-a)^2}{384}\bigvee_a^b (f''). \end{aligned}$$

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